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Open two-point Newton-Cotes integral inequalities for differentiable convex functions via Riemann-Liouville fractional integrals

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Abstract

In this paper, some open two-point Newton-Cotes type integral inequalities for functions whose first derivatives are convex via Riemann-Liouville fractional integrals are established. Our finding generalize some already known results. In order to illustrate the efficiency of our main results, some applications are given.

Keywords: Riemann-Liouville fractional integrals, convex functions, Hölder inequality, discrete power mean inequality.
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1. Introduction

Let I be a real interval

Definition 1.1. [21] A function $f : I \to \mathbb{R}$ is said to be convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

Convexity plays an important role in many fields, such as economics, finance, optimization, and game theory.

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This concept has a close relationship in the development of the theory of inequalities, which represents an important and powerful tool in the study of the properties of solutions of differential and integral equations as well as in the estimation of error of integration by quadrature methods.

The fundamental inequality for convex functions is the so-called Hermite-Hadamard inequality, which can be stated as follows: For every convex function f on the interval $[a,b]$ with $a<b$, we have

$$f\left(\tfrac{a+b}{2}\right) \leq \tfrac{1}{b-a}\int_a^b f(x)\,dx \leq \tfrac{f(a)+f(b)}{2}. \tag{1.2}$$

If the function f is concave, then (1.2) holds in the reverse direction see [10, 11].

The large family of quadratures related to inequality (1.2) is known as the n-point Newton-Cotes quadrature rule see [13, 16, 17, 19]. The downside of these methods is that the error term involves a high-order derivative and requires a lot of differentiation and computation.

In [7], Dragomir and Agarwal obtained the following closed two point Newton-Cotes type inequlities for differentiable convex functions called trapezoid inequality or Dragomir and Agarwal type inequalities

Theorem 1.2. *Let $f: I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentible mapping on $I^\circ, a,b \in I^\circ$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\tfrac{f(b)+f(a)}{2} - \tfrac{1}{b-a}\int_a^b f(u)\,du\right| \leq \tfrac{b-a}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right).$$

Theorem 1.3. *Let $f: I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentible mapping on $I^\circ, a,b \in I^\circ$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$, where $q,p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\left|\tfrac{f(a)+f(b)}{2} - \tfrac{1}{b-a}\int_a^b f(u)\,du\right| \leq \tfrac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\tfrac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

In [15], Latif and Dragomir gave the following trapezoid type inequalities for differentiable convex functions

Theorem 1.4. *Let $f: I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentible mapping on $I^\circ, a,b \in I^\circ$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\left|\tfrac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2} - \tfrac{1}{b-a}\int_a^b f(u)\,du\right| \\ \leq\ & \tfrac{b-a}{96}\left(\left|f'(a)\right| + 4\left|f'\left(\tfrac{3a+b}{4}\right)\right| + 2\left|f'\left(\tfrac{a+b}{2}\right)\right| + 4\left|f'\left(\tfrac{a+3b}{4}\right)\right| + \left|f'(b)\right|\right).\end{aligned}$$

Theorem 1.5. *Let $f: I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentible mapping on $I^\circ, a,b \in I^\circ$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$, where $q,p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\begin{aligned}&\left|\tfrac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2} - \tfrac{1}{b-a}\int_a^b f(u)\,du\right| \\ \leq\ & \tfrac{b-a}{16(p+1)^{\frac{1}{p}}}\left(\left(\tfrac{|f'(a)|^q+\left|f'\left(\frac{3a+b}{4}\right)\right|^q}{2}\right)^{\frac{1}{q}} + \left(\tfrac{\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right. \\ &\left. + \left(\tfrac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+3b}{4}\right)\right|^q}{2}\right)^{\frac{1}{q}} + \left(\tfrac{\left|f'\left(\frac{a+3b}{4}\right)\right|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right).\end{aligned}$$

Theorem 1.6. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentible mapping on $I^\circ, a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$, where $p \geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\tfrac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2} - \tfrac{1}{b-a}\int_a^b f(u)\,du\right| \\
\leq\; & \tfrac{b-a}{32}\left(\left(\tfrac{|f'(a)|^q+2\left|f'\left(\frac{3a+b}{4}\right)\right|^q}{3}\right)^{\frac{1}{q}} + \left(\tfrac{2\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{3}\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\tfrac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+2\left|f'\left(\frac{a+3b}{4}\right)\right|^q}{3}\right)^{\frac{1}{q}} + \left(\tfrac{2\left|f'\left(\frac{a+3b}{4}\right)\right|^q+|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

In [9], Guessab and Schmeisser proved the following companion of Ostrowski's inequality

Theorem 1.7. *Let $f : I \to \mathbb{R}$ satisfy the Lipschitz condition i.e.$|f(u) - f(v)| \leq M\,|u - v|$, then for each $x \in \left[a, \frac{a+b}{2}\right]$ we have*

$$\left|\tfrac{f(x)+f(a+b-x)}{2} - \tfrac{1}{b-a}\int_a^b f(t)\,dt\right| \leq (b-a)\left[\tfrac{1}{8} + 2\tfrac{\left(x-\frac{3a+b}{4}\right)^2}{(b-a)^2}\right] M.$$

Alomari et al. [1] discussed the following general two point Newton-Cotes for differentiable convex functions known as companion Ostrowski's inequalities

Theorem 1.8. *Let $f : [a, b] \subset I \to \mathbb{R}$ be an absolutely continuous mapping on I° with $a < b$, such that $f' \in L^1[a, b]$. If $|f'|$ is convex on $[a, b]$, then for all $x \in \left[a, \frac{a+b}{2}\right]$, we have*

$$\begin{aligned}
&\left|\tfrac{f(x)+f(a+b-x)}{2} - \tfrac{1}{b-a}\int_a^b f(u)\,du\right| \\
\leq\; & \tfrac{(x-a)^2}{6(b-a)}\left(\left|f'(a)\right| + \left|f'(b)\right|\right) + \tfrac{8(x-a)^2+3(a+b-2x)^2}{24(b-a)}\left(\left|f'(x)\right| + \left|f'(a+b-x)\right|\right).
\end{aligned}$$

Theorem 1.9. *Let $f : [a, b] \subset I \to \mathbb{R}$ be an absolutely continuous mapping on I° with $a < b$, such that $f' \in L^1[a, b]$. If $|f'|^q$ is convex function on $[a, b]$ where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for $x \in \left[a, \frac{a+b}{2}\right]$ we have*

$$\begin{aligned}
&\left|\tfrac{f(x)+f(a+b-x)}{2} - \tfrac{1}{b-a}\int_a^b f(u)\,du\right| \\
\leq\; & \left(\tfrac{1}{p+1}\right)^{\frac{1}{p}}\left(\tfrac{(x-a)^2}{b-a}\left(\left(\tfrac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}} + \left(\tfrac{|f'(a+b-x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right)\right. \\
&\left. + \tfrac{(a+b-2x)^2}{2(b-a)}\left(\tfrac{|f'(x)|^q+|f'(a+b-x)|^q}{2}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Fractional calculus is a branch of mathematical analysis whose study grows out of the classical definitions of integral and derivative operators of non-integer order and provide an excellent tool for the description of the memory and hereditary properties of various materials and processes.

Nowadays, fractional calculus has aroused the interest of many researchers and has become a very useful new mathematical method and a powerful tool due to its wide field of applications in different scientific fields such as physical and biological sciences as well as engineering sciences see [6, 12, 22, 23].

In the last decade various improvments, extensions and generalizations of certains integral inequalities have been obtained via different fractional integration operators see [2, 3, 5, 8, 18, 20, 25, 26].

Definition 1.10. [14] Let $f \in L^1[a,b]$. The Riemann-Liouville fractional integrals $I^{\alpha}_{a^+}f$ and $I^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$I^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

$$I^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$, is the gamma function and $I^0_{a^+}f(x) = I^0_{b^-}f(x) = f(x)$.

By using the above definition, sarikaya et al. [24], established the analogue fractional of inequality (1.2)

Theorem 1.11. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $\in L^1[a,b]$. If f is a convex function on $[a,b]$, then for $\alpha > 0$ the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(I^{\alpha}_{a^+}f(b) + I^{\alpha}_{b^-}f(a)\right) \leq \frac{f(a)+f(b)}{2}. \tag{1.3}$$

In the same paper the authors proved the following fractional closed two point Newton-Cotes type inequalities for differentiable convex functions

Theorem 1.12. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(b)+f(a)}{2} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(I^{\alpha}_{a^+}f(b) + I^{\alpha}_{b^-}f(a)\right)\right| \leq \frac{b-a}{2(\alpha+1)}\left(1 - \frac{1}{2^{\alpha}}\right)\left(\left|f'(a)\right| + \left|f'(b)\right|\right).$$

Motivated by the above results, in this paper we first establish a new integral identity, and then we derive some new open two-point Newton-Cotes type nequalities for fonctions whose first derivatives are convex via Riemann-Liouville fractional integrals.

2. Main results

We start this section with the following lemma, which represents a partial result.

Lemma 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $[a,b]$ with $a < b$. If $f' \in L^1[a,b]$, then the following equality holds for all $x \in \left(a, \frac{a+b}{2}\right)$*

$$\begin{aligned}\Lambda^{\alpha}(a,x,b;f) = (b-a)^{\alpha}\Bigg(&\left(\frac{x-a}{b-a}\right)^{\alpha+1}\int_0^1 t^{\alpha}f'((1-t)a+tx)\,dt \\ &- \left(\frac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\int_0^1 (1-t)^{\alpha} f'\left((1-t)x + t\frac{a+b}{2}\right)dt \\ &+ \left(\frac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\int_0^1 t^{\alpha}f'\left((1-t)\frac{a+b}{2} + t(a+b-x)\right)dt \\ &- \left(\frac{x-a}{b-a}\right)^{\alpha+1}\int_0^1 (1-t)^{\alpha} f'((1-t)(a+b-x)+tb)\,dt\Bigg),\end{aligned}$$

where

$$\begin{aligned}&\Lambda^{\alpha}(a,x,b;f) = \frac{2^{\alpha}(x-a)^{\alpha}+(a+b-2x)^{\alpha}}{2^{\alpha}(b-a)}\left(f(x) + f(a+b-x)\right) \\ &-\frac{\Gamma(\alpha+1)}{b-a}\left(I^{\alpha}_{x^-}f(a) + I^{\alpha}_{x^+}f\left(\tfrac{a+b}{2}\right) + I^{\alpha}_{(b+a-x)^-}f\left(\tfrac{a+b}{2}\right) + I^{\alpha}_{(a+b-x)^+}f(b)\right).\end{aligned} \tag{2.1}$$

Proof. Let

$$I=(x-a)^{\alpha+1}I_1-\left(\tfrac{a+b}{2}-x\right)^{\alpha+1}I_2+\left(\tfrac{a+b}{2}-x\right)^{\alpha+1}I_3-(x-a)^{\alpha+1}I_4, \tag{2.2}$$

where

$$\begin{aligned} I_1 &= \int_0^1 t^{\alpha} f'\left((1-t)a+tx\right)dt,\\ I_2 &= \int_0^1 (1-t)^{\alpha} f'\left((1-t)x+t\tfrac{a+b}{2}\right)dt,\\ I_3 &= \int_0^1 t^{\alpha} f'\left((1-t)\tfrac{a+b}{2}+t(a+b-x)\right)dt \end{aligned}$$

and

$$I_4=\int_0^1 (1-t)^{\alpha} f'\left((1-t)(a+b-x)+tb\right)dt.$$

Integrating by parts I_1, then using the change of variable, we get

$$\begin{aligned} I_1 &= \tfrac{1}{x-a}t^{\alpha}f\left((1-t)a+tx\right)\Big|_{t=0}^{t=1}-\tfrac{\alpha}{x-a}\int_0^1 t^{\alpha-1}f\left((1-t)a+tx\right)dt\\ &= \tfrac{1}{x-a}f(x)-\tfrac{\alpha}{(x-a)^{\alpha+1}}\int_a^x (u-a)^{\alpha-1}f(u)\,du\\ &= \tfrac{1}{x-a}f(x)-\tfrac{1}{(x-a)^{\alpha+1}}\Gamma(\alpha+1)I_{x^-}^{\alpha}f(a). \end{aligned} \tag{2.3}$$

Similarly, we have

$$I_2=-\tfrac{2}{a+b-2x}f(x)+\left(\tfrac{2}{a+b-2x}\right)^{\alpha+1}\Gamma(\alpha+1)I_{x^+}^{\alpha}f(\tfrac{a+b}{2}), \tag{2.4}$$

$$I_3=\tfrac{2}{a+b-2x}f(a+b-x)-\left(\tfrac{2}{a+b-2x}\right)^{\alpha+1}\Gamma(\alpha+1)I_{(b+a-x)^-}^{\alpha}f(\tfrac{a+b}{2}) \tag{2.5}$$

and

$$I_4=-\tfrac{1}{x-a}f(a+b-x)+\tfrac{1}{(x-a)^{\alpha+1}}\Gamma(\alpha+1)I_{(a+b-x)^+}^{\alpha}f(b). \tag{2.6}$$

Using (2.3)-(2.6) in (2.2), and then multiplying the resulting equality by $\frac{1}{b-a}$, we get the desired result. □

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be differentiable mapping on $[a,b]$ with $a<b$, such that $f'\in L^1[a,b]$. If $|f'|$ is convex, then for all $x\in\left(a,\frac{a+b}{2}\right)$ the following fractional inequality holds*

$$\begin{aligned} &\left|\Lambda^{\alpha}(a,x,b;f)\right|\\ \leq\ & \tfrac{(b-a)^{\alpha}}{(\alpha+1)(\alpha+2)}\Bigg(\left(\tfrac{x-a}{b-a}\right)^{\alpha+1}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)+2\left(\tfrac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\left|f'\left(\tfrac{a+b}{2}\right)\right|\\ &+(\alpha+1)\left(\left(\tfrac{x-a}{b-a}\right)^{\alpha+1}+\left(\tfrac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\right)\left(\left|f'(x)\right|+\left|f'(a+b-x)\right|\right)\Bigg). \end{aligned}$$

where $\Lambda^{\alpha}(a,x,b;f)$ is defined by (2.1).

Proof. From Lemma 2.1, properties of modulus and convexity of $|f'|$, we have

$$\begin{aligned}
&\left|\Lambda^{\alpha}(a,x,b;f)\right| \\
\leq\ & (b-a)^{\alpha}\left(\left(\tfrac{x-a}{b-a}\right)^{\alpha+1}\int_0^1 t^{\alpha}\left|f'((1-t)a+tx)\right|dt\right. \\
&+\left(\tfrac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\int_0^1 (1-t)^{\alpha}\left|f'\left((1-t)x+t\tfrac{a+b}{2}\right)\right|dt \\
&+\left(\tfrac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\int_0^1 t^{\alpha}\left|f'\left((1-t)\tfrac{a+b}{2}+t(a+b-x)\right)\right|dt \\
&\left.+\left(\tfrac{x-a}{b-a}\right)^{\alpha+1}\int_0^1 (1-t)^{\alpha}\left|f'((1-t)(a+b-x)+tb)\right|dt\right) \\
\leq\ & (b-a)^{\alpha}\left(\left(\tfrac{x-a}{b-a}\right)^{\alpha+1}\int_0^1 t^{\alpha}\left((1-t)\left|f'(a)\right|+t\left|f'(x)\right|\right)dt\right. \\
&+\left(\tfrac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\int_0^1 (1-t)^{\alpha}\left((1-t)\left|f'(x)\right|+t\left|f'\left(\tfrac{a+b}{2}\right)\right|\right)dt \\
&+\left(\tfrac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\int_0^1 t^{\alpha}\left((1-t)\left|f'\left(\tfrac{a+b}{2}\right)\right|+t\left|f'(a+b-x)\right|\right)dt \\
&\left.+\left(\tfrac{x-a}{b-a}\right)^{\alpha+1}\int_0^1 (1-t)^{\alpha}\left((1-t)\left|f'(a+b-x)\right|+t\left|f'(b)\right|\right)dt\right) \\
=\ & (b-a)^{\alpha}\left(\left(\tfrac{x-a}{b-a}\right)^{\alpha+1}\left|f'(a)\right|\int_0^1 t^{\alpha}(1-t)\,dt+\left(\tfrac{x-a}{b-a}\right)^{\alpha+1}\left|f'(x)\right|\int_0^1 t^{\alpha+1}dt\right. \\
&+\left(\tfrac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\left|f'(x)\right|\int_0^1 (1-t)^{\alpha+1}dt+\left(\tfrac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\left|f'\left(\tfrac{a+b}{2}\right)\right|\int_0^1 t(1-t)^{\alpha}dt \\
&+\left(\tfrac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\left|f'\left(\tfrac{a+b}{2}\right)\right|\int_0^1 t^{\alpha}(1-t)\,dt+\left(\tfrac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\left|f'(a+b-x)\right|\int_0^1 t^{\alpha+1}dt \\
&\left.+\left(\tfrac{x-a}{b-a}\right)^{\alpha+1}\left|f'(a+b-x)\right|\int_0^1 (1-t)^{\alpha+1}dt+\left(\tfrac{x-a}{b-a}\right)^{\alpha+1}\left|f'(b)\right|\int_0^1 t(1-t)^{\alpha}dt\right) \\
=\ & \tfrac{(b-a)^{\alpha}}{(\alpha+1)(\alpha+2)}\left(\left(\tfrac{x-a}{b-a}\right)^{\alpha+1}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)+2\left(\tfrac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\left|f'\left(\tfrac{a+b}{2}\right)\right|\right. \\
&\left.+(\alpha+1)\left(\left(\tfrac{x-a}{b-a}\right)^{\alpha+1}+\left(\tfrac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\right)\left(\left|f'(x)\right|+\left|f'(a+b-x)\right|\right)\right).
\end{aligned}$$

The proof is completed. □

Corollary 2.3. *In Theorem 2.2, if we take $\alpha = 1$, we obtain*

$$\begin{aligned} & \left| \frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\ \leq\; & \frac{(x-a)^2}{6(b-a)}\left(\left|f'(a)\right| + \left|f'(b)\right|\right) + \frac{(a+b-2x)^2}{12(b-a)}\left|f'\left(\frac{a+b}{2}\right)\right| \\ & + \frac{4(x-a)^2+(a+b-2x)^2}{12(b-a)}\left(\left|f'(x)\right| + \left|f'(a+b-x)\right|\right). \end{aligned}$$

Remark 2.4. Corollary 2.3 will be reduced to Theorem 1 from [15], if we choose $x = \frac{3a+b}{4}$.

Corollary 2.5. *In Theorem 2.2, if we use the convexity of $|f'|$ i.e.$\left|f'\left(\frac{a+b}{2}\right)\right| \leq \frac{|f'(a)|+|f'(b)|}{2}$, we get*

$$\begin{aligned} & \left|\Lambda^{\alpha}(a,x,b;f)\right| \\ \leq\; & \frac{2^{\alpha+1}(x-a)^{\alpha+1}+(a+b-2x)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(\alpha+2)(b-a)} \\ & \times\left(\left|f'(a)\right| + (\alpha+1)\left|f'(x)\right| + (\alpha+1)\left|f'(a+b-x)\right| + \left|f'(b)\right|\right). \end{aligned}$$

Corollary 2.6. *In Corollary 2.5, if we take $\alpha = 1$, we obtain*

$$\begin{aligned} & \left| \frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\ \leq\; & \frac{4(x-a)^2+(a+b-2x)^2}{24(b-a)}\left(\left|f'(a)\right| + 2\left|f'(x)\right| + 2\left|f'(a+b-x)\right| + \left|f'(b)\right|\right). \end{aligned}$$

Corollary 2.7. *In Theorem 2.2, if we use the convexity of $|f'|$ i.e.$\left|f'\left(\frac{a+b}{2}\right)\right| \leq \frac{|f'(x)|+|f'(a+b-x)|}{2}$, we get*

$$\begin{aligned} & \left|\Lambda^{\alpha}(a,x,b;f)\right| \\ \leq\; & \frac{(x-a)^{\alpha+1}}{(\alpha+1)(\alpha+2)(b-a)}\left(\left|f'(a)\right| + \left|f'(b)\right|\right) \\ & + \frac{2^{\alpha+1}(\alpha+1)(x-a)^{\alpha+1}+(\alpha+2)(a+b-2x)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(\alpha+2)(b-a)}\left(\left|f'(x)\right| + \left|f'(a+b-x)\right|\right). \end{aligned}$$

Remark 2.8. Corollary 2.7 will be reduced to Theorem 5 from [1], if we take $\alpha = 1$.

Theorem 2.9. *Let $f : [a,b] \to \mathbb{R}$ be differentiable mapping on $[a,b]$ with $a < b$, such that $f' \in L^1[a,b]$. If $|f'|^q$ is convex where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for all $x \in \left(a, \frac{a+b}{2}\right)$ the following fractional inequality holds*

$$\begin{aligned} & \left|\Lambda^{\alpha}(a,x,b;f)\right| \\ \leq\; & \left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{(x-a)^{\alpha+1}}{b-a}\left(\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}} + \left(\frac{|f'(a+b-x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right)\right. \\ & \left. + \frac{(a+b-2x)^{\alpha+1}}{2^{\alpha+1}(b-a)}\left(\left(\frac{|f'(x)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}} + \left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a+b-x)|^q}{2}\right)^{\frac{1}{q}}\right)\right). \end{aligned}$$

where $\Lambda^{\alpha}(a,x,b;f)$ is defined by (2.1).

Proof. From Lemma 2.1, properties of modulus and Hölder's inequality, we have

$$\begin{aligned} & \left|\Lambda^{\alpha}(a,x,b;f)\right| \\ \leq\; & (b-a)^{\alpha}\left(\left(\frac{x-a}{b-a}\right)^{\alpha+1}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'((1-t)a+tx)\right|^q dt\right)^{\frac{1}{q}}\right. \end{aligned}$$

$$+\left(\tfrac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\left(\int_0^1(1-t)^{p\alpha}\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)\,x+t\tfrac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)\,\tfrac{a+b}{2}+t\,(a+b-x)\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{x-a}{b-a}\right)^{\alpha+1}\left(\int_0^1(1-t)^{p\alpha}\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)\,(a+b-x)+tb\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg).$$

Since $|f'|^q$ is convex, we have

$$\begin{aligned}
&|\Lambda^{\alpha}(a,x,b;f)|\\
\leq\;&\left(\tfrac{1}{p\alpha+1}\right)^{\frac{1}{p}}\Bigg(\tfrac{(x-a)^{\alpha+1}}{b-a}\left(\left|f'(a)\right|^q\int_0^1(1-t)\,dt+\left|f'(x)\right|^q\int_0^1 tdt\right)^{\frac{1}{q}}\\
&+\tfrac{(a+b-2x)^{\alpha+1}}{2^{\alpha+1}(b-a)}\left(\left|f'(x)\right|^q\int_0^1(1-t)\,dt+\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\int_0^1 tdt\right)^{\frac{1}{q}}\\
&+\tfrac{(a+b-2x)^{\alpha+1}}{2^{\alpha+1}(b-a)}\left(\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\int_0^1(1-t)\,dt+\left|f'(a+b-x)\right|^q\int_0^1 tdt\right)^{\frac{1}{q}}\\
&+\tfrac{(x-a)^{\alpha+1}}{b-a}\left(\left|f'(a+b-x)\right|^q\int_0^1(1-t)\,dt+\left|f'(b)\right|^q\int_0^1 tdt\right)^{\frac{1}{q}}\Bigg)\\
=\;&\left(\tfrac{1}{p\alpha+1}\right)^{\frac{1}{p}}\Bigg(\tfrac{(x-a)^{\alpha+1}}{b-a}\left(\left(\tfrac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}+\left(\tfrac{|f'(a+b-x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right)\\
&+\tfrac{(a+b-2x)^{\alpha+1}}{2^{\alpha+1}(b-a)}\left(\left(\tfrac{|f'(x)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\tfrac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a+b-x)|^q}{2}\right)^{\frac{1}{q}}\right)\Bigg).
\end{aligned}$$

The proof is completed. □

Corollary 2.10. *In Theorem 2.9, if we take $\alpha=1$, we obtain*

$$\begin{aligned}
&\left|\tfrac{f(x)+f(a+b-x)}{2}-\tfrac{1}{b-a}\int_a^b f(u)\,du\right|\\
\leq\;&\left(\tfrac{1}{p+1}\right)^{\frac{1}{p}}\Bigg(\tfrac{(x-a)^2}{b-a}\left(\left(\tfrac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}+\left(\tfrac{|f'(a+b-x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right)\\
&+\tfrac{(a+b-2x)^2}{4(b-a)}\left(\left(\tfrac{|f'(x)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\tfrac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a+b-x)|^q}{2}\right)^{\frac{1}{q}}\right)\Bigg).
\end{aligned}$$

Remark 2.11. Corollary 2.10 will be reduced to Theorem 2 from [15], if we choose $x=\frac{3a+b}{4}$.

Corollary 2.12. *In Theorem 2.9, if we use the convexity of $|f'|$ i.e. $\left|f'\left(\frac{a+b}{2}\right)\right|^q\leq\frac{|f'(x)|^q+|f'(a+b-x)|^q}{2}$, we get*

$$|\Lambda^{\alpha}(a,x,b;f)|$$

$$\leq \left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{(x-a)^{\alpha+1}}{b-a}\left(\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(a+b-x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right)\right.$$
$$\left.+\frac{(a+b-2x)^{\alpha+1}}{2^{\alpha+1}(b-a)}\left(\left(\frac{3|f'(x)|^q+|f'(a+b-x)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(x)|^q+3|f'(a+b-x)|^q}{4}\right)^{\frac{1}{q}}\right)\right).$$

Corollary 2.13. *In Corollary 2.12, if we take $\alpha=1$, we obtain*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{(x-a)^2}{b-a}\left(\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(a+b-x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right)\right.$$
$$\left.+\frac{(a+b-2x)^2}{4(b-a)}\left(\left(\frac{3|f'(x)|^q+|f'(a+b-x)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(x)|^q+3|f'(a+b-x)|^q}{4}\right)^{\frac{1}{q}}\right)\right).$$

Corollary 2.14. *In Corollary 2.12, if we use the discrete power mean inequality i.e. $a^r+b^r\leq 2^{1-r}(a+b)^r$ for all $a,b>0$ and $0\leq r\leq 1$ (see [4]), we obtain*

$$|\Lambda^\alpha(a,x,b;f)|$$
$$\leq \left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{(x-a)^{\alpha+1}}{b-a}\left(\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(a+b-x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right)\right.$$
$$\left.+\frac{(a+b-2x)^{\alpha+1}}{2^\alpha(b-a)}\left(\frac{|f'(x)|^q+|f'(a+b-x)|^q}{2}\right)^{\frac{1}{q}}\right).$$

Remark 2.15. Corollary 2.14 will be reduced to Theorem 6 from [1], if we take $\alpha=1$.

Theorem 2.16. *Let $f:[a,b]\to\mathbb{R}$ be differentiable mapping on $[a,b]$ with $a<b$, such that $f'\in L^1[a,b]$. If $|f'|^q$ is convex where $q\geq 1$, then for all $x\in\left(a,\frac{a+b}{2}\right)$ the following fractional inequality holds*

$$|\Lambda^\alpha(a,x,b;f)|$$
$$\leq \frac{(x-a)^{\alpha+1}}{(\alpha+1)(b-a)}\left(\left(\frac{|f'(a)|^q+(\alpha+1)|f'(x)|^q}{\alpha+2}\right)^{\frac{1}{q}}+\left(\frac{(\alpha+1)|f'(a+b-x)|^q+|f'(b)|^q}{\alpha+2}\right)^{\frac{1}{q}}\right)$$
$$+\frac{(a+b-2x)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)}$$
$$\times\left(\left(\frac{(\alpha+1)|f'(x)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{\alpha+2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+(\alpha+1)|f'(a+b-x)|^q}{\alpha+2}\right)^{\frac{1}{q}}\right).$$

where $\Lambda^\alpha(a,x,b;f)$ is defined by (2.1) *and $x\in\left(a,\frac{a+b}{2}\right)$.*

Proof. From Lemma 2.1, properties of modulus and power mean inequality, we have

$$|\Lambda^\alpha(a,x,b;f)|$$
$$\leq (b-a)^\alpha\left(\left(\frac{x-a}{b-a}\right)^{\alpha+1}\left(\left(\int_0^1 t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^\alpha\left|f'((1-t)a+tx)\right|^q dt\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\int_0^1(1-t)^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^\alpha\left|f'((1-t)(a+b-x)+tb)\right|^q dt\right)^{\frac{1}{q}}\right)$$
$$+\left(\frac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\left(\left(\int_0^1 t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^\alpha\left|f'\left((1-t)\frac{a+b}{2}+t(a+b-x)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1 (1-t)^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)^\alpha \left|f'\left((1-t)x+t\tfrac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg)\Bigg).$$

Since $|f'|^q$ is convex, we have

$$\begin{aligned}
&|\Lambda^\alpha(a,x,b;f)| \\
\leq\ & (b-a)^\alpha\left(\tfrac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \\
&\times\Bigg(\left(\tfrac{x-a}{b-a}\right)^{\alpha+1}\Bigg(\left(|f'(a)|^q\int_0^1 t^\alpha(1-t)\,dt+|f'(x)|^q\int_0^1 t^{\alpha+1}dt\right)^{\frac{1}{q}} \\
&+\left(|f'(a+b-x)|^q\int_0^1(1-t)^{\alpha+1}dt+|f'(b)|^q\int_0^1 t(1-t)^\alpha dt\right)^{\frac{1}{q}}\Bigg) \\
&+\left(\tfrac{a+b-2x}{2(b-a)}\right)^{\alpha+1}\Bigg(\left(|f'(x)|^q\int_0^1(1-t)^{\alpha+1}dt+\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\int_0^1 t(1-t)^\alpha dt\right)^{\frac{1}{q}} \\
&+\left(\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\int_0^1 t^\alpha(1-t)\,dt+|f'(a+b-x)|^q\int_0^1 t^{\alpha+1}dt\right)^{\frac{1}{q}}\Bigg) \\
=\ & \tfrac{(x-a)^{\alpha+1}}{(\alpha+1)(b-a)}\left(\left(\tfrac{|f'(a)|^q+(\alpha+1)|f'(x)|^q}{\alpha+2}\right)^{\frac{1}{q}}+\left(\tfrac{(\alpha+1)|f'(a+b-x)|^q+|f'(b)|^q}{\alpha+2}\right)^{\frac{1}{q}}\right) \\
&+\tfrac{(a+b-2x)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)} \\
&\times\left(\left(\tfrac{(\alpha+1)|f'(x)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{\alpha+2}\right)^{\frac{1}{q}}+\left(\tfrac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+(\alpha+1)|f'(a+b-x)|^q}{\alpha+2}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

The proof is completed. □

Corollary 2.17. *In Theorem 2.16, if we take $\alpha=1$, we obtain*

$$\begin{aligned}
&\left|\tfrac{f(x)+f(a+b-x)}{2}-\tfrac{1}{b-a}\int_a^b f(u)\,du\right| \\
\leq\ & \tfrac{(x-a)^2}{2(b-a)}\left(\left(\tfrac{|f'(a)|^q+2|f'(x)|^q}{3}\right)^{\frac{1}{q}}+\left(\tfrac{2|f'(a+b-x)|^q+|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right) \\
&+\tfrac{(a+b-2x)^2}{8(b-a)}\left(\left(\tfrac{2|f'(x)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{3}\right)^{\frac{1}{q}}+\left(\tfrac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+2|f'(a+b-x)|^q}{3}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Remark 2.18. Corollary 2.17 will be reduced to Theorem 3 from [15], if we choose $x=\frac{3a+b}{4}$.

Corollary 2.19. *In Theorem 2.16, if we use the convexity of $|f'|$ i.e.$\left|f'\left(\frac{a+b}{2}\right)\right|^q\leq\frac{|f'(x)|^q+|f'(a+b-x)|^q}{2}$, we get*

$$\begin{aligned}
&|\Lambda^\alpha(a,x,b;f)| \\
\leq\ & \tfrac{(x-a)^{\alpha+1}}{(\alpha+1)(b-a)}\left(\left(\tfrac{|f'(a)|^q+(\alpha+1)|f'(x)|^q}{\alpha+2}\right)^{\frac{1}{q}}+\left(\tfrac{(\alpha+1)|f'(a+b-x)|^q+|f'(b)|^q}{\alpha+2}\right)^{\frac{1}{q}}\right)
\end{aligned}$$

$$+\tfrac{(a+b-2x)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(b-a)}\left(\left(\tfrac{(2\alpha+3)|f'(x)|^q+|f'(a+b-x)|^q}{2(\alpha+2)}\right)^{\frac{1}{q}}+\left(\tfrac{|f'(x)|^q+(2\alpha+3)|f'(a+b-x)|^q}{2(\alpha+2)}\right)^{\frac{1}{q}}\right).$$

Corollary 2.20. *In Corollary 2.19, if we take $\alpha=1$, we obtain*

$$\begin{aligned}&\left|\tfrac{f(x)+f(a+b-x)}{2}-\tfrac{1}{b-a}\int_a^b f(u)\,du\right|\\ \leq\ &\tfrac{(x-a)^2}{2(b-a)}\left(\left(\tfrac{|f'(a)|^q+2|f'(x)|^q}{3}\right)^{\frac{1}{q}}+\left(\tfrac{2|f'(a+b-x)|^q+|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right)\\ &+\tfrac{(a+b-2x)^2}{8(b-a)}\left(\left(\tfrac{5|f'(x)|^q+|f'(a+b-x)|^q}{6}\right)^{\frac{1}{q}}+\left(\tfrac{|f'(x)|^q+5|f'(a+b-x)|^q}{6}\right)^{\frac{1}{q}}\right).\end{aligned}$$

Corollary 2.21. *In Corollary 2.19, if we use the discrete power mean inequality, we obtain*

$$\begin{aligned}&|\Lambda^{\alpha}(a,x,b;f)|\\ \leq\ &\tfrac{(x-a)^{\alpha+1}}{(\alpha+1)(b-a)}\left(\left(\tfrac{|f'(a)|^q+(\alpha+1)|f'(x)|^q}{\alpha+2}\right)^{\frac{1}{q}}+\left(\tfrac{(\alpha+1)|f'(a+b-x)|^q+|f'(b)|^q}{\alpha+2}\right)^{\frac{1}{q}}\right)\\ &+\tfrac{(a+b-2x)^{\alpha+1}}{2^{\alpha}(\alpha+1)(b-a)}\left(\tfrac{|f'(x)|^q+|f'(a+b-x)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 2.22. *In Corollary 2.21, if we take $\alpha=1$, we obtain*

$$\begin{aligned}&\left|\tfrac{f(x)+f(a+b-x)}{2}-\tfrac{1}{b-a}\int_a^b f(u)\,du\right|\\ \leq\ &\tfrac{(x-a)^2}{2(b-a)}\left(\left(\tfrac{|f'(a)|^q+2|f'(x)|^q}{3}\right)^{\frac{1}{q}}+\left(\tfrac{2|f'(a+b-x)|^q+|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right)\\ &+\tfrac{(a+b-2x)^2}{4(b-a)}\left(\tfrac{|f'(x)|^q+|f'(a+b-x)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

3. Applications

Let Υ be the partition of the points $a=x_0<x_1<...<x_n=b$ of the interval $[a,b]$, and consider the quadrature formula

$$\int_a^b f(u)\,du=\lambda(f,\Upsilon)+R(f,\Upsilon),$$

where

$$\lambda(f,\Upsilon)=\sum_{i=0}^{n-1}\tfrac{(x_{i+1}-x_i)}{2}\left(f\left(\tfrac{5x_i+x_{i+1}}{6}\right)+f\left(\tfrac{x_i+5x_{i+1}}{6}\right)\right)$$

and $R(f,\Upsilon)$ denotes the associated approximation error.

Proposition 3.1. *Let $n\in\mathbb{N}$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $0\leq a<b$ and $f'\in L^1[a,b]$. If $|f'|$ is convex function, we have*

$$\begin{aligned}&|R(f,\Upsilon)|\leq\sum_{i=0}^{n-1}\tfrac{5(x_{i+1}-x_i)^2}{36}\\ &\times\left(\tfrac{|f'(x_i)|+10\left|f'\left(\frac{5x_i+x_{i+1}}{6}\right)\right|+8\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|+10\left|f'\left(\frac{x_i+5x_{i+1}}{6}\right)\right|+|f'(x_{i+1})|}{30}\right).\end{aligned}$$

Proof. Applying Corollary 2.3 with $x = \frac{5x_i+x_{i+1}}{6}$ on the subintervals $[x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$ of the partition Υ, we get

$$\left| \frac{f\left(\frac{5x_i+x_{i+1}}{6}\right)+f\left(\frac{x_i+5x_{i+1}}{6}\right)}{2} - \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(u)\,du \right| \tag{3.1}$$
$$\leq \frac{5(x_{i+1}-x_i)}{36}\left(\frac{|f'(x_i)|+10\left|f'\left(\frac{5x_i+x_{i+1}}{6}\right)\right|+8\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|+10\left|f'\left(\frac{x_i+5x_{i+1}}{6}\right)\right|+|f'(x_{i+1})|}{30}\right).$$

Multiplying both sides of (3.1) by $(x_{i+1} - x_i)$, then adding the resulting inequalities for all $i = 0, 1, ..., n-1$, and using the triangle inequality, we get the desired result. □

3.1. Application to special means

For arbitrary real numbers $a, b, a_1, a_2, ..., a_n$ we have:
The Arithmetic mean: $A(a_1, a_2, ..., a_n) = \frac{a_1+a_2+...+a_n}{n}$.
The p-Logarithmic mean: $L_p(a, b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $a, b > 0, a \neq b$ and $p \in \mathbb{R}\setminus\{-1, 0\}$.

Proposition 3.2. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, then we have*

$$\left|A^2(a,a,b) + A^2(a,b,b) - 2L_2^2(a,b)\right|$$
$$\leq \frac{b-a}{9}\left(2\left(\frac{a^q+2\left(\frac{2a+b}{3}\right)^q}{3}\right)^{\frac{1}{q}} + 2\left(\frac{2\left(\frac{a+2b}{3}\right)^q+b^q}{3}\right)^{\frac{1}{q}} + \left(\frac{\left(\frac{2a+b}{3}\right)^q+\left(\frac{a+2b}{3}\right)^q}{2}\right)^{\frac{1}{q}}\right).$$

Proof. The assertion follows from Corollary 2.22, with $x = \frac{2a+b}{3}$ and $q \geq 2$, applied to the function $f(x) = x^2$. □

References

[1] M. W. Alomari, M. E. Özdemir and H. Kavurmac, *On companion of Ostrowski inequality for mappings whose first derivatives absolute value are convex with applications.* Miskolc Math. Notes **13** (2012), no. 2, 233–248. 1, 2.8, 2.15

[2] D. Baleanu, A. Kashuri, P.O. Mohammed, and B. Meftah, *General Raina fractional integral inequalities on coordinates of convex functions. Adv. Difference Equ.* 2021, Paper no. 82, 23 pp. 1

[3] S. Belarbi and Z. Dahmani, *On some new fractional integral inequalities.* JIPAM. J. Inequal. Pure Appl. Math. **10** (2009), no. 3, Article 86, 5 pp. 1

[4] P. S. Bullen, *Handbook of means and their inequalities.* Revised from the 1988 original [P. S. Bullen, D. S. Mitrinović and P. M. Vasić, Means and their inequalities, Reidel, Dordrecht; MR0947142]. Mathematics and its Applications, 560. Kluwer Academic Publishers Group, Dordrecht, 2003. 2.14

[5] Z. Dahmani, *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration.* Ann. Funct. Anal. **1** (2010), no. 1, 51–58. 1

[6] L. Debnath, *Recent applications of fractional calculus to science and engineering.* Int. J. Math. Math. Sci. **54** (2003) 3413–3442. 1

[7] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula.* Appl. Math. Lett. **11** (1998), no. 5, 91–95. 1

[8] D. Foukrach and B. Meftah, *Some new generalized result of Gronwall-Bellman-Bihari type inequality with some singularity.* Filomat. **34** (2020), no. 10, 3299–3310. 1

[9] A. Guessab and G. Schmeisser, *Sharp integral inequalities of the Hermite-Hadamard type.* J. Approx. Theory **115** (2002), no. 2, 260–288. 1

[10] J. Hadamard, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl., **58** (1893), 171-215. 1

[11] C. Hermite, *Sur deux limites d'une intégrale définie*, Mathesis **3** (1883), 82. 1

[12] C. Ionescu, A. Lopes, D. Copot, J. A. T. Machado and J. H. T. Bates, *The role of fractional calculus in modeling biological phenomena: a review.* Commun. Nonlinear Sci. Numer. Simul. **51** (2017) 141–159. 1

[13] H. Kalsoom and S. Hussain, *Some Hermite-Hadamard type integral inequalities whose n-times differentiable functions are s-logarithmically convex functions.* Punjab Univ. J. Math. (Lahore) **51** (2019), no. 10, 65–75. 1

[14] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations. North-Holland Mathematics Studies*, 204. Elsevier Science B.V., Amsterdam, 2006. 1.10

[15] M. A. Latif and S. S. Dragomir, *New inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value are convex with applications.* [Paging previously given as 24–39]. Acta Univ. M. Belii Ser. Math. **21** (2013), 27–42. 1, 2.4, 2.11, 2.18

[16] M. A. Latif, S. S. Dragomir and E. Momoniat, *On Hermite-Hadamard type integral inequalities for n-times differentiable m- and* (α, m)*-logarithmically convex functions.* Filomat **30** (2016), no. 11, 3101–3114. 1

[17] B. Meftah, *Some new Ostrowski's inequalities for functions whose* n^{th} *derivatives are logarithmically convex.* Ann. Math. Sil. **32** (2017), no. 1, 275-284. 1

[18] B. Meftah and A. Souahi, *Fractional Hermite-Hadamard type inequalities for functions whose derivatives are extended* s-(α, m)*-preinvex*, Int. J. Optim. Control. Theor. Appl. IJOCTA **9** (2019), no. 1, 73-81. 1

[19] B. Meftah, M. Merad, N. Ouanas and A. Souahi, *Some new Hermite-Hadamard type inequalities for functions whose* n^{th} *derivatives are convex.* Acta Comment. Univ. Tartu. Math. **23** (2019), no. 2, 163-178. 1

[20] K. S. Nisar, G. Rahman, A. Khan, A. Tassaddiq and M. S. Abouzaid, *Certain generalized fractional integral inequalities.* AIMS Math. **5** (2020), no. 2, 1588–1602. 1

[21] J. E. Pečarić, F. Proschan and Y. L. Tong, *Convex functions, partial orderings, and statistical applications. Mathematics in Science and Engineering*, 187. Academic Press, Inc., Boston, MA, 1992. 1.1

[22] I. Podlubny, Fractional differential equations. *An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.* Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, 1999. 1

[23] J. Sabatier, O. P. Agrawal and J. A. T. Machado (Eds.), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering.* Springer, Dordrecht, 2007. 1

[24] M. Z. Sarikaya, E. Set, H. Yaldiz and N. Başak, *Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities.* Mathematical and Computer Modelling, **57** (2013), no. 9-10, 2403-2407. 1

[25] M. Z. Sarikaya and H. Yaldiz, *On generalized Hermite-Hadamard type integral inequalities involving Riemann-Liouville fractional integrals.* Nihonkai Math. J. **25** (2014), no. 2, 93–104. 1

[26] M. Z. Sarikaya and H. Yildirim, *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals.* Miskolc Math. Notes **17** (2016), no. 2, 1049–1059.

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