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# Category Of Grey Sets

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**Abstract**

In this paper, we introduce the concept of morphisms between two grey sets and defined a new category, namely, GSet, of grey sets and grey morphisms. We investigate some categorical notions such product, coproduct, pullback, and pushout. Additionally, we study equalizer and coequalizer for pairs of grey morphisms and show that any grey set has an injective hull.

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## 1. Introduction

Grey system is one of the most important scientific achievements in the field of how to use uncertain information, which was presented by Julang Deng [6]. His research focused on the prediction and controlling economic and fuzzy systems. He encountered many inaccurate systems, some known and some unknown, whose properties could not be adequately explained by fuzzy mathematics or statistics and probability. To solve these systems optimally, Deng published an article titled of "The Controlling Problems of Grey Systems" in 1982, introducing grey systems theory. The major advantage of the grey systems theory is its need for a low volume of data [7]. In fact, grey systems theory has been posited as an effective method for solving the problems with discrete data and imperfect information [6]. Incomplete information is fundamentally "grey", representing partially known and partially unknown information. Black represents unknown information, while white represents completely known information. Grey system stands for the system with partially known and partially unknown information. Liu et al. [12] presented some new concepts, frameworks, and models that the scientific applications of new models of grey systems can be used to solve various problems in the social sciences, Engineering including Air Flight, Air Warfare, Information, Metallurgy, Petroleum, Chemical Industry, Electric Power, Electronics, Lighting Industries, Energy Sources,

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Transportation, Pharmacy, Health, Agriculture, Forestry, Geography, Water Resources, Earthquake Found science, Meteorology, Environmental protection, Architecture, Behavioral sciences, Management sciences, Economics, Politics, Law, Logic, Military sciences, etc. These applications have yielded significant economic and social benefits in society and show the wide range of applications of grey systems theory, especially in situations where available information is incomplete or the data collected are inaccurate. Darvishi et al. [5] studied the existing approaches of ordering interval grey numbers in the context of decision-making by surveying existing definitions. We recall that a category is a common language for discussing classes of sets, groups, modules, topological spaces, etc. Fore more, see [1],[2] and [4]. Several categories have been investigate in various articles.

Goguen[8], [9], and [10] initiated the study of category *Set(V)* of $V$-sets for partially ordered set$V$, which any $V$-set is a function $A : X \longrightarrow V$ from a set $X$, and a morphism between two $V$-sets is a function $f : X \longrightarrow Y$ such that $A \leq Bf$. The category *Set(V)* becomes the category *Set* if $V$ is a singleton and the *fuzzy subsets* category if $V = [0, 1]$. Barr [3] studied the category *Set ([0,1])*, which he called *Fuz([0,1])*. Walker et al. [14] investigated two categories of fuzzy subsets and quotient category of fuzzy sets. Also, Harding et al. [11] studied the 2- category of fuzzy sets and relations. In [13], a new category of fuzzy sets and its properties were defined and studeid.

In this paper, we define the category **GSet**, where objects are grey sets, and morphisms are grey functions between two objects. We show the existence of the product and coproduct of grey sets. We also investigate the notions of pullaback and pushout in this category, study equalizer and coequalizer for morphisms, characterize injective grey sets, and explore the injective hull of any grey set.

## 2. Category GSet

In this section, we recall grey numbers and grey sets, define morphisms between two grey sets, and introduce the category **GSet** of grey sets and grey morphisms.

We recall from [17], a *grey numbers* is a number with clear upper and lower boundaries but which has an unknown position within the boundaries. A grey number is expressed mathematically as $g^{\pm} \in [g^{-}, g^{+}] = \{g^{-} \leq t \leq g^{+}\}$ where $g^{\pm}$ is a grey number, $t$ is information, and $g^{-}$ and $g^{+}$ are the lower and upper limits of the information.

If $g^{-} = g^{+}$, then $g^{\pm}$ is called a white number.

We recall the following definition for grey numbers from [17]. It is shown a new representation of grey numbers to consider continuous and discrete grey numbers.

Let $\mathbb{R}$ be set of real numbers and $g^{\pm}$ be a union set of closed or open intervals $g^{\pm} \in \bigcup_{i=1}^{n}[a_i^{-}, a_i^{+}]$, which $i = 1, 2, 3, \cdots, n$, $n$ is an integer and $0 < n < \infty$, $a_i^{-}, a_i^{+} \in \mathbb{R}$ and $a_{i-1}^{-} \leq a_i^{-} \leq a_i^{+} \leq a_{i+1}^{+}$. For any interval $[a_i^{-}, a_i^{+}]$, $p_i$ is probability for $g^{\pm} \in [a_i^{-}, a_i^{+}]$, if the following conditions hold for

(i) $p_i > 0$ if and only if $g^{\pm} \in [a_i^{-}, a_i^{+}]$

(ii) $p_i = 0$ if and only if $g^{\pm} \notin [a_i^{-}, a_i^{+}]$

(iii)$\Sigma_{i=1}^{n} p_i = 1$

then we call $g$ is a grey number represented by $g^{\pm}$. $g^{-}{=}inf_{a_i^{-} \in g^{\pm}} a_i^{-}$ and $g^{+}{=}sup_{a_i^{+} \in g^{\pm}} a_i^{+}$ are called as the lower and upper of $g^{\pm}$.

**Theorem 2.1.** *[15, Theorem1] The following properties hold for* $g^{\pm}$*:*
(i) $g^{\pm}$ *is a continnuous grey number* $g^{\pm} = [a_1^{-}, a_n^{+}]$ *iff* $a_i^{-} = a_{i+1}^{+} (\forall i > 1)$ *or* $n = 1$.
(ii) $g^{\pm}$ *is a discrete grey number* $g^{\pm} = \{a_1, a_2, \cdots, a_n\}$ *iff* $a_i = a_i^{-} = a_i^{+}$

Now, let $U$ denote a universe of discourse. For a set $A \subseteq U$, if the characteristic function value of $x$ with respect to $A$ can be expressed with a single white number $v \in [0, 1]$, $\chi_A : U \longrightarrow [0, 1]$, then $A$ is a *white set*.

For a set $A \subseteq U$, if the characteristic function value of $x$ with respect to $A$ can be expressed with a grey number $g_A^{\pm}(x) \in \bigcup_{i=1}^{n}[a_i^-, a_i^+] \in D[0,1]^{\pm}$

$$\chi_A : U \longrightarrow D[0,1]^{\pm}$$

then $A$ is a *grey set*, where $D[0,1]^{\pm}$ refers to the set of all grey numbers within the interval $[0,1]$. Similar to fuzzy set, a grey set $A$ is shown to
$A = \frac{g^{\pm}(x_1)}{x_1} + \frac{g^{\pm}(x_2)}{x_2} + \cdots + \frac{g^{\pm}(x_n)}{x_n}$ for $x_i \in A$. The grey set will be denoted by $A = (U, \chi_A)$.

The special case of the grey set is the white set, and any fuzy set is special case of white set. Replacing the characteristic function with a fuzzy membership function, the white set become a fuzzy set. In the following, we recall grey lattice operation from [16].
Consider grey numbers $x^{\pm} = [x^-, x^+]$ (lower $x^-$ and upper $x^+$) and $y^{\pm} = [y^-, y^+]$ (lower $y^-$ and upper $y^+$). The *Join* and *Meet* of these grey numbers are defined as $x^{\pm} \vee y^{\pm} = [\ \min\ (x^-, y^-), \max\ (x^+, y^+)]$ and $x^{\pm} \wedge y^{\pm} = [\ \max\ (x^-, y^-), \min\ (x^+, y^+)]$, respectively.
Now according to definition Join and Meet, the partial order $\preceq$ on grey set $(X, \chi_A)$ is shown as below:

$$x^{\pm} \preceq y^{\pm} \Longleftrightarrow x^+ \leq y^+ and\ y^- \leq x^-$$

**Definition 2.2.** Let $X = (U, \chi_A)$ and $Y = (U', \chi_{A'})$ be two grey sets which $\chi_A : U \longrightarrow D[0,1]^{\pm}$ and $\chi_{A'} : U' \longrightarrow D[0,1]^{\pm}$. A grey morphism between grey sets $X$ and $Y$ is ordinary function $f : U \longrightarrow U'$ such that (upper) lower $\chi_A(x) \leq$ (upper) lower $\chi_B f(x)$, for shortly, $\chi_A^{\pm}(x) \leq \chi_{A'}^{\pm} f(x)$.

The composition of two grey morphisms , which is composition of two functions is assotative. The category of grey sets and grey morphismss between is denoted **GSet**.

Recall from [1], a concrete category over $X$ is a pair $(A, U)$, where $A$ is a category and $U : A \longrightarrow X$ is a faithful functor. Some times $U$ is called the forgetfull functor of the concrte category and $X$ is called the base category for $(A, U)$. Any concrete category over category *Set* is called a construct.

It is clear that category *GSet* is construct category.

**Example 2.3.** Let $A = \{a, b, c, d\}$ and $B = \{e, f, g, h, i\}$. Define $\chi_A(a) = [0.1, 0.15]$, $\chi_A(b) = [0.1, 0.23]$, $\chi_A(c) = \chi_A(d) = [0.3, 0.3]$, $\chi_B(e) = [0.1, 0.26]$, $\chi_B(f) = [0.3, 0.34]$, $\chi_B(g) = [0.6, 0.6]$ and $\chi_A(d) = [0.7, 0.81]$. We define function from $X$ to $Y$ such that $f(a) = f(b) = e$, $f(c) = i$ and $f(d) = h$. It is clear that $f$ is morphism between two objects $(A, \chi_A)$ and $(B, \chi_B)$.

**Lemma 2.4.** *Any morphism in the category GSet is a monomorphism if and only if it is one-to-one.*

*Proof.* We claim in the category GSet, any grey monomorphism is one-to-one. For this, consider grey monomorphism $f : (U, \chi_A) \longrightarrow (V, \chi_B)$. Suppose that $f(x) = f(y)$ and $x \neq y$ for any $x, y \in U$. Define morphism, $g : (\{*\}, \chi_C) \longrightarrow (U, \chi_A)$ such that $g(*) = x$ and $\chi_C(*) = \chi_A(x)$. Also define morphism $h : (\{*\}, \chi_D) \longrightarrow (U, \chi_A)$ such that $h(*) = y$ and $\chi_D(*) = \chi_A(y)$. We have $f(g(*)) = f(x) = f(y) = f(h(*))$. So $g(*) = h(*)$ since $f$ is monomorphism. Thus $x = y$. The converse is clear. □

**Lemma 2.5.** *Any grey morphism in the category GSet is epimorphism if and only if it is surjective.*

*Proof.* Consider epimorphism $f : (U, \chi_A) \longrightarrow (V, \chi_B)$. Define morphism $g : (V, \chi_B) \longrightarrow (V \cup \{*_1, *_2\}, \chi_C)$ such that

$$g(x) = \begin{cases} x & x \in Imf \\ *_1 & otherwise \end{cases}$$

which $\chi_C(x) = \chi_B(x)$ for any $x \in V$ and $\chi_C(x) = \{0,1\}$ for $x \in \{*_1, *_2\}$. Also define morphism $h : (V, \chi_B) \longrightarrow (V \cup \{*_1, *_2\}, \chi_C)$ such that

$$h(x) = \begin{cases} x & x \in Imf \\ *_2 & otherwise \end{cases}$$

which $\chi_C(x) = \chi_B(x)$ for any $x \in V$ and $\chi_C(x) = \{0,1\}$ for $x \in \{*_1, *_2\}$. For any $u \in U$, $g(f(u)) = f(u) = h(f(u))$. So $g = h$ since $f$ is an epimorphism. Suppose that the morphism $f$ is not surjective. So there exists $x \in V$ such that $x \notin Im(f)$. Therefore $*_1 = g(x) = h(x) = *_2$, which is contradicts. The converse is clear.

□

It is well-known that category GSet is balanced by two lemmas, 2.4 and 2.5.

**Proposition 2.6.** *A map $f : (U, \chi_A) \longrightarrow (U', \chi_{A'})$in Gset is an isomorphisms if and only if it is one-to-one, onto and $\chi^{\pm}_A(x) = \chi^{\pm}_{A'} f(x)$.*

*Proof.* Let map $f$ be isomorphism, then it is one- to- one and onto. Since $f$ is isomorphism there exists $f^{-1}$. So, we have $\chi^{\pm}_A(x) \leq \chi^{\pm}_{A'} f(x)$ and $\chi^{\pm}_{A'}(x) \leq \chi^{\pm}_A f^{-1}(x)$. Hence, $\chi^{\pm}_A(x) = \chi^{\pm}_{A'} f(x)$. The converse is clear. □

## 3. Main Result

In this section, we study the existence of product and coproduct in the category *GSet*. Additionally, we investigate some categorical concepts such as pushout, pullback, equalizer, and injective objects in the category *GSet*, and we demonstrate that any grey set has an injective hull.

**Proposition 3.1.** *The category Gset has products.*

*Proof.* Consider a family $(X_i, \chi_i)_{i \in J}$ of grey sets. We claim $(X, \chi)$ is the product of this family, which $X$ is the product $X_i$'s and $\chi$ is the meet of $\chi_i$, i.e $\wedge\chi^{\pm}_i = \{\wedge\chi^{-}_i, \wedge\chi^{+}_i\}$, for $i \in I$ and $\pi_j : (X, \chi) \longrightarrow (X_j, \chi_j)$ is projection map. We have $\chi^{\pm}(x) = (\wedge_{i \in J}\chi^{\pm}_i)(x) = \wedge_{i \in J}(\chi^{\pm}_i(x)) \leq \chi^{\pm}_{j\in}(x_j) = \chi^{\pm}_j(\pi_j(x))$. So, $\pi_j \in GSet$. Now, suppose that there exists a map $m_j : (Y, \tau) \longrightarrow (X_j, \chi_j)$, by universal properties of product of sets, there exists unique function $\varphi : (Y, \tau) \longrightarrow (\prod_{i \in J} X_j, \chi)$ such that $\pi_j\varphi = m_j$. Also, we have $\tau^{\pm} \leq \chi^{\pm}_j m_j = \chi^{\pm}_j \pi_j \varphi = \chi^{\pm}_j \varphi$, hence $\varphi \in GSet$.

□

**Proposition 3.2.** *The category GSet has coproducts.*

*Proof.* For a family $(X_i, \chi_i)_{i \in J}$ of grey sets, we claim $(X, \chi)$ is the coproduct, which $X^{\pm}$ is disjoint union of $X_i$'s and $\chi^{\pm}$ is the disjoint union of $\chi^{\pm}_j$. For this, consider $q_j : (X_j, \chi_j) \longrightarrow (\coprod_{i \in J} X_i, \coprod_{i \in J} \chi_i)$, which $q_j : X_j \longrightarrow \coprod_{i \in J} X_i$ is inclusion maps, such that $\chi^{\pm}_j(x_j) = (\coprod_{i \in J} \chi^{\pm}_i q_j(x_j)$. It is clear that $q_j$ is a map in *GSet*. If there exists any grey map such $\psi_j : (X_j, \chi_j) \longrightarrow (Y, \varsigma)$ in *GSet*, by the prpperty of coproduct, there exists a unique function $\tau : \coprod_{i \in J} \chi_i) \longrightarrow Y$ such that $\tau q_j = \psi_j$. Also, since $\chi^{\pm}_j \leq \varsigma^{\pm}\psi_j$, we have $\coprod_{i \in J} \chi_i \leq \varsigma^{\pm}\tau$. Hence $\tau \in GSet$.

□

We denote the constant characteristic function at a point such as $m$ by the $\chi_m$.

**Proposition 3.3.** *In the category Gset, the empty set with the constant characteristic function $\chi_0$ is the initial object and singelton set with the constant characteristic function $\chi_1$ is the terminal object.*

*Proof.* Consider arbitrary grey set $(U, \chi_A)$. We have from the category $Set$, the empty set is initial object. So there exists a unique function $f : \emptyset \longrightarrow U$. It is clear that f is a grey morphism. For the terminal object, consider arbitrary grey set $(U, \chi_A)$, consider grey set $(\{*\}, \chi_*\})$, with characteristic function to grey number with upper bound 1 and lower bound zero. It is clear there exists one function $f : A \longrightarrow \{*\}$ and so $f \in$ GSet. $\square$

It is clear that an empty grey set is a free object, since it is an initial object in the category GSet by [1, Example 8.23].

**Proposition 3.4.** *In the category GSet , the equalizer of two maps* $f, g : (X, \chi_1) \longrightarrow (Y, \chi_2)$ *is the pair* $(E, \chi_E)$*, which* $E = \{x \in X | f(x) = g(x)\}$*,* $\chi_E : E \longrightarrow D[0, 1]^{\pm}$ *is the restriction of* $\chi_1$ *on* $E$ *and the inclusion map* $i : (E, \chi_E) \longrightarrow (X, \chi_1)$*.*

*Proof.* Consider map $i : (E, \chi_E) \longrightarrow (X, \chi_1)$ which $i : E \longrightarrow X$ is inclusion function such that $E = \{x \in X|f(x) = g(x)\}$. Also $\chi_E : E \longrightarrow D[0, 1]^{\pm}$. I t is clear that $i$ is a grey map in the category $GSet$. If there exists any grey map $l : (K, \chi_K) \longrightarrow (X, \chi_1)$ such that $fl = gl$. For any $x \in K$, $l(x)\ inE$, since $fl(x) = gl(x)$. So, we define $m : K \longrightarrow E$ satisfying $im = l$. It is clear that $\chi^{\pm}_K \leq \chi^{\pm}_E m$. Thus $m$ is a grey map in the category $GSet$. We claim $m$ is unique, otherwise there exists a map $m' : (K, \chi_K) \longrightarrow (E, \chi_E)$ such that $im' = l$. Thus $im = im'$ and so $m = m'$. $\square$

**Proposition 3.5.** *In the category GSet, the coequalizer of two maps* $f, g : (X, \chi_1) \longrightarrow (Y, \chi_2)$ *is the pair* $(K, \chi_K)$*, which* $K = \dfrac{Y}{\theta}$*, which* $\theta$ *is the equivalence relation generated by of* $\{(f(x), g(x))|x \in X\}$ *and* $\chi^{\pm}_K(k) = sup\{\chi^{\pm}_2(y)|\pi(y) = k\}$*, which* $\pi : Y \longrightarrow K$ *is natural function.*

*Proof.* By assumption, since for any $y \in Y$, $\chi^{\pm}_2(y) \leq \chi^{\pm}_K\pi(y)$, we can conclude that $\pi$ is a grey map in the category $GSet$. Now, suppose that there exists a map $h : (Y, \chi_2) \longrightarrow (W, \chi_W)$ such that $hf = hg$. We recall from category $Set$, there exists a unique function $\alpha : K \longrightarrow W$ satisfying $\alpha\pi = h$. We show that $\alpha$ is a grey map in $Gset$. Consider $y = \pi^{-1}(k)$ for $k \in K$. Since $\chi^{\pm}_2(y) \leq \chi^{\pm}_W h(y) = \chi^{\pm}_W \alpha\pi(y) = \chi^{\pm}_W \alpha(k)$. So $\chi^{\pm}_K(k) \leq \chi^{\pm}_W \alpha(k)$, and hence $\alpha \in GSet$. $\square$

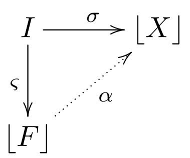
An object $S$ is called a seperator if for distinct morphisms $f, g : A \longrightarrow B$, there exists a morphisms $h : S \longrightarrow A$ such that $goh \neq foh$.

*Remark* 3.6. In the category $GSet$, a non-empty set with constant characteristic function $\chi_1$ is seperator object.

An object $C$ is called co-separator if for distinct morphisms $f, g : B \to A$, there exists a morphisms $h : A \longrightarrow C$ such that $hog \neq hof$.

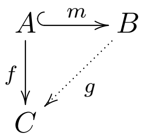
*Remark* 3.7. In the category $GSet$, the co-separators are sets with at least two elements and the constant characteristic function $\chi_1$.

Recall that in a category $\mathcal{C}$, the object $F \in C$ is called a free object in $\mathcal{C}$, if there exists $I$ in the category $Set$ and a function $\varsigma : I \longrightarrow \lfloor F \rfloor$ such that for any $X \in \mathcal{C}$ and every function $\sigma : I \longrightarrow \lfloor X \rfloor$, there exists exactly $\alpha \in Mor_{\mathcal{C}}(F, X)$ such that $\alpha\varsigma = \sigma$ in the category $Set$, i.e. the following diagram is commutative.




*Remark* 3.8. It is clear that $(\emptyset, \chi_0)$ is a free object, since it is an initial object in $GSet$ by [1, Example 8.23].

In the concrete category, an object $C$ is called injective if for any embedding $m : A \longrightarrow B$ and any morphism $f : A \longrightarrow C$, there exists a morphism $g : B \longrightarrow C$ such that the following diagram is commutative.




**Lemma 3.9.** *In the category GSet, the non-empty grey sets with constant characteristic function $\chi_1$ are percisely injective objects.*

*Proof.* Let $f : (A, \chi_A) \longrightarrow (B, \chi_B)$ be a monomorphism grey map and $g : (A, \chi_A) \longrightarrow (X, \chi_c)$ be arbitrary map in the category $GSet$, where $\chi_1$ is constant function. We show that there exists a map $h : (B, \chi_B) \longrightarrow (X, \chi_C)$ such that $hg = f$. Recall that in the category $Set$ any non-empty set is injective, so there exists a map $h : B \longrightarrow X$ such that extend $g$. We show that $g$ is a grey map in the category **GSet**. By definition of $X_C$, the result $\chi_B \leq \chi_C h$ is clear. So, $h \in GSet$.

□

*Remark* 3.10. In the category $GSet$, any non-empty object $(A, \chi_1)$ can be embedded into injective object.

An injecetive hull of $A$ is an extension $m : A \longrightarrow B$ such that $B$ is injective and $m$ is essential.

**Lemma 3.11.** *In the category GSet, any object has a injective hull.*

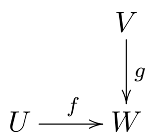
*Proof.* It is clear that the injective hull of non-empty grey set, is itself. If $A$ is empty, then $(\{*\}, \chi_1)$ is injective hull of grey set $A$.

□

Given two morphisms $f : A \longrightarrow C$ and $g : B \longrightarrow C$ in a given category C, a pullback of pair $(f, g)$ is a pair $(P; (\rho_1; \rho_2))$ with $\rho_A : P \longrightarrow A$, $\rho_B : P \longrightarrow A$ such that $f\rho_A = g\rho_B$ and satisfies the following universal property:
For any pair $(P'; (\rho'_A; \rho'_B))$ with $\rho'_A : P' \longrightarrow A$, $\rho'_B : P' \longrightarrow A$ such that $f\rho'_A = g\rho'_B$ there exists a unique morphism $\theta : P' \longrightarrow P$ such that $\rho_A\theta = \rho'_A$ and $\rho_B\theta = \rho'_B$. Next, we show the pullback for a pair $(f, g)$ in the category GSet.

**Theorem 3.12.** *In the category GSet, there is a pullback of any morphisms $f : (U, \chi_A) \longrightarrow (W, \chi_C)$ and $g : (V, \chi_B) \longrightarrow (W, \chi_C)$.*

*Proof.* Consider grey sets $(U, \chi_A)$, $(V, \chi_B)$, $(W, \chi_C)$ and morphisms $f : (U, \chi_A) \longrightarrow (W, \chi_C)$ and $g : (V, \chi_B) \longrightarrow (W, \chi_C)$. Consider folowing diagram.




In the category $Set$, the pullback of this diagram is set $T = \{(x, y) \in U \times V | f(x) = g(y)\}$ and functions $\rho_U : T \longrightarrow U$, $\rho_V : T \longrightarrow V$, which $\rho_U(a, b) = a$ and $\rho_V(a, b) = b$, for $a, b \in U \times V$ such that $f\rho_U = g\rho_V$. Define characteristic function $\chi$ such that $\chi^+_D(a, b) = (\chi_A(a) \wedge \chi_B(b))^+$ and $\chi^-_D(a, b) = (\chi_A(a) \wedge \chi_B(b))^-$ for any $(a, b) \in T$.

We claim $\rho_U$ and $\rho_V$ are in the category GSet. It is clear that for any $a \in U$ and $b \in V$, $\chi^+_D(a, b) = (\chi_A(a) \wedge \chi_B(b))^+ \leq \chi^+_A(a) = \chi^+_A\rho_U(a, b)$ and $\chi^-_A\rho_U(a, b) = \chi^-_A(a) \leq (\chi_A(a) \wedge \chi_B(b))^- = \chi^-_D(a, b)$. Hence

$\rho_U \in$ GSet. Similarly we can show $\rho_V \in$ GSet. Now consider grey object $(T', \chi_{D'})$ such that $ff' = gg'$. From category *Set*, there exists exactly one function $h : T' \longrightarrow T$ such that $\rho_U h = g'$ and $\rho_V h = f'$. It is easy to check that $h \in$ GSet. □

Given two morphisms $f : A \longrightarrow B$ and $g : A \longrightarrow C$ in a given category C, a pushout of pair $(f, g)$ is a pair $(Q; (\alpha; \beta))$ with $\beta g = \alpha f$ that satisfies the following universal property:
For every triple $(Q'; (\alpha'; \beta'))$ with $\beta' g = \alpha' f$, there exists a unique morphism $\theta : Q \longrightarrow Q'$ such that $\theta\beta = \beta'$ and $\theta\alpha = \alpha'$.

**Theorem 3.13.** *In category GSet, there is a pushout of any morphisms* $f : (U, \chi_A) \longrightarrow (V, \chi_B)$ *and* $g : (U, \chi_A) \longrightarrow (W, \chi_C)$.

*Proof.* Let $(U, \chi_A)$, $(V, \chi_B)$ and $(W, \chi_C)$ be grey sets and $f : (U, \chi_A) \longrightarrow (V, \chi_B)$ and $g : (U, \chi_A) \longrightarrow (W, \chi_C)$ be two grey morphisms. Consider the following diagram

$$\begin{array}{ccc} U & \xrightarrow{g} & W \\ {\scriptstyle f}\big\downarrow & & \\ V & & \end{array}$$

In the category *Set*, pushout of this diagram is the pair $(Z, (h, k))$ such that $hg = kf$ which $Z = (V \sqcup W)/\theta$, $\theta$ is the equivalence relation on $V \sqcup W$ generated by all pairs $H = \{(u_V f(x), u_W g(x)) : x \in U\}$, $h = \pi u_W : W \to Z$, $k = \pi u_V : V \to Z$, $\pi : U \sqcup W \to Z$ is the natural epimorphism and $u_V : V \to V \sqcup W, u_W : W \to V \sqcup W$ are coproduct injections. Consider $(Z, \chi_Q)$, which for any $[x]_\theta \in Z$,

$$\chi_Q^+([x]_\theta) = Sup \begin{cases} \chi_B^+(v) : \{k(v) = [x]_\theta\} & x \in V \\ \chi_C^+(w) : \{h(w) = [x]_\theta\} & x \in W \end{cases}$$

and

$$\chi_Q^-([x]_\theta) = Inf \begin{cases} \chi_B^-(v) : \{k(v) = [x]_\theta\} & x \in V \\ \chi_C^-(w) : \{h(w) = [x]_\theta\} & x \in W \end{cases}$$

We show that $k$ and $h$ are morphisms in category GSet. It is clear that $\chi_B^+(x) \leq \chi_Q^+([x]_\theta) = \chi_Q^+ k(x)$ and $\chi_Q^- k(x) = \chi_Q^-([x]_\theta) \leq \chi_B^-(x)$ for any $x \in V$, so $k \in$ GSet. Similarly, we can show $h \in$ GSet. Now consider arbitray grey set $(Z', \chi_{Q'})$. If there exists grey morphisms $k' : V \longrightarrow Z'$ and $h' : W \longrightarrow Z'$ such that $k'f = h'g$, it is follow from categery Set, there exists a unique function $\varphi : Z \longrightarrow Z'$ such that $\varphi k = k'$ and $\varphi h = h'$. We show that $\varphi \in$ GSet. For any $[x]_\theta \in Z$, if $x \in V$, let $v = (k)^{-1}([x]_\theta)$. We have $\chi_B^+(v) \leq \chi_{Q'}^+ k'(v) = \chi_{Q'}^+ \varphi k(v) = \chi_{Q'}^+ \varphi([x]_\theta)$ and $\chi_{Q'}^- \varphi([x]_\theta) = \chi_{Q'}^- \varphi k(v) = \chi_{Q'}^- k'(v) \leq \chi_B^-(v)$.

Thus $\chi_Q^+([x]_\theta) \leq \chi_{Q'}^+ \varphi([x]_\theta)$ and $\chi_{Q'}^- \varphi([x]_\theta) \leq \chi_Q^-([x]_\theta)$. If $x \in W$, we result by similar method. So, $\varphi \in$ GSet. □

**Lemma 3.14.** *In the category GSet, any grey set with constant characteristic function* $\chi_0$ *is percisely projective grey set.*

*Proof.* Consider surjective grey map $\pi : (A, \chi_A) \longrightarrow (B, \chi_B)$ and arbitrary grey map $f : (P, \chi_0) \longrightarrow (B, \chi_B)$, which $\chi_0$. In the category *Set*, there exists a function $g : C \longrightarrow A$ such that $\pi g = f$. Obviously, $\chi^{\pm}(p) \leq \chi_A^{\pm} g(p)$, for any $p \in P$. So $g \in GSet$ and $(P, \chi_0)$ is projective in the category $GSet$. □

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