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Approximation of Fixed Point of Multivalued Mean Nonexpansive Mappings in CAT(0) spaces

Mujahid Abbas[a], Khushdil Ahmad[a,*], Khurram Shabbir[a]

[a] *Department of Mathematics, Government College University, Katchery Road 54000, Lahore Pakistan.*

Abstract

The aim of this paper is to present the convergence results to approximate the fixed points of multivalued mean nonexpansive mappings in CAT(0) spaces. Strong and Δ-convergence results are established for these mappings using F-iterative scheme. Moreover, a numerical example is given to show the convergence behavior of the iterative scheme for multivalued mean nonexpansive mappings. The results which we derived are generalization of many results existing in literature.

Keywords: Multivalued mean nonexpansive mapping, F-Iteration, CAT(0) Space.
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1. Introduction

Browder [9] and Gohde [16] proved that every nonexpansive mapping on a closed convex and bounded subset of uniformly convex Banach space has a fixed point. Kirk initiated fixed point theory for single and multivalued mapping in the framework of CAT(0) spaces which possess a nonlinear structure, for details we refer to [21, 19, 13, 26, 6]).
Kirk and Panyank [20] defined the concept of Δ-convergence in CAT(0) space. Dhompongsa and Panyank [14] obtained Δ-convergence of different iterative schemes in such spaces.

Zhang [32] was the first who introduced the mean nonexpansive mapping in Banach spaces and proved the existence and uniqueness of fixed points for such mappings using normal structure. Afterwards, these mappings were extensively studied see for example, [30], [31] and references mentioned therein.

Browder's demiclosedness principle [8] is one of the fundamental results in the theory of nonexpansive mappings which is given as follows:
Let C be a nonempty closed and convex subset of a uniformly convex Banach space $\mathcal{X}$ and $\mathcal{F}: C \to \mathcal{X}$ a nonexpansive mapping, then $I - \mathcal{F}$ is demiclosed at each $f \in \mathcal{X}$, that is, for any sequence $\{c_n\}$ in C, $c_n \rightharpoonup c$ and $(I - \mathcal{F})c_n \to y$ imply that $(I - \mathcal{F})c = f$, where the symbol $\rightharpoonup$ stands for a weak convergence.

*Corresponding author
Email address: khushdilahmad834@gmail.com (Khushdil Ahmad)

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This principle also holds in spaces satisfying the Opial's condition. It is well known that the demiclosedness principle plays a key role in studying the asymptotic behavior of nonexpansive mapping (See [8, 29, 24, 15]).

The purpose of this paper is to study the existence of fixed points of multivalued mean nonexpansive mappings and to obtain the demiclosed principle for such mappings in CAT(0) spaces. Moreover, we proved a Δ-convergence and strong convergence of F- iterative scheme for multivalued mean nonexpansive mappings. Finally, we present a numerical example to illustrate the convergence behavior of different iterative schemes.

2. Preliminaries

Let $\mathcal{X}$ be a metric space and C a nonempty subset of $\mathcal{X}$. A subset C is called proximinal if for each $t \in \mathcal{X}$, there exists an element $c \in C$ such that

$$d(t,c) = d(t,C) = \inf\{d(t,w) : w \in C\} \tag{2.1}$$

where $d(t,C)$ is the distance of the point t from the set C. We denote the family of nonempty closed bounded subsets of $\mathcal{X}$ by $\mathcal{FB}(\mathcal{X})$, the family of nonempty bounded proximinal subsets of $\mathcal{X}$ by $\Lambda(\mathcal{X})$, and the family of nonempty compact subsets of $\mathcal{X}$ by $\kappa(\mathcal{X})$. Define the Hausdroff distance, $H(.,.)$ on $\mathcal{FB}(\mathcal{X})$ by

$$H(P,Q) = \max\{\sup_{p\in P} d(p,Q), \sup_{q\in Q} d(q,P)\}. \tag{2.2}$$

The mapping H is called Pompeiu-Hausdorff metric induced by d.

An element w^* is fixed point of a multivalued mapping $\mathcal{F} : \mathcal{X} \longrightarrow \mathcal{FB}(\mathcal{X})$, if $w^* \in \mathcal{F}w^*$. The set $F(\mathcal{F})$ denotes the set of all fixed points of $\mathcal{F}$. A multivalued mapping $\mathcal{F} : \mathcal{X} \longrightarrow \mathcal{FB}(\mathcal{X})$ is called:

1. nonexpansive if
$$H(\mathcal{F}w, \mathcal{F}v) \leq d(w,v), \quad \forall \quad w,v \in \mathcal{X}.$$
2. quasi-nonexpansive if $F(\mathcal{F}) \neq \emptyset$ and for any $w^* \in F(\mathcal{F})$, we have
$$H(\mathcal{F}w, \mathcal{F}w^*) \leq d(w,w^*), \quad \forall \quad w \in \mathcal{X}.$$
3. satisfies the Condition (C) which was given in [3], if for all $w, v \in \mathcal{X}$
$$\frac{1}{2}d(w,\mathcal{F}w) \leq d(w,v) \quad \text{implies that} \quad H(\mathcal{F}w,\mathcal{F}v) \leq d(w,v). \tag{2.3}$$

Chen et al., [11] gave the concept of multivalued mean nonexpansive mapping in Banach space. We present the multivalued mean nonexpansive mapping in the frame work of CAT(0) Space which is as follows:

Definition 2.1. Let C be a nonempty subset of $\mathcal{X}$. A mapping $\mathcal{F} : C \to C$ is said to be mean nonexpansive if there exists nonnegative real numbers a and b satisfying $a + b \leq 1$ such that for all $w, v \in C$, the following holds:

$$d(\mathcal{F}w, \mathcal{F}v) \leq ad(w,v) + bd(w,\mathcal{F}v) \quad . \tag{2.4}$$

Obviously, nonexpansive mappings are mean nonexpansive mappings. A mean nonexpansive mapping is not necessarily continuous and hence not nonexpansive.

Example 2.2. [33] Let $\mathcal{F} : [0,1] \to [0,1]$ be a mapping defined by

$$\mathcal{F}w = \begin{cases} \frac{1-w}{3}, & w \in [0,1] \text{ is rational number;} \\ \frac{1+w}{5} & w \in [0,1] \text{ is irrational number.} \end{cases}$$

Then $\mathcal{F}$ is mean nonexpansive with $a = \frac{1}{3}$ and $b = \frac{2}{3}$. Note that, $\mathcal{F} : [0,1] \to [0,1]$ is not continuous at any point in $[0,1]$ except $w = \frac{1}{4}$; the fixed point of $\mathcal{F}$.

Nakprasit [25] gave an example of a mean nonexpansive mapping which is not Suzuki-generalized nonexpansive and showed that increasing mean nonexpansive mappings are Suzuki-generalized nonexpansive mappings.

Recall that, a geodesic path joining two points w and v in a metric space $\mathcal{X}$ is a map ϕ from a closed interval $[0, f] \subset \mathbb{R}$ to $\mathcal{X}$ such that $\phi(0) = w$, $\phi(f) = v$ and $d(\phi(t), \phi(t')) = |t - t'|$ for all $t, t' \in [0, f]$. In particular, $f = d(w, v)$. The image of ϕ is called the geodesic or metric segment joining w and v. If the image is unique, then it is denoted by $[w, v]$.

The space $(\mathcal{X}, d)$ is called the geodesic space if any two points of $\mathcal{X}$ are joined by a geodesic and $\mathcal{X}$ is said to be uniquely geodesic if for each $w, v \in \mathcal{X}$, there is exactly one metric segment which joins w and v. A subset C of $\mathcal{X}$ is called convex if C contains every geodesic segment joining any two of its points. A geodesic triangle $\triangle(w_1, w_2, w_3)$ in a geodesic metric space $(\mathcal{X}, d)$ consists of three points in $\mathcal{X}$ with w_1, w_2 and w_3 are the vertices of $\triangle$ and a geodesic segments between them are the sides of $\triangle$. A comparison triangle for $\triangle(w_1, w_2, w_3)$ in $(\mathcal{X}, d)$ is a triangle $\overline{\triangle}(w_1, w_2, w_3)$=$\triangle(\overline{w_1}, \overline{w_2}, \overline{w_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d(w_1, w_2) = d_{\mathbb{E}^2}(\overline{w_1}, \overline{w_2})$, $d(w_1, w_3) = d_{\mathbb{E}^2}(\overline{w_1}, \overline{w_2})$ and $d(w_2, w_3) = d_{\mathbb{E}^2}(\overline{w_2}, \overline{w_3})$.

Suppose that $\triangle$ is a geodesic triangle in E and $\overline{\triangle}$ is a comparison triangle for $\triangle$. A geodesic space is said to be a CAT(0) space, if all geodesic triangles of appropriate size satisfy the following comparison axiom called CAT(0) inequality:

$$d(u, v) \leq d_{\mathbb{E}^2}(\overline{u}, \overline{v}), \quad \text{for all } \ u, v \in \triangle, \quad \overline{u}, \overline{v} \in \overline{\triangle}.$$

For more details, we refer [7].

Lemma 2.3. *[10] Let $(\mathcal{X}, d)$ be a CAT(0) space. For $w, v \in \mathcal{X}$ and $t \in [0, 1]$, there exists a unique $\sigma \in [w, v]$ such that*

$$d(w, \sigma) = (1 - t)d(w, v) \text{ and } d(v, \sigma) = td(w, v).$$

We denote the unique point $\sigma \in [w, v]$ by $(1 - t)w \oplus tv$.

Lemma 2.4. *[14] Let $(\mathcal{X}, d)$ be a CAT(0) space. For $w, v, \sigma \in \mathcal{X}$ and $t \in [0, 1]$, we have the following inequalities:*

i) $d((1 - t)w \oplus tv, \sigma) \leq (1 - t)d(w, \sigma) + td(v, \sigma)$.
ii) $d((1 - t)w \oplus tv, \sigma)^2 \leq (1 - t)d(w, v)^2 + td(v, \sigma)^2 - t(1 - t)d(w, v)^2$.

Now we recall the concepts of asymptotic center and Δ-convergence in the frame work of CAT(0) spaces.

Definition 2.5. [4, 12] Let C be a bounded subset of a CAT(0) space $\mathcal{X}$ and $\{\eta_n\}$ a bounded sequence in $\mathcal{X}$, then:

i) Define a mapping $r(., \{w_n\}) : \mathcal{X} \to \mathbb{R}^+$ by

$$r(w, \{w_n\}) = \limsup_{n \to \infty} d(w_n, w).$$

For each $w \in \mathcal{X}$, the value $r(w, \{w_n\})$ is called asymptotic radius of $\{w_n\}$ at w.

ii) The asymptotic radius of $\{w_n\}$ relative to C is the number r given by

$$r = \inf\{r(w, \{w_n\}); w \in C\}.$$

We denote the asymptotic radius of $\{w_n\}$ relative to C by $r(C, \{w_n\})$.

iii) The asymptotic center of $\{w_n\}$ relative to C is the set $A(\{w_n\})$ of points in $\mathcal{X}$ for which the following holds:

$$r(C, \{w_n\}) = r(w, \{w_n\}),$$

that is,

$$A(\{w_n\}) = \{w \in \mathcal{X} : r(w, \{w_n\}) = r\}.$$

The set of all asymptotic centers of $\{w_n\}$ with respect to C is denoted by $A(C, \{w_n\})$. If the asymptotic radius and the asymptotic center are taken with respect to $\mathcal{X}$, then they are simply denoted by $r(\{w_n\})$ and $A(\{w_n\})$, respectively.
It is well-known that in uniformly convex Banach spaces as well as CAT(0) spaces; bounded sequences have unique asymptotic center with respect to closed and convex subset.

Definition 2.6. [27] A sequence $\{w_n\}$ in a CAT(0) space $\mathcal{X}$ is said to be $\triangle$-convergent to $w \in \mathcal{X}$ if w is the unique asymptotic center of every subsequence of $\{w_n\}$. In this case we write it as $\triangle - \lim_n w_n = w$ and called it $\triangle$-limit of $\{w_n\}$.

Definition 2.7. [28] A Banach space B is said to satisfy Opial's condition if for each weakly convergent sequence $\{w_n\}$ to $w \in C$,

$$\liminf_{n\to\infty} \|w_n - w\| < \liminf_{n\to\infty} \|w_n - v\|$$

holds, for $w \in C$ with $w \neq v$.

If for the given $\{w_n\} \subset \mathcal{X}$ such that $\{w_n\}$ $\triangle$-converges to w, we take $v \in \mathcal{X}$ such that $w \neq v$, then by the uniqueness of the asymptotic center, we have $\limsup_{n\to\infty} d(w_n, v) < \limsup_{n\to\infty} d(w_n, v)$. Thus, every CAT(0) space satisfies the Opial's condition.

Lemma 2.8. *[14] If C is a closed convex subset of a complete CAT(0) space and $\{w_n\}$ is a bounded sequence in C, then the asymptotic center of $\{w_n\}$ in C.*

Lemma 2.9. *[20] Every bounded sequence in a complete CAT(0) space admits a $\triangle$-convergent subsequence.*

Lemma 2.10. *[22] Let $\mathcal{X}$ be a complete CAT(0) Space and $w^* \in \mathcal{X}$. Suppose that $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0, 1)$ and $\{w_n\}$, $\{v_n\}$ are two sequences in $\mathcal{X}$ such that*

$$\begin{aligned} \limsup_{n\to\infty} d(w_n, w^*) &\leq d, \ \limsup_{n\to\infty} d(v_n, w^*) \leq d \text{ and} \\ \limsup_{n\to\infty} d(t_n w_n + (1 - t_n)v_n, w^*) &= d \text{ hold for some } d \geq 0. \end{aligned}$$

Then

$$\lim_{n\to\infty} d(w_n, v_n) = 0.$$

Theorem 2.11. *[13] Let C be a nonempty, closed and convex subset of a CAT(0) space $\mathcal{X}$ and $\mathcal{F} : C \longrightarrow \mathcal{FB}(C)$ a multivalued nonexpansive mapping. If $\Delta - \lim_n w_n = w$ and $\lim_{n\to\infty} d(w_n, \mathcal{F}w_n) = 0$, then w is fixed point of $\mathcal{F}$*

Definition 2.12. Let C be a nonempty subset of a CAT(0) space $\mathcal{X}$ and $\{w_n\}$ a sequence in $\mathcal{X}$. Then $\{w_n\}$ is called a Fejér monotone sequence with respect to C if for all $w \in C$ and $n \in \mathbb{N}$, we have

$$d(w_{n+1}, w) \leq d(w_n, w).$$

Proposition 2.13. *[17] Let C be a nonempty subset of $\mathcal{X}$ and $\{w_n\}$ a sequence in $\mathcal{X}$. Assume that $\mathcal{F} : C \to \mathcal{FB}(C)$ is any nonlinear mapping and the sequence $\{w_n\}$ is Fejér monotone with respect to C, then we have the following:*

1. *$\{w_n\}$ is bounded.*

2. The sequence $\{d(w_n, w^*)\}$ *is decreasing and converges for all* $w^* \in F(\mathcal{F})$.

3. $\lim_{n\to\infty} d(w_n, F(\mathcal{F}))$ *exists.*

Let C be a nonempty convex subset of CAT(0) space $\mathcal{X}$ and $\mathcal{F} : C \to \mathcal{FB}(C)$ with $w^* \in F(\mathcal{F})$. Then, Mann, Ishikawa and S-itrative process in setting of CAT(0) space are given in [23] and [2], respectively.

1. The S-iterative scheme is defined as:

$$\begin{aligned} v_n &= (1-\vartheta_n)w_n \oplus \vartheta_n u_n, \\ w_{n+1} &= (1-\theta_n)u_n \oplus \theta_n u'_n, \end{aligned} \tag{2.5}$$

where $u_n \in \mathcal{F}w_n, u'_n \in \mathcal{F}v_n$ and $d(u_n, u'_n) \leq H(\mathcal{F}w_n, \mathcal{F}v_n)$ and $d(u_{n+1}, u'_n) \leq H(\mathcal{F}w_{n+1}, \mathcal{F}v_n)$ and $\theta_n, \vartheta_n \in (0, 1)$.

2. The process of Abbas & Nazir for multivalued mapping is defined as follows:
Take $w_1 \in C$,

$$\begin{cases} u_n &= (1-\sigma_n)w_n \oplus \sigma_n p_n, \\ v_n &= (1-\vartheta_n)p_n \oplus \vartheta q_n, \\ w_{n+1} &= (1-\theta_n)r_n \oplus \theta_n q_n, \end{cases} \tag{2.6}$$

where $p_n \in \mathcal{F}w_n$, $q_n \in \mathcal{F}u_n$ and $r_n \in \mathcal{F}v_n$ such that $d(p_n, q_n) \leq H(\mathcal{F}w_n, \mathcal{F}u_n), d(r_n, q_n) \leq H(\mathcal{F}u_n, \mathcal{F}v_n)$, and $d(w_{n+1}, q_n) \leq H(\mathcal{F}w_{n+1}, \mathcal{F}v_n)$ and $\theta_n, \vartheta_n\sigma_n \in (0, 1)$.

3. The process of Thakur et.al for multivalued mapping is defined as;

$$\begin{cases} u_n &= (1-\sigma_n)w_n \oplus \sigma_n p_n, \\ v_n &= (1-\vartheta_n)u_n \oplus \vartheta q_n, \\ w_{n+1} &= (1-\theta_n)q_n \oplus \theta_n r_n, \end{cases} \tag{2.7}$$

where $p_n \in \mathcal{F}w_n$, $q_n \in \mathcal{F}u_n$ and $r_n \in \mathcal{F}v_n$ such that $d(p_n, q_n) \leq H(\mathcal{F}w_n, \mathcal{F}u_n), d(r_n, q_n) \leq H(\mathcal{F}u_n, \mathcal{F}v_n)$, and $d(w_{n+1}, q_n) \leq H(\mathcal{F}w_{n+1}, \mathcal{F}v_n)$ and $\theta_n, \vartheta_n\sigma_n \in (0, 1)$.

4. The process of M-iteration for multivalued mapping is defined as;

$$\begin{cases} u_n &= (1-\theta_n)w_n \oplus \theta_n p_n, \\ v_n &= q_n, \\ w_n &= r_n, \end{cases} \tag{2.8}$$

where $p_n \in \mathcal{F}w_n, q_n \in \mathcal{F}v_n, r_n \in \mathcal{F}u_n$, and $d(p_n, q_n) \leq H(\mathcal{F}w_n, \mathcal{F}u_n), d(p_n, r_n) \leq H(\mathcal{F}w_n, \mathcal{F}v_n)$, $d(q_n, r_n) \leq H(\mathcal{F}u_n, \mathcal{F}v_n)$ and $d(w_{n+1}, r_n) \leq H(\mathcal{F}w_{n+1}, \mathcal{F}v_n)$ and $\theta_n \in (0, 1)$.

Recently, Ali et.al., [5] introduced a new iterative scheme known as F-iterative scheme.
In this paper, we define an F-iterative scheme for multivalued mapping in the framework of $CAT(0)$ space is as follows:

$$\begin{cases} h_n &= (1-\theta_n)w_n \oplus \theta_n q_n, \\ u_n &= r_n, \\ v_n &= s_n, \\ w_{n+1} &= t_n, \end{cases} \tag{2.9}$$

where $q_n \in \mathcal{F}w_n, r_n \in \mathcal{F}h_n, s_n \in \mathcal{F}u_n, t_n \in \mathcal{F}v_n$ and $d(q_n, r_n) \leq H(\mathcal{F}w_n, \mathcal{F}h_n), d(r_n, s_n) \leq H(\mathcal{F}h_n, \mathcal{F}u_n)$, $d(s_n, t_n) \leq H(\mathcal{F}u_n, \mathcal{F}v_n)$ and $d(w_{n+1}, t_n) \leq H(\mathcal{F}w_{n+1}, \mathcal{F}v_n)$ and $\theta_n \in (0, 1)$.

3. Convergence Results

Theorem 3.1. *Let $(\mathcal{X}, d)$ be a CAT(0) space and let $\mathcal{F} : \mathcal{X} \to \mathcal{FB}(\mathcal{X})$ be a multivalued mean nonexpansive mapping* (2.4) *with $a + b \leq 1$, then $\mathcal{F}$ has a fixed point.*

Note that the proof of above theorem is same as Theorem 3.1 given in [11] which is proved in the context of Banach space. Hence, that is valid for metric spaces as well as in CAT(0) spaces.

Theorem 3.2. *Let C be a nonempty closed and convex subset of CAT(0) space $\mathcal{X}$ and $\mathcal{F} : C \to \mathcal{FB}(C)$ a multivalued mean nonexpansive mapping with $a + b \leq 1$. If $F(\mathcal{F}) \neq \emptyset$, then $F(\mathcal{F})$ is closed and convex.*

Proof. First, we show that $F(\mathcal{F})$ is closed.
Let $\{w_n\}$ be a sequence in $F(\mathcal{F})$ such that $\{w_n\}$ converges to some $w^* \in C$. We show that $w^* \in F(\mathcal{F})$ as follows:
Consider that

$$\begin{aligned} d(w_n, \mathcal{F}w^*) =& H(\mathcal{F}w_n, \mathcal{F}w^*) \\ \leq& ad(w_n, w^*) + bd(w_n, \mathcal{F}w^*) \\ d(w_n, \mathcal{F}w^*) - bd(w_n, \mathcal{F}w^*) \leq& ad(w_n, w^*) \\ d(w_n, \mathcal{F}w^*) \leq& \frac{a}{1-b} d(w_n, w^*). \end{aligned}$$

As, $a + b \leq 1$ implies $a \leq 1 - b$, so we have

$$d(w_n, \mathcal{F}w^*) \leq d(w_n, w^*). \tag{3.1}$$

So, $\lim_{n\to\infty} d(w_n, w^*) = 0$. By sandwich theorem, we obtain that

$$\lim_{n\to\infty} d(w_n, \mathcal{F}w^*) = 0.$$

By uniqueness of limit, we have $w^* \in \mathcal{F}w^*$. Hence, $F(\mathcal{F})$ is closed.
Next, we show $F(\mathcal{F})$ is convex. If $w, v \in F(\mathcal{F})$, then we have

$$\begin{aligned} d(w, \mathcal{F}v) =& H(\mathcal{F}w, \mathcal{F}v) \\ \leq& ad(w, v) + bd(w, \mathcal{F}v) \\ \leq& \frac{a}{1-b} d(w, v). \end{aligned}$$

As, $a + b \leq 1$ implies $a \leq 1 - b$, so we have

$$d(w, \mathcal{F}v) \leq d(w, v). \tag{3.2}$$

$$\begin{aligned} d(v, \mathcal{F}w) =& H(\mathcal{F}v, \mathcal{F}w) \\ \leq& ad(v, w) + bd(v, \mathcal{F}w) \\ \leq& \frac{a}{1-b} d(v, w). \end{aligned}$$

As, $a + b \leq 1$ implies $a \leq 1 - b$, we get

$$d(w, \mathcal{F}v) \leq d(w, v). \tag{3.3}$$

Using (3.2) and (3.3), we obtain that

$$\begin{aligned} d(w, v) \leq& d(w, \mathcal{F}v) + d(v, \mathcal{F}v) \\ \leq& d(w, v) + d(v, v) \\ \leq& d(w, v). \end{aligned} \tag{3.4}$$

It follows from (3.2) and (3.3) that $d(w, \mathcal{F}v) = d(w, v)$ and $d(v, \mathcal{F}w) = d(w, v)$, respectively. Indeed, if $d(w, \mathcal{F}v) < d(w, v)$ and $d(v, \mathcal{F}w) < d(w, v)$, then the inequality in (3.4) becomes strictly less than, which leads to the contradiction, $d(w, v) < d(w, v)$. Hence we have that $d(w, v) = d(w, v)$ and the result follows. □

Corollary 3.3. *Let C be a nonempty closed and convex subset of CAT(0) space $\mathcal{X}$ and $\mathcal{F} : C \to \mathcal{FB}(C)$ a multivalued nonexpansive mapping, if $F(\mathcal{F}) \neq \emptyset$, then $F(\mathcal{F})$ is closed and convex.*

Proof. Take $a = 1$ and $b = 0$ in Theorem 3.2 we get our required result. □

Theorem 3.4. *Let C be a nonempty closed and convex subset of complete CAT(0) space, $\mathcal{F} : C \to \mathcal{FB}(C)$ a multivalued mean nonexpansive mapping with $b < 1$, $\{w_n\}$ a bounded sequence in C such that $\lim_{n\to\infty} d(w_n, q_n) = 0$, where $q_n \in \mathcal{F}w_n$, If $\Delta - \lim_{n\to\infty} w_n = w^*$, then $w^* \in F(\mathcal{F})$.*

Proof. Since, $\{w_n\}$ is a bounded sequence in C, it follows from Lemma 2.8 that $\{w_n\}$ has an asymptotic center in C. Also, $\Delta - \lim_{n\to\infty} w_n = w^*$ gives $A(\{w_n\}) = \{w^*\}$.
Now , we have

$$\begin{aligned} d(w_n, \mathcal{F}w^*) \leq& d(w_n, q_n) + d(q_n, \mathcal{F}w^*) \\ \leq& d(w_n, q_n) + H(\mathcal{F}w_n, \mathcal{F}w^*) \\ \leq& d(w_n, q_n) + ad(w_n, w^*) + bd(w_n, \mathcal{F}w^*) \\ \leq& \frac{1}{1-b}\big[d(w_n, q_n) + ad(w_n, w^*)\big]. \end{aligned}$$

On taking lim sup as $n \to \infty$ on the both sides, we have

$$\begin{aligned} \limsup_{n\to\infty} d(w_n, \mathcal{F}w^*) &= \frac{1}{1-b}\big[\limsup_{n\to\infty} d(w_n, q_n) + a \limsup_{n\to\infty} d(w_n, w^*)\big] \\ r(\mathcal{F}w^*, \{w_n\}) = \limsup_{n\to\infty} d(w_n, \mathcal{F}w^*) &\leq \limsup_{n\to\infty} d(w_n, w^*) = r(w^*, \{w_n\}). \end{aligned}$$

As $\{w_n\}$ has an asymptotic center in C, we have $w^* \in F(\mathcal{F})$. □

Corollary 3.5. *Let C be a nonempty closed convex subset of a complete CAT(0) space $\mathcal{X}$, $\mathcal{F} : C \to \mathcal{FB}(C)$ a multivalued nonexpansive mapping and $\{w_n\}$ a bounded sequence in C such that $\lim_{n\to\infty} d(w_n, q_n) = 0$. If $\Delta - \lim_{n\to\infty} w_n = w^*$ then $w^* \in F(\mathcal{F})$.*

Lemma 3.6. *Let C be a nonempty closed and convex subset of a complete CAT(0) space $\mathcal{X}$ and $\mathcal{F} : C \to \mathcal{FB}(C)$ a multivalued mean nonexpansive mappings. Assume that $F(\mathcal{F}) \neq \emptyset$ and the sequence $\{w_n\}$ is defined by (2.9), then*

a. $\lim_{n\to\infty} d(w_n, w^*)$ *exists. for all* $w^* \in F(\mathcal{F})$.

b. $\lim_{n\to\infty} d(w_n, F(\mathcal{F}))$ *exists.*

Proof. Let $w^* \in F(\mathcal{F})$, then from (2.9), we have

$$\begin{aligned} d(w_{n+1}, w^*) =& d(t_n, w^*) \\ \leq& H(\mathcal{F}v_n, \mathcal{F}w^*) \\ \leq& ad(v_n, w^*) + bd(v_n, \mathcal{F}w^*) \\ \leq& ad(v_n, w^*) + bd(v_n, w^*) + bd(v_n, \mathcal{F}w^*) \\ \leq& [a + b]d(v_n, w^*) \\ \leq& d(v_n, w^*), \end{aligned} \tag{3.5}$$

$$\begin{aligned} d(v_n, w^*) =& d(s_n, w^*) \\ \leq& H(\mathcal{F}u_n, \mathcal{F}w^*) \\ \leq& ad(u_n, w^*) + bd(u_n, \mathcal{F}w^*) \\ \leq& ad(u_n, w^*) + bd(u_n, w^*) + bd(w^*, \mathcal{F}w^*) \\ \leq& [a+b]d(u_n, w^*) \\ \leq& d(u_n, w^*), \end{aligned} \tag{3.6}$$

$$\begin{aligned} d(u_n, w^*) =& d(r_n, w^*) \\ \leq& H(\mathcal{F}h_n, \mathcal{F}w^*) \\ \leq& ad(h_n, w^*) + bd(h_n, \mathcal{F}w^*) \\ \leq& ad(h_n, w^*) + bd(h_n, w^*) + bd(w^*, \mathcal{F}w^*) \\ \leq& [a+b]d(h_n, w^*) \\ \leq& d(h_n, w^*), \end{aligned} \tag{3.7}$$

$$\begin{aligned} d(h_n, w^*) =& d((1-\theta_n)w_n \oplus \theta_n q_n, w^*) \\ \leq& (1-\theta_n)d(w_n, w^*) \oplus \theta_n d(q_n, w^*) \\ \leq& (1-\theta_n)d(w_n, w^*) \oplus \theta_n H(\mathcal{F}w_n, \mathcal{F}w^*) \\ \leq& (1-\theta_n)d(w_n, w^*) \oplus \theta_n [ad(w_n, w^*) + bd(w_n, \mathcal{F}w^*)] \\ \leq& (1-\theta_n)d(w_n, w^*) \oplus \theta_n [a+b]d(w_n, w^*) \\ \leq& d(w_n, w^*). \end{aligned} \tag{3.8}$$

Using (3.8) in (3.7), we obtain that

$$d(u_n, w^*) \leq d(w_n, w^*). \tag{3.9}$$

From (3.9) and (3.6), we have

$$d(v_n, w^*) \leq d(w_n, w^*). \tag{3.10}$$

Using (3.10) in (3.5), we get that

$$d(w_{n+1}, w^*) \leq d(w_n, w^*). \tag{3.11}$$

Now, (3.11) implies that $\{w_n\}$ is Fejer monotone with respect to $w^* \in F(\mathcal{F})$. Thus by Proposition 2.13, we have that $\{w_n\}$ is bounded. Therefore, $\lim_{n\to\infty} d(w_n, w^*)$ exists for all $w^* \in F(\mathcal{F})$ and $\lim_{n\to\infty} d(w_n, F(\mathcal{F}))$ exists. □

Lemma 3.7. *Let C be a nonempty closed and convex subset of a complete CAT(0) space $\mathcal{X}$ and $\mathcal{F} : C \to \mathcal{FB}(C)$ a multivalued mean nonexpansive mappings. If $w^* \in F(\mathcal{F}) \neq \emptyset$ and the sequence $\{w_n\}$ is defined by (2.9), then* $\lim_{n\to\infty} d(w_n, q_n) = 0$.

Proof. From the above Lemma 3.6 we have $\lim_{n\to\infty} d(w_n, w^*)$ exists for each $w^* \in F(\mathcal{F})$. Suppose that

$$\lim_{n\to\infty} d(w_n, w^*) = d, \tag{3.12}$$

for some $d \geq 0$. If $d = 0$, then we are done.
Suppose that $d > 0$. It is clear from (3.10), (3.9) and (3.8) that

$$\begin{aligned} d(v_n, w^*) \leq& d(w_n, w^*), \\ d(u_n, w^*) \leq& d(w_n, w^*), \\ d(h_n, w^*) \leq& d(w_n, w^*). \end{aligned}$$

On taking lim sup as $n \to \infty$ on the both sides of the above inequalities, we get

$$\limsup_{n\to\infty} d(v_n, w^*) \leq d, \tag{3.13}$$

$$\limsup_{n\to\infty} d(u_n, w^*) \leq d, \tag{3.14}$$

$$\limsup_{n\to\infty} d(h_n, w^*) \leq d. \tag{3.15}$$

As,

$$\begin{aligned}\limsup_{n\to\infty} d(s_n, w^*) &\leq \limsup_{n\to\infty} H(\mathcal{F}u_n, \mathcal{F}w^*)\\ &\leq \limsup_{n\to\infty} \left[ad(u_n, w^*) + bd(u_n, \mathcal{F}w^*)\right]\\ &\leq \limsup_{n\to\infty} \left[(a+b)d(u_n, w^*)\right]\\ &\leq \limsup_{n\to\infty} d(u_n, w^*)\\ &\leq d. \end{aligned} \tag{3.16}$$

Similarly,

$$\limsup_{n\to\infty} d(r_n, w^*) \leq d, \tag{3.17}$$

$$\limsup_{n\to\infty} d(t_n, w^*) \leq d, \tag{3.18}$$

$$\limsup_{n\to\infty} d(q_n, w^*) \leq d. \tag{3.19}$$

Now,

$$\begin{aligned} d(w_{n+1}, w^*) =& d(t_n, w^*)\\ \leq& H(\mathcal{F}v_n, \mathcal{F}w^*)\\ \leq& ad(v_n, w^*) + bd(v_n, \mathcal{F}w^*)\\ \leq& [a+b]d(v_n, w^*)\\ \leq& d(v_n, w^*). \end{aligned}$$

On taking lim inf as $n \to \infty$ on the both sides, we have

$$d \leq \liminf_{n\to\infty} d(v_n, w^*). \tag{3.20}$$

So, from (3.20) and (3.13), we get

$$\lim_{n\to\infty} d(v_n, w^*) = d.$$

Also, from (3.7), (3.6) and (3.5) we have

$$d(w_{n+1}, w^*) \leq d(h_n, w^*).$$

On taking lim inf as $n \to \infty$ on the both sides, we get

$$d \leq \liminf_{n\to\infty} d(h_n, w^*). \tag{3.21}$$

By (3.21) and (3.15), we have

$$\lim_{n\to\infty} d(h_n, w^*) = d.$$

Note that,

$$
\begin{aligned}
d &= \lim_{n\to\infty} d(h_n, w^*) \\
&= \lim_{n\to\infty} d((1-\theta_n)w_n \oplus \theta_n q_n, w^*) \\
&\leq \lim_{n\to\infty} \big[(1-\theta_n)d(w_n, w^*) \oplus \theta_n d(q_n, w^*)\big] \\
&\leq \lim_{n\to\infty} \big[(1-\theta_n)d(w_n, w^*) \oplus \theta_n H(\mathcal{F}w_n, \mathcal{F}w^*)\big] \\
&\leq \lim_{n\to\infty} \big[(1-\theta_n)d(w_n, w^*) \oplus \theta_n \{ad(w_n, w^*) + bd(w_n, \mathcal{F}w^*)\}\big] \\
&\leq \lim_{n\to\infty} \big[(1-\theta_n)d(w_n, w^*) \oplus \theta_n (a+b)d(w_n, w^*) \\
&\leq \lim_{n\to\infty} \big[(1-\theta_n)d(w_n, w^*) \oplus \theta_n d(w_n, w^*) \\
&\leq d(w_n, w^*) \\
&\leq d.
\end{aligned} \tag{3.22}
$$

Hence,

$$
\lim_{n\to\infty} d((1-\theta_n)w_n \oplus \theta_n q_n, w^*) = d. \tag{3.23}
$$

So, from (3.23), (3.19), (3.12) and Lemma 2.10, we obtain that

$$
\lim_{n\to\infty} d(w_n, q_n) = 0. \tag{3.24}
$$

□

Theorem 3.8. *Let C be a nonempty closed and convex subset of a complete CAT(0) space and $\mathcal{F} : C \to \mathcal{FB}(C)$ a multivalued mean nonexpansive mapping such that $b < 1$. Suppose that $F(\mathcal{F}) \neq \emptyset$ and $\{w_n\}$ is a sequence given by* (2.9) *and $\Delta - \lim_n w_n = w$ and $\lim_{n\to\infty} d(w_n, q_n) = 0$ hold, then $w \in F(\mathcal{F})$.*

Proof. Since, $\Delta - \lim_n w_n = w$. From the Lemma 3.7, we have $\lim_{n\to\infty} d(w_n, q_n) = 0$, where $q_n \in \mathcal{F}w_n$. Now, for $a \in \mathcal{F}w$, we have

$$
\begin{aligned}
d(a, w_n) &\leq d(w, q_n) + d(q_n, w_n) \\
\limsup_{n\to\infty} d(a, w_n) &\leq \limsup_{n\to\infty} [d(a, q_n) + d(q_n, w_n)] \\
&\leq \limsup_{n\to\infty} d(a, q_n) + \limsup_{n\to\infty} d(q_n, w_n) \\
&\leq \limsup_{n\to\infty} d(a, q_n) \\
&\leq \limsup_{n\to\infty} H(\mathcal{F}w, \mathcal{F}w_n) \\
&\leq \limsup_{n\to\infty} [ad(w, w_n) + bd(w, \mathcal{F}w_n) \\
&\leq \limsup_{n\to\infty} d(w, w_n).
\end{aligned}
$$

By the uniqueness of the asymptotic center, we get $a = w$, where $a \in \mathcal{F}w$. □

Theorem 3.9. *Let C be a nonempty closed and convex subset of a complete CAT(0) space and $\mathcal{F} : C \to \mathcal{FB}(C)$ a multivalued mean nonexpansive mapping such that $b < 1$. Suppose $F(\mathcal{F}) \neq \emptyset$ and $\{w_n\}$ is a sequence given by* (2.9)*, then sequence $\{w_n\}$ Δ-converges to a fixed point of $\mathcal{F}$.*

Proof. It follows from Lemma 3.6 that $\lim_{n\to\infty} d(w_n, w^*)$ exists for each $w^* \in F(\mathcal{F})$.
So, $\{w_n\}$ is bounded and from Lemma 3.7,we have $\lim_{n\to\infty} d(w_n, q_n) = 0$ where $q_n \in \mathcal{F}w_n$. Let $W_\Delta(w_n = \bigcup A_c(z_n)$, where the union taken overall subsequences $\{z_n\}$ of $\{w_n\}$. We have to prove that $\{w_n\}$ is $\Delta-$

convergent to a fixed point of $\mathcal{F}$. First, we show $W_\Delta(w_n) \subset F(\mathcal{F})$. Let $a \in W_\Delta(w_n)$, then there exists a subsequence $\{a_n\}$ of $\{w_n\}$ such that $A(\{a_n\}) = \{a\}$. By Lemma 2.8 and Lemma 2.9, there exists a subsequence $\{b_n\}$ of $\{a_n\}$ such that $\Delta - \lim_n b_n = b$ and $b \in C$. Since $\lim_{n\to\infty} d(w_n, q_n) = 0$ and $\{b_n\}$ is also a subsequence of $\{w_n\}$, we have that $\lim_{n\to\infty} d(b_n, \mathcal{F}b_n) = 0$. By the above Theorem 3.8, we have $b \in \mathcal{F}b$ and hence $b \in F(\mathcal{F})$.
Now, we claim that $b = a$. If $b \neq a$, then we have

$$\begin{aligned}
\limsup_{n\to\infty} d(b_n, b) &< \limsup_{n\to\infty} d(b_n, a)\\
&\leq \limsup_{n\to\infty} d(a_n, a)\\
&\leq \limsup_{n\to\infty} d(a_n, b)\\
&\leq \limsup_{n\to\infty} d(w_n, b)\\
&= \limsup_{n\to\infty} d(b_n, b),
\end{aligned}$$

a contradiction and hence $b = a \in F(\mathcal{F})$.

We now show that $W_\Delta(w_n)$ is a singleton set. Assume that $\{a_n\}$ be a subsequence of $\{w_n\}$. From Lemma 2.8 and Lemma 2.9, there exists a subsequence $\{b_n\}$ of $\{a_n\}$ such that $\Delta - \lim_n b_n = b$. Assume that $A(\{a_n\}) = \{a\}$ and $A(\{w_n\}) = \{w\}$. As we have proved that $b = a$ and hence it is enough to show that $b = w$. By Lemma 3.6 $\{d(w_n, w^*)\}$ is convergent. By uniqueness of asymptotic center, we have

$$\begin{aligned}
\limsup_{n\to\infty} d(b_n, b) &< \limsup_{n\to\infty} d(b_n, w)\\
&\leq \limsup_{n\to\infty} d(w_n, w)\\
&< \limsup_{n\to\infty} d(w_n, b)\\
&= \limsup_{n\to\infty} d(b_n, b),
\end{aligned}$$

a contradiction that $b \neq w$ and hence $b = w \in F(\mathcal{F})$. Hence the result follows. □

Theorem 3.10. *Suppose that the assumption in the Theorem* 3.9 *holds, then the sequence* $\{w_n\}$ *defined by* (2.9) *converges strongly to* $w^* \in F(\mathcal{F})$ *if and only if* $\liminf_{n\to\infty} d(w_n, F(\mathcal{F})) = 0$, *where* $d(w_n, F(\mathcal{F})) = \inf\{d(w_n, w^*) : w^* \in F(\mathcal{F})\}$.

Proof. Suppose that the sequence $\{w_n\}$ converges strongly to $w^* \in F(\mathcal{F})$. Then, we have $\lim_{n\to\infty} d(w_n, w^*) = 0$ and $0 \leq d(w_n, w^*) \leq d(w_n, F(\mathcal{F}))$. It follows that $\lim_{n\to\infty} d(w_n, F(\mathcal{F})) = 0$ and hence $\liminf_{n\to\infty} d(w_n, F(\mathcal{F})) = 0$.

Conversely, Suppose that $\liminf_{n\to\infty} d(w_n, F(\mathcal{F})) = 0$. Then, from Lemma 3.7, we obtain that $\lim_{n\to\infty} d(w_n, F(\mathcal{F}))$ 0. Suppose that $\{w_{n_k}\}$ is any arbitrary subsequence of $\{w_n\}$ and $\{a_k\}$ is a sequence in $F(\mathcal{F})$ such that for all $n \geq 1$, we have

$$d(w_{n_k}, a_k) \leq \frac{1}{2^k}.$$

It follows from (3.11) that

$$d(w_{n_k+1}, a_k) \leq \frac{1}{2^k},$$

which implies that

$$\begin{aligned}
d(a_{k+1}, a_k) \leq& d(a_{k+1}, w_{n+1}) + d(w_{n+1}, a_k)\\
<& \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}.
\end{aligned}$$

This shows that $\{a_k\}$ is Cauchy sequence in $F(\mathcal{F})$. Also by the Theorem 3.2, $F(\mathcal{F})$ is closed subset of $\mathcal{X}$. Thus $\{a_k\}$ is convergent sequence in $F(\mathcal{F})$. Let $\lim_{n\to\infty} a_k = a^*$, then $a^* \in F(\mathcal{F})$ and we have

$$d(w_{n_k}, a^*) \leq d(w_{n_k}, a_k) + d(a_k, a^*) \to 0 \text{ as } n \to \infty,$$

which gives

$$\lim_{n\to\infty} d(w_{n_k}, a^*) = 0.$$

As, $\lim_{n\to\infty} d(w_n, a^*)$ exists, we conclude that the sequence $\{w_n\}$ converges strongly to $a^* \in F(\mathcal{F})$. $\square$

4. Numerical Example

Now, we present an example of CAT(0) space which is not Hilbert space.

Example 4.1. Consider a metric space l^2 which is defined as:

$$d(a,b) = \sqrt{\sum (a_n - b_n)^2},$$

where $a = (a_1, a_2, a_3, \cdots)$ and $b = (b_1, b_2, b_3, \cdots)$.
From [18] we can say that l^2 is a CAT(0) space and not a Hilbert space. For more related example see [1].

The following example illustrate that the mapping $\mathcal{F}$ is multivalued mean nonexpansive mappings but does not satisfy the Condition (C).

Example 4.2. Let $C = [1,4]$ and $\mathcal{X}$ a CAT (0) space equipped with a standard norm given by $d(w,v) = |w - v|$ and $\mathcal{F} : C \to \mathcal{FB}(C)$ is a multivalued mapping defined by

$$\mathcal{F}(w) = \begin{cases} [1, \frac{w+1}{2}] & if; \;\; 1 \leq w < 2 \\ \{1\} & if \;\; 2 \leq w \leq 4. \end{cases}$$

We have to show the following assertions.

(1) $\mathcal{F}$ does not satisfy the condition (C).

(2) $\mathcal{F}$ is multivalued Mean nonexpansive mapping.

To prove (1), we need to show that $\mathcal{F}$ does not satisfy that $\frac{1}{2}d(w, \mathcal{F}w) \leq d(w,v)$ implies $H(\mathcal{F}w, \mathcal{F}v) \leq d(w,v)$ for all $w, v \in C$.
Let $w \in \frac{3}{2}$ and $v \in \frac{7}{3}$, then we have

$$\begin{aligned} \frac{1}{2}d(w, \mathcal{F}w) =& \frac{1}{2}\left|\frac{3}{2} - \frac{\frac{3}{4}+1}{2}\right| \\ =& \frac{1}{2}\left|\frac{3}{2} - \frac{5}{4}\right| = \frac{1}{2}\left|\frac{1}{4}\right| \\ =& \frac{1}{8} = 0.125, \end{aligned} \tag{4.1}$$

and

$$d(w,v) = \left|\frac{3}{2} - \frac{7}{3}\right| = \frac{5}{6} = 0.8\overline{33}.$$

Also,

$$\begin{aligned} H(\mathcal{F}w, \mathcal{F}v) =& H(\mathcal{F}(\frac{3}{4}), \mathcal{F}(\frac{7}{3})) = H(\frac{5}{3}, \{1\}) \\ =& \frac{5}{4} = 1.25. \end{aligned} \tag{4.2}$$

So, from (4.1) and (4.2), we have $\frac{1}{2}d(w,\mathcal{F}w)\leq d(w,v)$ and from (4.2) and (4.2), we get that $H(\mathcal{F}w,\mathcal{F}v)>d(w,v)$. Hence $\mathcal{F}$ does not satisfy the condition (C).

Now, to prove (2) we have to show that $H(\mathcal{F}w,\mathcal{F}v)\leq ad(w,v)+bd(w,\mathcal{F}v)$.
For this we consider the following three cases. Choose, $a=\frac{2}{5}$ and $b=\frac{3}{5}$.
Case:1 Let $w,v\in[1,2)$, then

$$\begin{aligned}
ad(w,v)+bd(w,\mathcal{F}v) =&\frac{2}{5}\Big|w-v\Big|+\frac{3}{5}\Big|w-(\frac{v+1}{2})\Big|\\
\geq&\frac{1}{5}\Big|2w-2v-3(w-(\frac{v+1}{2}))\Big|\\
\geq&\frac{1}{5}\Big|\frac{4w-v+3}{2}\Big|\\
\geq&\frac{1}{2}\Big|w-v\Big|\\
=&H(\mathcal{F}w,\mathcal{F}v).
\end{aligned}$$

Case:2 Let $w,v\in[2,4]$. Then it is trivial.
Case:3 Let $w\in[1,2)$ and $v\in[2,4]$, then we have

$$\begin{aligned}
ad(w,v)+bd(w,\mathcal{F}v) =&\frac{2}{5}\Big|w-v\Big|+\frac{3}{5}\Big|w-1\Big|\\
\geq&\frac{1}{5}\Big|2w-2v-3w+3\Big|\\
\geq&\frac{1}{5}\Big|w+2v-3\Big|\\
\geq&\frac{1}{2}|w-1|\\
=&H(\mathcal{F}w,\mathcal{F}v).
\end{aligned}$$

Hence, in all cases of (2), $H(\mathcal{F}w,\mathcal{F}v)\leq ad(w,v)+bd(w,\mathcal{F}v)$.

Now, we compare the convergence of the iterative scheme (2.9) with other schemes. In table (1) we discuss the convergence of different iterative schemes. We choose the stopping criteria $\|w_n-w*\|<10^{-9}$ and fix the parameters $\alpha_n=\sqrt{\frac{n+1}{5n+1}}$, $\beta_n=\frac{1}{\sqrt{2n+5}}$ and $\gamma_n=\sqrt{\frac{n+3}{6n+3}}$, where w^* is a fixed point of the problem. It is shown that, F-iterative process converges faster than some other known iterative schemes for multivalued mean nonexpansive mappings.

We illustrate the convergence behaviour of other iterative schemes with the F-iterative scheme for different choices of parameters. For this, we choose the initial point $w_1=3.78$. Observe that, for different choices of parameters the F-iterative scheme (2.9) converges faster than other iterative schemes to the fixed point of multivalued mean nonexpansive mappings.

5. Conclusion

In this paper, we presented the multivalued version of F-iterative scheme and showed that it is faster than the other known iterative schemes in the setting of CAT (0) spaces. Further, we established the $\triangle$-convergence and strong convergence results for multivalued mean nonexpansive mappings. The proposed mapping and results generalize and extend various results of [25, 32, 33].

Iteration	Abbas	Thakur	M	F
1	3.7800000000	3.7800000000	3.7800000000	3.7800000000
2	1.9993232470	2.0198546017	1.5336139235	1.2668069618
3	1.3592255223	1.3741379167	1.1024258343	1.0256064586
4	1.1291303652	1.1372540561	1.0196603782	1.0024575473
5	1.0464183366	1.0503522233	1.0037737595	1.0002358600
6	1.0166859435	1.0184719232	1.0007243635	1.0000226364
7	1.0059980760	1.0067765021	1.0001390397	1.0000021725
8	1.0021561211	1.0024859881	1.0000266883	1.0000002085
9	1.0007750583	1.0009119951	1.0000051227	1.0000000200
10	1.0002786092	1.0003345692	1.0000009833	1.0000000019
11	1.0001001513	1.0001227381	1.0000001887	1.0000000002
12	1.0000360013	1.0000450270	1.0000000362	1.0000000000
13	1.0000129413	1.0000165183	1.0000000070	1.0000000000
14	1.0000046520	1.0000060598	1.0000000013	1.0000000000
15	1.0000016723	1.0000022231	1.0000000003	1.0000000000

Table 1: Convergence Behavior of Iterative schemes for multivalued Mean nonexpansive Mapping

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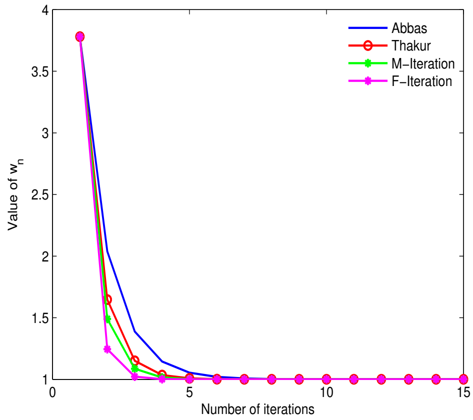
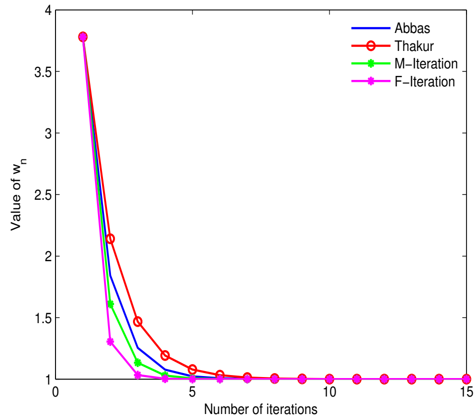
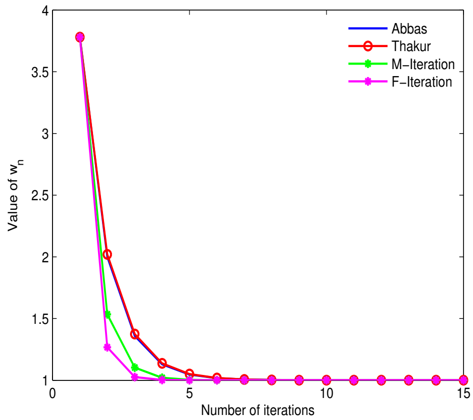


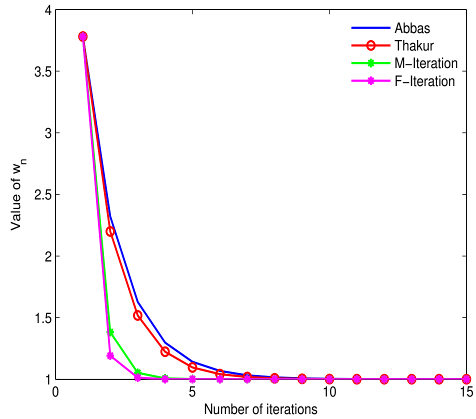
(a) $\alpha_n = \frac{n}{n+7}$, $\beta_n = \frac{n}{n+10}$,$\gamma_n = \frac{n}{n+1}$



(b) $\alpha_n = \frac{1}{\sqrt{n+7}}$, $\beta_n = \frac{1}{\sqrt{n+3}}$, $\gamma_n = \frac{1}{\sqrt{n+1}}$



(c) $\alpha_n = \sqrt{\frac{n+1}{5n+1}}$, $\beta_n = \frac{1}{\sqrt{2n+5}}$, $\gamma_n = \sqrt{\frac{n+3}{6n+3}}$



(d) $\alpha_n = \frac{n^2}{n^2+n+1}$, $\beta_n = \frac{n}{n^2+1}$, $\gamma_n = \frac{2n}{n^2+3}$

Figure 1: Comparison of Iteration Processes for different choices of parameters

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