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Formulas of the solutions of a solvable system of nonlinear difference equations

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Abstract

Consider the following general three dimensional system of difference equations

$$\begin{cases} x_{n+1} = f^{-1}\left(\frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)[a_n(g(y_{n-2}))^q+b_n g(y_n)g(y_{n-1})]}\right), \\ y_{n+1} = g^{-1}\left(\frac{h(z_n)h(z_{n-1})(g(y_{n-1}))^q}{g(y_n)[c_n(h(z_{n-2}))^r+d_n h(z_n)h(z_{n-1})]}\right), \\ z_{n+1} = h^{-1}\left(\frac{f(x_n)f(x_{n-1})(h(z_{n-1}))^r}{h(z_n)[s_n(f(x_{n-2}))^p+t_n f(x_n)f(x_{n-1})]}\right), \end{cases} \tag{1}$$

where $n \in \mathbb{N}_0$, $p, q, r \in \mathbb{N}$, $f, g, h : D \to \mathbb{R}$ are continuous one-to-one functions on $D \subseteq \mathbb{R}$, the coefficients $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(c_n)_{n\in\mathbb{N}_0}$, $(d_n)_{n\in\mathbb{N}_0}$, $(s_n)_{n\in\mathbb{N}_0}$, $(t_n)_{n\in\mathbb{N}_0}$ are non-zero real numbers and the initial values x_{-i}, y_{-i}, z_{-i}, $i = 0, 1, 2$, are real numbers. We will give explicit formulas for well-defined solutions of the aforementioned system in both variable and constant cases of the coefficients. As an application, we will deduce the formulas of the solutions of the particular system obtained from the general one by taking $f(x) = g(x) = h(x) = x$.

Keywords: System of difference equations, Well-defined solutions, Closed-form of the solutions, Generalized Fibonacci sequence.
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1. Introduction and preliminaries

Our aim in the present work is to solve in closed form the following system of difference equations

$$\begin{cases} x_{n+1} = f^{-1}\left(\frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)[a_n(g(y_{n-2}))^q + b_n g(y_n)g(y_{n-1})]}\right), \\ y_{n+1} = g^{-1}\left(\frac{h(z_n)h(z_{n-1})(g(y_{n-1}))^q}{g(y_n)[c_n(h(z_{n-2}))^r + d_n h(z_n)h(z_{n-1})]}\right), \\ z_{n+1} = h^{-1}\left(\frac{f(x_n)f(x_{n-1})(h(z_{n-1}))^r}{h(z_n)[s_n(f(x_{n-2}))^p + t_n f(x_n)f(x_{n-1})]}\right), \end{cases} \tag{1.1}$$

where $n \in \mathbb{N}_0$, p, q, $r \in \mathbb{N}$, f, g, $h : D \to \mathbb{R}$ are continuous one-to-one functions on $D \subseteq \mathbb{R}$, the coefficients $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(c_n)_{n\in\mathbb{N}_0}$, $(d_n)_{n\in\mathbb{N}_0}$, $(s_n)_{n\in\mathbb{N}_0}$, $(t_n)_{n\in\mathbb{N}_0}$ are non-zero real numbers and the initial values $x_{-i}, y_{-i}, z_{-i}, i = 0, 1, 2$, are real numbers. We will give explicit formulas of well-defined solutions in both variable and constant cases of the coefficients. As an example, we will apply the obtained results, on the system of difference equations

$$x_{n+1} = \frac{y_n y_{n-1} x_{n-1}^p}{x_n(a_n y_{n-2}^q + b_n y_n y_{n-1})}, \; y_{n+1} = \frac{z_n z_{n-1} y_{n-1}^q}{y_n(c_n z_{n-2}^r + d_n z_n z_{n-1})}, \; z_{n+1} = \frac{x_n x_{n-1} z_{n-1}^r}{z_n(s_n x_{n-2}^p + t_n x_n x_{n-1})},$$

which is obtained from the general one by taking $f(x) = g(x) = h(x) = x$.
Our work can be seen as a continuation and an extension to the three dimensional case of the two dimensional system of difference equations studied in [21].

Difference equations are a very active field of research, and many contributions with different models of such equations are published, see for example [8], [9], [10], [11], [12],[13], [14], [18], [24], [34], [35]. This type of equations is a very important tool in the comprehension of the behavior of phenomenons defined by discrete models in many scientific areas and as consequence a phenomenon will be more easy to understand if the corresponding model is solvable explicitly.
The following is a short list of papers which concern some solvable models: [1], [2], [3], [4], [5], [6], [7],[15], [16], [17], [19], [22], [23], [25], [26], [27], [28], [29], [30], [31], [32], [33].
To solve our system, we will transform it via some convenable change of variables to a linear system which can be solved explicitly. After that, we will write the solutions of our system using those of the corresponding linear system.
The following very well known lemma will be used in resolution of the corresponding linear system.

Lemma 1.1. *Let $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ be two sequences of real numbers and consider the third order linear difference equation*

$$r_{n+3} = \alpha_n r_n + \beta_n, \quad n \in \mathbb{N}_0.$$

Then for $i = 0, 1, 2$ and $n \in \mathbb{N}_0$ we have

$$r_{3n+i} = \left[\prod_{j=0}^{n-1} \alpha_{3j+i}\right] y_i + \sum_{r=0}^{n-1} \left[\prod_{j=r+1}^{n-1} \alpha_{3j+i}\right] \beta_{3r+i}.$$

Moreover, if $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ are constants (i.e., $\alpha_n = \alpha$, $\beta_n = \beta$), then

$$r_{3n+i} = r_i + \beta n, \; \alpha = 1, \; r_{3n+i} = \alpha^n r_i + \left(\frac{\alpha^n - 1}{\alpha - 1}\right)\beta, \; \alpha \neq 1,$$

where, $\prod_{i=j}^{k} a_i = 1$ and $\sum_{i=j}^{k} a_i = 0$, for all $k < j$.

We will see in next section that the following generalized Fibonacci sequence is used in the formulas of the solutions of our System. Let $\{F_{k,n}\}_{n=0}^{\infty}$ be the sequence defined by

$$F_{k,n+2} = F_{k,n+1} + kF_{k,n}, \quad F_{k,0} = F_{k,1} = 1, \; k \in \{p, q, r\}. \tag{1.2}$$

The following definition, is devoted to the notion of well-defined solutions of System (1.1).

Definition 1.2. A solution $\{x_n, y_n, z_n\}_{n\geq -2}$ of System (1.1) is said to be well-defined if for all $n \in \mathbb{N}_0$, we have

$$f(x_n)\left[a_n(g(y_{n-2}))^q + b_n g(y_n)g(y_{n-1})\right] \neq 0,$$
$$g(y_n)\left[c_n(h(z_{n-2}))^r + d_n h(z_n)h(z_{n-1})\right] \neq 0,$$
$$h(z_n)\left[s_n(f(x_{n-2}))^p + t_n f(x_n)f(x_{n-1})\right] \neq 0,$$
$$\frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)\left[a_n(g(y_{n-2}))^q + b_n g(y_n)g(y_{n-1})\right]} \in D_{f^{-1}},$$
$$\frac{h(z_n)h(z_{n-1})(g(y_{n-1}))^q}{g(y_n)\left[c_n(h(z_{n-2}))^r + d_n h(z_n)h(z_{n-1})\right]} \in D_{g^{-1}},$$
$$\frac{f(x_n)f(x_{n-1})(h(z_{n-1}))^r}{h(z_n)\left[s_n(f(x_{n-2}))^p + t_n f(x_n)f(x_{n-1})\right]} \in D_{h^{-1}}.$$

2. Explicit formulas for well-defined solutions of System (1.1)

Let $\{x_n, y_n, z_n\}_{n\geq -2}$ is a well-defined solution to System (1.1). Since f, g and h are assumed to be one to one, then it follows from (1.1) that

$$f(x_{n+1}) = \frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)(a_n(g(y_{n-2}))^q + b_n g(y_n)g(y_{n-1}))},$$
$$g(y_{n+1}) = \frac{h(z_n)h(z_{n-1})(g(y_{n-1}))^q}{g(y_n)(c_n(h(z_{n-2}))^r + d_n h(z_n)h(z_{n-1}))},$$
$$h(z_{n+1}) = \frac{f(x_n)f(x_{n-1})(h(z_{n-1}))^r}{h(z_n)(s_n(f(x_{n-2}))^p + t_n f(x_n)f(x_{n-1}))}.$$

From which, we get

$$\frac{(f(x_{n-1}))^p}{f(x_{n+1})f(x_n)} = a_n\frac{(g(y_{n-2}))^q}{g(y_n)g(y_{n-1})} + b_n,$$
$$\frac{(g(y_{n-1}))^q}{g(y_{n+1})g(y_n)} = c_n\frac{(h(z_{n-2}))^r}{h(z_n)h(z_{n-1})} + d_n,$$
$$\frac{(h(z_{n-1}))^r}{h(z_{n+1})h(z_n)} = s_n\frac{(f(x_{n-2}))^p}{f(x_n)f(x_{n-1})} + t_n.$$

Consider the following change of variables

$$u_n = \frac{(f(x_{n-2}))^p}{f(x_n)f(x_{n-1})}, \quad v_n = \frac{(g(y_{n-2}))^q}{g(y_n)g(y_{n-1})}, \quad w_n = \frac{(h(z_{n-2}))^r}{h(z_n)h(z_{n-1})}, \tag{2.1}$$

then, System (1.1) is transformed to the following linear system

$$u_{n+1} = a_n v_n + b_n,\ v_{n+1} = c_n w_n + d_n,\ w_{n+1} = s_n u_n + t_n,\ n \in \mathbb{N}_0. \tag{2.2}$$

From (2.2), we have

$$\begin{aligned} u_{n+3} = a_{n+2}v_{n+2} + b_{n+2} &= a_{n+2}\left[c_{n+1}w_{n+1} + d_{n+1}\right] + b_{n+2} \\ &= \left[a_{n+2}c_{n+1}w_{n+1} + a_{n+2}d_{n+1}\right] + b_{n+2} \\ &= a_{n+2}c_{n+1}\left[s_n u_n + t_n\right] + a_{n+2}d_{n+1} + b_{n+2} \\ &= a_{n+2}c_{n+1}s_n u_n + a_{n+2}c_{n+1}t_n + a_{n+2}d_{n+1} + b_{n+2}. \end{aligned}$$

$$\begin{aligned}v_{n+3} = c_{n+2}w_{n+2} + d_{n+2} &= c_{n+2}[s_{n+1}u_{n+1} + t_{n+1}] + d_{n+2}\\ &= c_{n+2}[s_{n+1}(a_n v_n + b_n) + t_{n+1}] + d_{n+2}\\ &= c_{n+2}[s_{n+1}a_n v_n + s_{n+1}b_n + t_{n+1}] + d_{n+2}\\ &= c_{n+2}s_{n+1}a_n v_n + c_{n+2}s_{n+1}b_n + c_{n+2}t_{n+1} + d_{n+2}.\end{aligned}$$

$$\begin{aligned}w_{n+3} = s_{n+2}u_{n+2} + t_{n+2} &= s_{n+2}[a_{n+1}v_{n+1} + b_{n+1}] + t_{n+2}\\ &= s_{n+2}[a_{n+1}(c_n w_n + d_n) + b_{n+1}] + t_{n+2}\\ &= s_{n+2}[a_{n+1}c_n w_n + a_{n+1}d_n + b_{n+1}] + t_{n+2}\\ &= s_{n+2}a_{n+1}c_n w_n + s_{n+2}a_{n+1}d_n + s_{n+2}b_{n+1} + t_{n+2}.\end{aligned}$$

That is, we have obtained the following three linear third order linear difference equations defined for all $n \in \mathbb{N}_0$ by

$$\begin{cases} u_{n+3} = a_{n+2}c_{n+1}s_n u_n + a_{n+2}c_{n+1}t_n + a_{n+2}d_{n+1} + b_{n+2},\\ v_{n+3} = c_{n+2}s_{n+1}a_n v_n + c_{n+2}s_{n+1}b_n + c_{n+2}t_{n+1} + d_{n+2},\\ w_{n+3} = s_{n+2}a_{n+1}c_n w_n + s_{n+2}a_{n+1}d_n + s_{n+2}b_{n+1} + t_{n+2}. \end{cases} \tag{2.3}$$

Using Lemma(1.1), we get for all $n \in \mathbb{N}_0$ and for $i = 0,\ 1,\ 2$ that

$$\begin{aligned}u_{3n+i} = &\left[\prod_{j=0}^{n-1} a_{3j+i+2}c_{3j+i+1}s_{3j+i}\right] u_i \\ &+ \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} a_{3j+i+2}c_{3j+i+1}s_{3j+i}\right]\left(a_{3r+i+2}c_{3r+i+1}t_{3r+i} + a_{3r+i+2}d_{3r+i+1} + b_{3r+i+2}\right),\end{aligned} \tag{2.4}$$

$$\begin{aligned}v_{3n+i} = &\left[\prod_{j=0}^{n-1} c_{3j+i+2}s_{3j+i+1}a_{3j+i}\right] v_i \\ &+ \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} c_{3j+i+2}s_{3j+i+1}a_{3j+i}\right]\left(c_{3r+i+2}s_{3r+i+1}b_{3r+i} + c_{3r+i+2}t_{3r+i+1} + d_{3r+i+2}\right),\end{aligned} \tag{2.5}$$

$$\begin{aligned}w_{3n+i} = &\left[\prod_{j=0}^{n-1} s_{3j+i+2}a_{3j+i+1}c_{3j+i}\right] w_i \\ &+ \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} s_{3j+i+2}a_{3j+i+1}c_{3j+i}\right]\left(s_{3r+i+2}a_{3r+i+1}d_{3r+i} + s_{3r+i+2}b_{3r+i+1} + t_{3r+i+2}\right).\end{aligned} \tag{2.6}$$

Now, from (2.1), (2.4), (2.5) and (2.6), it follows that for all $n \in \mathbb{N}_0$

$$u_{3n} = \left[\prod_{j=0}^{n-1} a_{3j+2}c_{3j+1}s_{3j}\right]\frac{(f(x_{-2}))^p}{f(x_0)f(x_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} a_{3j+2}c_{3j+1}s_{3j}\right]\left(a_{3r+2}c_{3r+1}t_{3r} + a_{3r+2}d_{3r+1} + b_{3r+2}\right), \tag{2.7}$$

$$\begin{aligned}u_{3n+1} = &\frac{\left[\prod_{j=0}^{n-1} a_{3j+3}c_{3j+2}s_{3j+1}\right]\left[a_0(g(y_{-2}))^q + b_0 g(y_0)g(y_{-1})\right]}{g(y_0)g(y_{-1})}\\ &+ \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} a_{3(j+1)}c_{3j+2}s_{3j+1}\right]\left(a_{3r+3}c_{3r+2}t_{3r+1} + a_{3(r+1)}d_{3r+2} + b_{3(r+1)}\right),\end{aligned} \tag{2.8}$$

$$u_{3n+2} = \frac{\left[\prod_{j=0}^{n-1} a_{3j+4}c_{3(j+1)}s_{3j+2}\right]\left[a_1c_0(h(z_{-2}))^r + a_1d_0h(z_0)h(z_{-1}) + b_1h(z_0)h(z_{-1})\right]}{h(z_0)h(z_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} a_{3j+4}c_{3(j+1)}s_{3j+2}\right]\left(a_{3r+4}c_{3(r+1)}t_{3r+2} + a_{3r+4}d_{3(r+1)} + b_{3r+4}\right), \quad (2.9)$$

$$v_{3n} = \frac{\left[\prod_{j=0}^{n-1} c_{3j+2}s_{3j+1}a_{3j}\right](g(y_{-2}))^q}{g(y_0)g(y_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} c_{3j+2}s_{3j+1}a_{3j}\right]\left(c_{3r+2}s_{3r+1}b_{3r} + c_{3r+2}t_{3r+1} + d_{3r+2}\right), \quad (2.10)$$

$$v_{3n+1} = \frac{\left[\prod_{j=0}^{n-1} c_{3(j+1)}s_{3j+2}a_{3j+1}\right]\left[c_0(h(z_{-2}))^r + d_0h(z_0)h(z_{-1})\right]}{h(z_0)h(z_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} c_{3(j+1)}s_{3j+2}a_{3j+1}\right]\left(c_{3(r+1)}s_{3r+2}b_{3r+1} + c_{3(r+1)}t_{3r+2} + d_{3(r+1)}\right), \quad (2.11)$$

$$v_{3n+2} = \frac{\left[\prod_{j=0}^{n-1} c_{3j+4}s_{3(j+1)}a_{3j+2}\right]\left[c_1s_0(f(x_{-2}))^p + c_1t_0f(x_0)f(x_{-1}) + d_1f(x_0)f(x_{-1})\right]}{f(x_0)f(x_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} c_{3j+4}s_{3(j+1)}a_{3j+2}\right]\left(c_{3r+4}s_{3(r+1)}b_{3r+2} + c_{3r+4}t_{3(r+1)} + d_{3r+4}\right), \quad (2.12)$$

and

$$w_{3n} = \frac{\left[\prod_{j=0}^{n-1} s_{3j+2}a_{3j+1}c_{3j}\right](h(z_{-2}))^r}{h(z_0)h(z_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} s_{3j+2}a_{3j+1}c_{3j}\right]\left(s_{3r+2}a_{3r+1}d_{3r} + s_{3r+2}b_{3r+1} + t_{3r+2}\right), \quad (2.13)$$

$$w_{3n+1} = \frac{\left[\prod_{j=0}^{n-1} s_{3(j+1)}a_{3j+2}c_{3j+1}\right]\left[s_0(f(x_{-2}))^p + t_0f(x_0)f(x_{-1})\right]}{f(x_0)f(x_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} s_{3(j+1)}a_{3j+2}c_{3j+1}\right]\left(s_{3(r+1)}a_{3r+2}d_{3r+1} + s_{3(r+1)}b_{3r+2} + t_{3(r+1)}\right), \quad (2.14)$$

$$w_{3n+2} = \frac{\left[\prod_{j=0}^{n-1} s_{3j+4}a_{3(j+1)}c_{3j+2}\right]\left[sa(g(y_{-2}))^q + sbg(y_0)g(y_{-1}) + tg(y_0)g(y_{-1})\right]}{g(y_0)g(y_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} s_{3j+4}a_{3(j+1)}c_{3j+2}\right]\left(s_{3r+4}a_{3(r+1)}d_{3r+2} + s_{3r+4}b_{3(r+1)} + t_{3r+4}\right). \quad (2.15)$$

If the coefficients are constant, that is

$$a_n = a,\ b_n = b,\ c_n = c,\ d_n = d,\ s_n = s,\ t_n = t$$

then the linear equations in (2.3) becomes

$$\begin{cases} u_{n+3} = acsu_n + act + ad + b, \\ v_{n+3} = csav_n + csb + ct + d, \\ w_{n+3} = sacw_n + sad + sb + t. \end{cases} \tag{2.16}$$

Again, from Lemma 1.1, we get for all $n \in \mathbb{N}_0$ and for $i = 0, 1, 2$

$$u_{3n+i} = \begin{cases} u_i + (act + ad + b)n, & acs = 1, \\ (acs)^n u_i + \left(\dfrac{(acs)^n - 1}{acs - 1}\right)(act + ad + b), & otherwise, \end{cases} \tag{2.17}$$

$$v_{3n+i} = \begin{cases} v_i + (csb + ct + d)n, & acs = 1, \\ (csa)^n v_i + \left(\dfrac{(csa)^n - 1}{csa - 1}\right)(csb + ct + d), & otherwise, \end{cases} \tag{2.18}$$

and

$$w_{3n+i} = \begin{cases} w_i + (sad + sb + t)n, & acs = 1, \\ (sac)^n w_i + \left(\dfrac{(sac)^n - 1}{sac - 1}\right)(sad + sb + t), & otherwise. \end{cases} \tag{2.19}$$

Using, (2.1), (2.17), (2.18) and (2.19), it follows that for all $n \in \mathbb{N}_0$

$$u_{3n} = \begin{cases} \dfrac{(f(x_{-2}))^p}{f(x_0)f(x_{-1})} + (act + ad + b)n, & acs = 1, \\ \dfrac{(acs)^n (f(x_{-2}))^p}{f(x_0)f(x_{-1})} + \left(\dfrac{(acs)^n - 1}{acs - 1}\right)(act + ad + b), & otherwise, \end{cases} \tag{2.20}$$

$$u_{3n+1} = \begin{cases} \dfrac{a(g(y_{-2}))^q + bg(y_0)g(y_{-1})}{g(y_0)g(y_{-1})} + (act + ad + b)n, & acs = 1, \\ \dfrac{(acs)^n (a(g(y_{-2}))^q + bg(y_0)g(y_{-1}))}{g(y_0)g(y_{-1})} + \left(\dfrac{(acs)^n - 1}{acs - 1}\right)(act + ad + b), & otherwise, \end{cases} \tag{2.21}$$

$$u_{3n+2} = \begin{cases} \dfrac{ac(h(z_{-2})^r + adh(z_0)h(z_{-1}) + bh(z_0)h(z_{-1})}{h(z_0)h(z_{-1})} + (act + ad + b)n, & acs = 1, \\ \dfrac{(acs)^n (ac(h(z_{-2})^r + adh(z_0)h(z_{-1})) + bh(z_0)h(z_{-1}))}{h(z_0)h(z_{-1})} + \left(\dfrac{(acs)^n - 1}{acs - 1}\right)(act + ad + b), & otherwise, \end{cases} \tag{2.22}$$

$$v_{3n} = \begin{cases} \dfrac{(g(y_{-2}))^q}{g(y_0)g(y_{-1})} + (csb + ct + d)n, & acs = 1, \\ \dfrac{(aa)^n (g(y_{-2}))^q}{g(y_0)g(y_{-1})} + \left(\dfrac{(acs)^n - 1}{acs - 1}\right)(csb + ct + d), & otherwise, \end{cases} \tag{2.23}$$

$$v_{3n+1} = \begin{cases} \dfrac{c(h(z_{-2})^r + dh(z_0)h(z_{-1})}{h(z_0)h(z_{-1})} + (csb + ct + d)n, & acs = 1, \\ \dfrac{(csa)^n (c(h(z_{-2})^r + dh(z_0)h(z_{-1}))}{h(z_0)h(z_{-1})} + \left(\dfrac{(csa)^n - 1}{csa - 1}\right)(csb + ct + d), & otherwise, \end{cases} \tag{2.24}$$

$$v_{3n+2}=\begin{cases}\dfrac{cs(f(x_{-2}))^p+ctf(x_0)f(x_{-1})+df(x_0)f(x_{-1})}{f(x_0)f(x_{-1})}+(csb+ct+d)n, & acs=1,\\ \dfrac{(csa)^n(cs(f(x_{-2}))^p+ctf(x_0)f(x_{-1})+df(x_0)f(x_{-1}))}{f(x_0)f(x_{-1})}+\left(\dfrac{(csa)^n-1}{csa-1}\right)(csb+ct+d), & otherwise,\end{cases} \tag{2.25}$$

$$w_{3n}=\begin{cases}\dfrac{(h(z_{-2})^r}{h(z_0)h(z_{-1})}+(sad+sb+t)n, & acs=1,\\ \dfrac{(sac)^n(h(z_{-2})^r}{h(z_0)h(z_{-1})}+\left(\dfrac{(sac)^n-1}{sac-1}\right)(sad+sb+t), & otherwise,\end{cases} \tag{2.26}$$

$$w_{3n+1}=\begin{cases}\dfrac{s(f(x_{-2}))^p+tf(x_0)f(x_{-1})}{f(x_0)f(x_{-1})}+(sad+sb+t)n, & acs=1,\\ \dfrac{(sac)^n(s(f(x_{-2}))^p+tf(x_0)f(x_{-1}))}{f(x_0)f(x_{-1})}+\left(\dfrac{(sac)^n-1}{sac-1}\right)(sad+sb+t), & otherwise,\end{cases} \tag{2.27}$$

$$w_{3n+2}=\begin{cases}\dfrac{sa(g(y_{-2}))^q+sbg(y_0)g(y_{-1})+tg(y_0)g(y_{-1})}{g(y_0)g(y_{-1})}+(sad+sb+t)n, & acs=1,\\ \dfrac{(sac)^n(sa(g(y_{-2}))^q+sbg(y_0)g(y_{-1})+tg(y_0)g(y_{-1}))}{g(y_0)g(y_{-1})}+\left(\dfrac{(sac)^n-1}{sac-1}\right)(sad+sb+t), & otherwise.\end{cases} \tag{2.28}$$

Now, using the fact that the functions f, g and h are one-to-one, it follows from (2.1) that

$$x_n=f^{-1}\left(\frac{(f(x_{n-2}))^p}{u_nf(x_{n-1})}\right),\ y_n=g^{-1}\left(\frac{(g(y_{n-2}))^q}{v_ng(y_{n-1})}\right),\ z_n=h^{-1}\left(\frac{(h(z_{n-2}))^r}{w_nh(z_{n-1})}\right), \tag{2.29}$$

From which we obtain,

$$x_0=f^{-1}\left(\frac{(f(x_{-2}))^p}{u_0f(x_{-1})}\right)=f^{-1}\left(\frac{(f(x_{-2}))^{pF_{p,0}}}{u_0^{F_{p,0}}(f(x_{-1}))^{F_{p,1}}}\right),$$

Moreover,

$$x_1=f^{-1}\left(\frac{(f(x_{-1}))^p}{u_1f(x_0)}\right)=f^{-1}\left(\frac{u_0f(x_{-1})(f(x_{-1}))^p}{u_1(f(x_{-2}))^p}\right)=f^{-1}\left(\frac{u_0(f(x_{-1}))^{p+1}}{u_1(f(x_{-2}))^p}\right)=f^{-1}\left(\frac{u_0^{F_{p,1}}(f(x_{-1}))^{F_{p,2}}}{u_1^{F_{p,0}}(f(x_{-2}))^{pF_{p,1}}}\right).$$

Similarly, we obtain

$$x_2=f^{-1}\left(\frac{u_1(f(x_{-2}))^{p^2+p}}{u_2u_0^{p+1}(f(x_{-1}))^{2p+1}}\right)=f^{-1}\left(\frac{u_1^{F_{p,1}}(f(x_{-2}))^{pF_{p,2}}}{u_2^{F_{p,0}}u_0^{F_{p,2}}(f(x_{-1}))^{F_{p,3}}}\right).$$

$$x_3=f^{-1}\left(\frac{u_2u_0^{2p+1}(f(x_{-1}))^{p^2+3p+1}}{u_3u_1^{p+1}(f(x_{-2}))^{2p^2+p}}\right)=f^{-1}\left(\frac{u_2^{F_{p,1}}u_0^{F_{p,3}}(f(x_{-1}))^{F_{p,4}}}{u_3^{F_{p,0}}u_1^{F_{p,2}}(f(x_{-2}))^{pF_{p,3}}}\right),$$

$$x_4=f^{-1}\left(\frac{u_3u_1^{2p+1}(f(x_{-2}))^{p^3+3p^2+p}}{u_4u_2^{p+1}u_0^{p^2+3p+1}(f(x_{-1}))^{3p^2+4p+1}}\right)=f^{-1}\left(\frac{u_3^{F_{p,1}}u_1^{F_{p,3}}(f(x_{-2}))^{pF_{p,4}}}{u_4^{F_{p,0}}u_2^{F_{p,2}}u_0^{F_{p,4}}(f(x_{-1}))^{F_{p,5}}}\right),$$

$$x_5=f^{-1}\left(\frac{u_4u_2^{2p+1}u_0^{3p^2+4p+1}(f(x_{-1}))^{p^3+6p^2+5p+1}}{u_5u_3^{p+1}u_1^{p^2+3p+1}(f(x_{-2}))^{3p^3+4p^2+p}}\right)=f^{-1}\left(\frac{u_4^{F_{p,1}}u_2^{F_{p,3}}u_0^{F_{p,5}}(f(x_{-1}))^{F_{p,6}}}{u_5^{F_{p,0}}u_3^{F_{p,2}}u_1^{F_{p,4}}(f(x_{-2}))^{pF_{p,5}}}\right),$$

$$x_6=f^{-1}\left(\frac{u_5u_3^{2p+1}u_1^{3p^2+4p+1}(f(x_{-2}))^{p^4+6p^3+5p^2+p}}{u_6u_4^{p+1}u_2^{p^2+3p+1}u_0^{p^3+6p^2+5p+1}(f(x_{-1}))^{4p^3+10p^2+6p+1}}\right)=f^{-1}\left(\frac{u_5^{F_{p,1}}u_3^{F_{p,3}}u_1^{F_{p,5}}(f(x_{-2}))^{pF_{p,6}}}{u_6^{F_{p,0}}u_4^{F_{p,2}}u_2^{F_{p,4}}u_0^{F_{p,6}}(f(x_{-1}))^{F_{p,7}}}\right),$$

$$x_7 = f^{-1}\left(\frac{u_6^{F_{p,1}} u_4^{F_{p,3}} u_2^{F_{p,5}} u_0^{F_{p,7}} (f(x_{-1}))^{F_{p,8}}}{u_7^{F_{p,0}} u_5^{F_{p,2}} u_3^{F_{p,4}} u_1^{F_{p,6}} (f(x_{-2}))^{pF_{p,7}}}\right),$$

$$x_8 = f^{-1}\left(\frac{u_7^{F_{p,1}} u_5^{F_{p,3}} u_3^{F_{p,5}} u_1^{F_{p,7}} (f(x_{-2}))^{pF_{p,8}}}{u_8^{F_{p,0}} u_6^{F_{p,2}} u_4^{F_{p,4}} u_2^{F_{p,6}} u_0^{F_{p,8}} (f(x_{-1}))^{F_{p,9}}}\right),$$

$$x_9 = f^{-1}\left(\frac{u_8^{F_{p,1}} u_6^{F_{p,3}} u_4^{F_{p,5}} u_2^{F_{p,7}} u_0^{F_{p,9}} (f(x_{-1}))^{F_{p,10}}}{u_9^{F_{p,0}} u_7^{F_{p,2}} u_5^{F_{p,4}} u_3^{F_{p,6}} u_1^{F_{p,8}} (f(x_{-2}))^{pF_{p,9}}}\right),$$

$$x_{10} = f^{-1}\left(\frac{u_9^{F_{p,1}} u_7^{F_{p,3}} u_5^{F_{p,5}} u_3^{F_{p,7}} u_1^{F_{p,9}} (f(x_{-2}))^{pF_{p,10}}}{u_{10}^{F_{p,0}} u_8^{F_{p,2}} u_6^{F_{p,4}} u_4^{F_{p,6}} u_2^{F_{p,8}} u_0^{F_{p,10}} (f(x_{-1}))^{F_{p,11}}}\right),$$

$$x_{11} = f^{-1}\left(\frac{u_{10}^{F_{p,1}} u_8^{F_{p,3}} u_6^{F_{p,5}} u_4^{F_{p,7}} u_2^{F_{p,9}} u_0^{F_{p,11}} (f(x_{-1}))^{F_{p,12}}}{u_{11}^{F_{p,0}} u_9^{F_{p,2}} u_7^{F_{p,4}} u_5^{F_{p,6}} u_3^{F_{p,8}} u_1^{F_{p,10}} (f(x_{-2}))^{pF_{p,11}}}\right),$$

$$x_{12} = f^{-1}\left(\frac{u_{11}^{F_{p,1}} u_9^{F_{p,3}} u_7^{F_{p,5}} u_5^{F_{p,7}} u_3^{F_{p,9}} u_1^{F_{p,11}} (f(x_{-2}))^{pF_{p,12}}}{u_{12}^{F_{p,0}} u_{10}^{F_{p,2}} u_8^{F_{p,4}} u_6^{F_{p,6}} u_4^{F_{p,8}} u_2^{F_{p,10}} u_0^{F_{p,12}} (f(x_{-1}))^{F_{p,13}}}\right).$$

By induction, it follows that

$$x_{6n} = f^{-1}\left(\frac{\left(\prod_{i=0}^{n-1} u_{3(2i)+1}^{F_{p,6(n-i)-1}} u_{3(2i+1)}^{F_{p,6(n-i)-3}} u_{3(2i+1)+2}^{F_{p,6(n-i)-5}}\right)(f(x_{-2}))^{pF_{p,6n}}}{\left(\prod_{i=0}^{n} u_{3(2i)}^{F_{p,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i)+2}^{F_{p,6(n-i)-2}} u_{3(2i+1)+1}^{F_{p,6(n-i)-4}}\right)(f(x_{-1}))^{F_{p,6n+1}}}\right), \tag{2.30}$$

$$x_{6n+1} = f^{-1}\left(\frac{\left(\prod_{i=0}^{n} u_{3(2i)}^{F_{p,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i)+2}^{F_{p,6(n-i)-1}} u_{3(2i+1)+1}^{F_{p,6(n-i)-3}}\right)(f(x_{-1}))^{F_{p,6n+2}}}{\left(\prod_{i=0}^{n} u_{3(2i)+1}^{F_{p,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i+1)}^{F_{p,6(n-i)-2}} u_{3(2i+1)+2}^{F_{p,6(n-i)-4}}\right)(f(x_{-2}))^{pF_{p,6n+1}}}\right), \tag{2.31}$$

$$x_{6n+2} = f^{-1}\left(\frac{\left(\prod_{i=0}^{n} u_{3(2i)+1}^{F_{p,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i+1)}^{F_{p,6(n-i)-1}} u_{3(2i+1)+2}^{F_{p,6(n-i)-3}}\right)(f(x_{-2}))^{pF_{p,6n+2}}}{\left(\prod_{i=0}^{n} u_{3(2i)}^{F_{p,6(n-i)+2}} u_{3(2i)+2}^{F_{p,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i+1)+1}^{F_{p,6(n-i)-2}}\right)(f(x_{-1}))^{F_{p,6n+3}}}\right), \tag{2.32}$$

$$x_{6n+3} = f^{-1}\left(\frac{\left(\prod_{i=0}^{n} u_{3(2i)}^{F_{p,6(n-i)+3}} u_{3(2i)+2}^{F_{p,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i+1)+1}^{F_{p,6(n-i)-1}}\right)(f(x_{-1}))^{F_{p,6n+4}}}{\left(\prod_{i=0}^{n} u_{3(2i)+1}^{F_{p,6(n-i)+2}} u_{3(2i+1)}^{F_{p,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i+1)+2}^{F_{p,6(n-i)-2}}\right)(f(x_{-2}))^{pF_{p,6n+3}}}\right), \tag{2.33}$$

$$x_{6n+4} = f^{-1}\left(\frac{\left(\prod_{i=0}^{n} u_{3(2i)+1}^{F_{p,6(n-i)+3}} u_{3(2i+1)}^{F_{p,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i+1)+2}^{F_{p,6(n-i)-1}}\right)(f(x_{-2}))^{pF_{p,6n+4}}}{\left(\prod_{i=0}^{n} u_{3(2i)}^{F_{p,6(n-i)+4}} u_{3(2i)+2}^{F_{p,6(n-i)+2}} u_{3(2i+1)+1}^{F_{p,6(n-i)}}\right)(f(x_{-1}))^{F_{p,6n+5}}}\right), \tag{2.34}$$

$$x_{6n+5} = f^{-1}\left(\frac{\left(\prod_{i=0}^{n} u_{3(2i)}^{F_{p,6(n-i)+5}} u_{3(2i)+2}^{F_{p,6(n-i)+3}} u_{3(2i+1)+1}^{F_{p,6(n-i)+1}}\right)(f(x_{-1}))^{F_{p,6(n+1)}}}{\left(\prod_{i=0}^{n} u_{3(2i)+1}^{F_{p,6(n-i)+4}} u_{3(2i+1)}^{F_{p,6(n-i)+2}} u_{3(2i+1)+2}^{F_{p,6(n-i)}}\right)(f(x_{-2}))^{pF_{p,6n+5}}}\right), \tag{2.35}$$

Similarly, by following the same steps we obtain

$$y_{6n} = g^{-1}\left(\frac{\left(\prod_{i=0}^{n-1} v_{3(2i)+1}^{F_{q,6(n-i)-1}} v_{3(2i+1)}^{F_{q,6(n-i)-3}} v_{3(2i+1)+2}^{F_{q,6(n-i)-5}}\right)(g(y_{-2}))^{qF_{q,6n}}}{\left(\prod_{i=0}^{n} v_{3(2i)}^{F_{q,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} v_{3(2i)+2}^{F_{q,6(n-i)-2}} v_{3(2i+1)+1}^{F_{q,6(n-i)-4}}\right)(g(y_{-1}))^{F_{q,6n+1}}}\right), \tag{2.36}$$

$$y_{6n+1} = g^{-1}\left(\frac{\left(\prod_{i=0}^{n} v_{3(2i)}^{F_{q,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} v_{3(2i)+2}^{F_{q,6(n-i)-1}} v_{3(2i+1)+1}^{F_{q,6(n-i)-3}}\right)(g(y_{-1}))^{F_{q,6n+2}}}{\left(\prod_{i=0}^{n} v_{3(2i)+1}^{F_{q,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} v_{3(2i+1)}^{F_{q,6(n-i)-2}} v_{3(2i+1)+2}^{F_{q,6(n-i)-4}}\right)(g(y_{-2}))^{qF_{q,6n+1}}}\right), \tag{2.37}$$

$$y_{6n+2} = g^{-1}\left(\frac{\left(\prod_{i=0}^{n} v_{3(2i)+1}^{F_{q,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} v_{3(2i+1)}^{F_{q,6(n-i)-1}} v_{3(2i+1)+2}^{F_{q,6(n-i)-3}}\right)(g(y_{-2}))^{qF_{q,6n+2}}}{\left(\prod_{i=0}^{n} v_{3(2i)}^{F_{q,6(n-i)+2}} v_{3(2i)+2}^{F_{q,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} v_{3(2i+1)+1}^{F_{q,6(n-i)-2}}\right)(g(y_{-1}))^{F_{q,6n+3}}}\right), \tag{2.38}$$

$$y_{6n+3} = g^{-1}\left(\frac{\left(\prod_{i=0}^{n} v_{3(2i)}^{F_{q,6(n-i)+3}} v_{3(2i)+2}^{F_{q,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} v_{3(2i+1)+1}^{F_{q,6(n-i)-1}}\right)(g(y_{-1}))^{F_{q,6n+4}}}{\left(\prod_{i=0}^{n} v_{3(2i)+1}^{F_{q,6(n-i)+2}} v_{3(2i+1)}^{F_{q,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} v_{3(2i+1)+2}^{F_{q,6(n-i)-2}}\right)(g(y_{-2}))^{qF_{q,6n+3}}}\right), \tag{2.39}$$

$$y_{6n+4} = g^{-1}\left(\frac{\left(\prod_{i=0}^{n} v_{3(2i)+1}^{F_{q,6(n-i)+3}} v_{3(2i+1)}^{F_{q,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} v_{3(2i+1)+2}^{F_{q,6(n-i)-1}}\right)(g(y_{-2}))^{qF_{q,6n+4}}}{\left(\prod_{i=0}^{n} v_{3(2i)}^{F_{q,6(n-i)+4}} v_{3(2i)+2}^{F_{q,6(n-i)+2}} v_{3(2i+1)+1}^{F_{q,6(n-i)}}\right)(g(y_{-1}))^{F_{q,6n+5}}}\right), \tag{2.40}$$

$$y_{6n+5} = g^{-1}\left(\frac{\left(\prod_{i=0}^{n} v_{3(2i)}^{F_{q,6(n-i)+5}} v_{3(2i)+2}^{F_{q,6(n-i)+3}} v_{3(2i+1)+1}^{F_{q,6(n-i)+1}}\right)(g(y_{-1}))^{F_{q,6(n+1)}}}{\left(\prod_{i=0}^{n} v_{3(2i)+1}^{F_{q,6(n-i)+4}} v_{3(2i+1)}^{F_{q,6(n-i)+2}} v_{3(2i+1)+2}^{F_{q,6(n-i)}}\right)(g(y_{-2}))^{qF_{q,6n+5}}}\right), \tag{2.41}$$

and

$$z_{6n} = h^{-1}\left(\frac{\left(\prod_{i=0}^{n-1} w_{3(2i)+1}^{F_{r,6(n-i)-1}} w_{3(2i+1)}^{F_{r,6(n-i)-3}} w_{3(2i+1)+2}^{F_{r,6(n-i)-5}}\right)(h(z_{-2}))^{rF_{r,6n}}}{\left(\prod_{i=0}^{n} w_{3(2i)}^{F_{r,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} w_{3(2i)+2}^{F_{r,6(n-i)-2}} w_{3(2i+1)+1}^{F_{r,6(n-i)-4}}\right)(h(z_{-1}))^{F_{r,6n+1}}}\right), \tag{2.42}$$

$$z_{6n+1} = h^{-1}\left(\frac{\left(\prod_{i=0}^{n} w_{3(2i)}^{F_{r,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} w_{3(2i)+2}^{F_{r,6(n-i)-1}} w_{3(2i+1)+1}^{F_{r,6(n-i)-3}}\right)(h(z_{-1}))^{F_{r,6n+2}}}{\left(\prod_{i=0}^{n} w_{3(2i)+1}^{F_{r,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} w_{3(2i+1)}^{F_{r,6(n-i)-2}} w_{3(2i+1)+2}^{F_{r,6(n-i)-4}}\right)(h(z_{-2}))^{rF_{r,6n+1}}}\right), \tag{2.43}$$

$$z_{6n+2} = h^{-1}\left(\frac{\left(\prod_{i=0}^{n} w_{3(2i)+1}^{F_{r,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} w_{3(2i+1)}^{F_{r,6(n-i)-1}} w_{3(2i+1)+2}^{F_{r,6(n-i)-3}}\right)(h(z_{-2}))^{rF_{r,6n+2}}}{\left(\prod_{i=0}^{n} w_{3(2i)}^{F_{r,6(n-i)+2}} w_{3(2i)+2}^{F_{r,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} w_{3(2i+1)+1}^{F_{r,6(n-i)-2}}\right)(h(z_{-1}))^{F_{r,6n+3}}}\right), \tag{2.44}$$

$$z_{6n+3} = h^{-1}\left(\frac{\left(\prod_{i=0}^{n} w_{3(2i)}^{F_{r,6(n-i)+3}} w_{3(2i)+2}^{F_{r,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} w_{3(2i+1)+1}^{F_{r,6(n-i)-1}}\right)(h(z_{-1}))^{F_{r,6n+4}}}{\left(\prod_{i=0}^{n} w_{3(2i)+1}^{F_{r,6(n-i)+2}} w_{3(2i+1)}^{F_{r,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} w_{3(2i+1)+2}^{F_{r,6(n-i)-2}}\right)(h(z_{-2}))^{rF_{r,6n+3}}}\right), \tag{2.45}$$

$$z_{6n+4}=h^{-1}\left(\frac{\left(\prod_{i=0}^{n} w_{3(2i)+1}^{F_{r,6(n-i)+3}} w_{3(2i+1)}^{F_{r,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} w_{3(2i+1)+2}^{F_{r,6(n-i)-1}}\right)(h(z_{-2}))^{rF_{r,6n+4}}}{\left(\prod_{i=0}^{n} w_{3(2i)}^{F_{r,6(n-i)+4}} w_{3(2i)+2}^{F_{r,6(n-i)+2}} w_{3(2i+1)+1}^{F_{r,6(n-i)}}\right)(h(z_{-1}))^{F_{r,6n+5}}}\right), \tag{2.46}$$

$$z_{6n+5}=h^{-1}\left(\frac{\left(\prod_{i=0}^{n} w_{3(2i)}^{F_{r,6(n-i)+5}} w_{3(2i)+2}^{F_{r,6(n-i)+3}} w_{3(2i+1)+1}^{F_{r,6(n-i)+1}}\right)(h(z_{-1}))^{F_{r,6(n+1)}}}{\left(\prod_{i=0}^{n} w_{3(2i)+1}^{F_{r,6(n-i)+4}} w_{3(2i+1)}^{F_{r,6(n-i)+2}} w_{3(2i+1)+2}^{F_{r,6(n-i)}}\right)(h(z_{-2}))^{rF_{r,6n+5}}}\right). \tag{2.47}$$

In summary we have the following result.

Theorem 2.1. *Let $\{x_n, y_n, z_n\}_{n\geq -2}$ be a well-defined solution of System (1.1). Then, for all $n \in \mathbb{N}_0$, the x_n-component (resp. the y_n component and of the z_n-component) are given by equations (2.30)-(2.35) for x_n (resp. equations (2.36)-(2.41) for y_n and equations (2.42)-(2.47) for the z_n), where the sequences $\{u_n\}_{n\in\mathbb{N}_0}$, $\{v_n\}_{n\in\mathbb{N}_0}$ and $\{w_n\}_{n\in\mathbb{N}_0}$ are defined by the formulas (2.7)-(2.15) in the case of variables coefficients and by formulas (2.20)-(2.28) in the case of constant coefficients.*

3. Formulas of the solutions of a particular System

In System (1.1), we let $f(x) = g(x) = h(x) = x,\ D = \mathbb{R} - \{0\}$ to end up with the following particular system

$$x_{n+1}=\frac{y_n y_{n-1} x_{n-1}^{p}}{x_n(a_n y_{n-2}^{q}+b_n y_n y_{n-1})},\ y_{n+1}=\frac{z_n z_{n-1} y_{n-1}^{q}}{y_n(c_n z_{n-2}^{r}+d_n z_n z_{n-1})},\ z_{n+1}=\frac{x_n x_{n-1} z_{n-1}^{r}}{z_n(s_n x_{n-2}^{p}+t_n x_n x_{n-1})}. \tag{3.1}$$

Clearly, we have

$$f^{-1}(x) = g^{-1}(x) = h^{-1}(x) = x.$$

In this case a solution $\{x_n, y_n, z_n\}_{n\geq -2}$ of System (3.1) is said to be well-defined if for all $n \in \mathbb{N}_0$, we have

$$x_n y_n z_n \left(a_n y_{n-2}^{q} + b_n y_n y_{n-1}\right)\left(c_n z_{n-2}^{r} + d_n z_n z_{n-1}\right)\left(s_n x_{n-2}^{p} + t_n x_n x_{n-1}\right) \neq 0.$$

As a result of Theorem (2.1), we have

Corollary 3.1. *Let $\{x_n, y_n, z_n\}_{n\geq -2}$ be a well-defined solution of System (3.1). Then, for $n = 0, 1, \ldots,$ we have*

$$x_{6n}=\frac{\prod_{i=0}^{n-1}\left(u_{3(2i)+1}^{F_{p,6(n-i)-1}}\right)\left(u_{3(2i+1)}^{F_{p,6(n-i)-3}}\right)\left(u_{3(2i+1)+2}^{F_{p,6(n-i)-5}}\right)x_{-2}^{pF_{p,6n}}}{\prod_{i=0}^{n}\left(u_{3(2i)}^{F_{p,6(n-i)}}\right)\prod_{i=0}^{n-1}\left(u_{3(2i)+2}^{F_{p,6(n-i)-2}}\right)\left(u_{3(2i+1)+1}^{F_{p,6(n-i)-4}}\right)x_{-1}^{F_{p,6n+1}}},$$

$$x_{6n+1}=\frac{\prod_{i=0}^{n}\left(u_{3(2i)}^{F_{p,6(n-i)+1}}\right)\prod_{i=0}^{n-1}\left(u_{3(2i)+2}^{F_{p,6(n-i)-1}}\right)\left(u_{3(2i+1)+1}^{F_{p,6(n-i)-3}}\right)x_{-1}^{F_{p,6n+2}}}{\prod_{i=0}^{n}\left(u_{3(2i)+1}^{F_{p,6(n-i)}}\right)\prod_{i=0}^{n-1}\left(u_{3(2i+1)}^{F_{p,6(n-i)-2}}\right)\left(u_{3(2i+1)+2}^{F_{p,6(n-i)-4}}\right)x_{-2}^{pF_{p,6n+1}}},$$

$$x_{6n+2}=\frac{\prod_{i=0}^{n}\left(u_{3(2i)+1}^{F_{p,6(n-i)+1}}\right)\prod_{i=0}^{n-1}\left(u_{3(2i+1)}^{F_{p,6(n-i)-1}}\right)\left(u_{3(2i+1)+2}^{F_{p,6(n-i)-3}}\right)x_{-2}^{pF_{p,6n+2}}}{\prod_{i=0}^{n}\left(u_{3(2i)}^{F_{p,6(n-i)+2}}\right)\left(u_{3(2i)+2}^{F_{p,6(n-i)}}\right)\prod_{i=0}^{n-1}\left(u_{3(2i+1)+1}^{F_{p,6(n-i)-2}}\right)x_{-1}^{F_{p,6n+3}}},$$

$$x_{6n+3}=\frac{\prod_{i=0}^{n}\left(u_{3(2i)}^{F_{p,6(n-i)+3}}\right)\left(u_{3(2i)+2}^{F_{p,6(n-i)+1}}\right)\prod_{i=0}^{n-1}\left(u_{3(2i+1)+1}^{F_{p,6(n-i)-1}}\right)x_{-1}^{F_{p,6n+4}}}{\prod_{i=0}^{n}\left(u_{3(2i)+1}^{F_{p,6(n-i)+2}}\right)\left(u_{3(2i+1)}^{F_{p,6(n-i)}}\right)\prod_{i=0}^{n-1}\left(u_{3(2i+1)+2}^{F_{p,6(n-i)-2}}\right)x_{-2}^{pF_{p,6n+3}}},$$

$$x_{6n+4}=\frac{\prod_{i=0}^{n}\left(u_{3(2i)+1}^{F_{p,6(n-i)+3}}\right)\left(u_{3(2i+1)}^{F_{p,6(n-i)+1}}\right)\prod_{i=0}^{n-1}\left(u_{3(2i+1)+2}^{F_{p,6(n-i)-1}}\right)x_{-2}^{pF_{p,6n+4}}}{\prod_{i=0}^{n}\left(u_{3(2i)}^{F_{p,6(n-i)+4}}\right)\left(u_{3(2i)+2}^{F_{p,6(n-i)+2}}\right)\left(u_{3(2i+1)+1}^{F_{p,6(n-i)}}\right)x_{-1}^{F_{p,6n+5}}},$$

$$x_{6n+5}=\frac{\prod_{i=0}^{n}\left(u_{3(2i)}^{F_{p,6(n-i)+5}}\right)\left(u_{3(2i)+2}^{F_{p,6(n-i)+3}}\right)\left(u_{3(2i+1)+1}^{F_{p,6(n-i)+1}}\right)x_{-1}^{F_{p,6(n+1)}}}{\prod_{i=0}^{n}\left(u_{3(2i)+1}^{F_{p,6(n-i)+4}}\right)\left(u_{3(2i+1)}^{F_{p,6(n-i)+2}}\right)\left(u_{3(2i+1)+2}^{F_{p,6(n-i)}}\right)x_{-2}^{pF_{p,6n+5}}},$$

$$y_{6n}=\frac{\prod_{i=0}^{n-1}\left(v_{3(2i)+1}^{F_{q,6(n-i)-1}}\right)\left(v_{3(2i+1)}^{F_{q,6(n-i)-3}}\right)\left(v_{3(2i+1)+2}^{F_{q,6(n-i)-5}}\right)y_{-2}^{qF_{q,6n}}}{\prod_{i=0}^{n}\left(v_{3(2i)}^{F_{q,6(n-i)}}\right)\prod_{i=0}^{n-1}\left(v_{3(2i)+2}^{F_{q,6(n-i)-2}}\right)\left(v_{3(2i+1)+1}^{F_{q,6(n-i)-4}}\right)y_{-1}^{F_{q,6n+1}}},$$

$$y_{6n+1}=\frac{\prod_{i=0}^{n}\left(v_{3(2i)}^{F_{q,6(n-i)+1}}\right)\prod_{i=0}^{n-1}\left(v_{3(2i)+2}^{F_{q,6(n-i)-1}}\right)\left(v_{3(2i+1)+1}^{F_{q,6(n-i)-3}}\right)y_{-1}^{F_{q,6n+2}}}{\prod_{i=0}^{n}\left(v_{3(2i)+1}^{F_{q,6(n-i)}}\right)\prod_{i=0}^{n-1}\left(v_{3(2i+1)}^{F_{q,6(n-i)-2}}\right)\left(v_{3(2i+1)+2}^{F_{q,6(n-i)-4}}\right)y_{-2}^{qF_{q,6n+1}}},$$

$$y_{6n+2}=\frac{\prod_{i=0}^{n}\left(v_{3(2i)+1}^{F_{q,6(n-i)+1}}\right)\prod_{i=0}^{n-1}\left(v_{3(2i+1)}^{F_{q,6(n-i)-1}}\right)\left(v_{3(2i+1)+2}^{F_{q,6(n-i)-3}}\right)y_{-2}^{qF_{q,6n+2}}}{\prod_{i=0}^{n}\left(v_{3(2i)}^{F_{q,6(n-i)+2}}\right)\left(v_{3(2i)+2}^{F_{q,6(n-i)}}\right)\prod_{i=0}^{n-1}\left(v_{3(2i+1)+1}^{F_{q,6(n-i)-2}}\right)y_{-1}^{F_{q,6n+3}}},$$

$$y_{6n+3}=\frac{\prod_{i=0}^{n}\left(v_{3(2i)}^{F_{q,6(n-i)+3}}\right)\left(v_{3(2i)+2}^{F_{q,6(n-i)+1}}\right)\prod_{i=0}^{n-1}\left(v_{3(2i+1)+1}^{F_{q,6(n-i)-1}}\right)y_{-1}^{F_{q,6n+4}}}{\prod_{i=0}^{n}\left(v_{3(2i)+1}^{F_{q,6(n-i)+2}}\right)\left(v_{3(2i+1)}^{F_{q,6(n-i)}}\right)\prod_{i=0}^{n-1}\left(v_{3(2i+1)+2}^{F_{q,6(n-i)-2}}\right)y_{-2}^{qF_{q,6n+3}}},$$

$$y_{6n+4}=\frac{\prod_{i=0}^{n}\left(v_{3(2i)+1}^{F_{q,6(n-i)+3}}\right)\left(v_{3(2i+1)}^{F_{q,6(n-i)+1}}\right)\prod_{i=0}^{n-1}\left(v_{3(2i+1)+2}^{F_{q,6(n-i)-1}}\right)y_{-2}^{qF_{q,6n+4}}}{\prod_{i=0}^{n}\left(v_{3(2i)}^{F_{q,6(n-i)+4}}\right)\left(v_{3(2i)+2}^{F_{q,6(n-i)+2}}\right)\left(v_{3(2i+1)+1}^{F_{q,6(n-i)}}\right)y_{-1}^{F_{q,6n+5}}},$$

$$y_{6n+5}=\frac{\prod_{i=0}^{n}\left(v_{3(2i)}^{F_{q,6(n-i)+5}}\right)\left(v_{3(2i)+2}^{F_{q,6(n-i)+3}}\right)\left(v_{3(2i+1)+1}^{F_{q,6(n-i)+1}}\right)y_{-1}^{F_{q,6(n+1)}}}{\prod_{i=0}^{n}\left(v_{3(2i)+1}^{F_{q,6(n-i)+4}}\right)\left(v_{3(2i+1)}^{F_{q,6(n-i)+2}}\right)\left(v_{3(2i+1)+2}^{F_{q,6(n-i)}}\right)y_{-2}^{qF_{q,6n+5}}},$$

$$z_{6n}=\frac{\prod_{i=0}^{n-1}\left(w_{3(2i)+1}^{F_{r,6(n-i)-1}}\right)\left(w_{3(2i+1)}^{F_{r,6(n-i)-3}}\right)\left(w_{3(2i+1)+2}^{F_{r,6(n-i)-5}}\right)z_{-2}^{rF_{r,6n}}}{\prod_{i=0}^{n}\left(w_{3(2i)}^{F_{r,6(n-i)}}\right)\prod_{i=0}^{n-1}\left(w_{3(2i)+2}^{F_{r,6(n-i)-2}}\right)\left(w_{3(2i+1)+1}^{F_{r,6(n-i)-4}}\right)z_{-1}^{F_{r,6n+1}}},$$

$$z_{6n+1}=\frac{\prod_{i=0}^{n}\left(w_{3(2i)}^{F_{r,6(n-i)+1}}\right)\prod_{i=0}^{n-1}\left(w_{3(2i)+2}^{F_{r,6(n-i)-1}}\right)\left(w_{3(2i+1)+1}^{F_{r,6(n-i)-3}}\right)z_{-1}^{F_{r,6n+2}}}{\prod_{i=0}^{n}\left(w_{3(2i)+1}^{F_{r,6(n-i)}}\right)\prod_{i=0}^{n-1}\left(w_{3(2i+1)}^{F_{r,6(n-i)-2}}\right)\left(w_{3(2i+1)+2}^{F_{r,6(n-i)-4}}\right)z_{-2}^{rF_{r,6n+1}}},$$

$$z_{6n+2}=\frac{\prod_{i=0}^{n}\left(w_{3(2i)+1}^{F_{r,6(n-i)+1}}\right)\prod_{i=0}^{n-1}\left(w_{3(2i+1)}^{F_{r,6(n-i)-1}}\right)\left(w_{3(2i+1)+2}^{F_{r,6(n-i)-3}}\right)z_{-2}^{rF_{r,6n+2}}}{\prod_{i=0}^{n}\left(w_{3(2i)}^{F_{r,6(n-i)+2}}\right)\left(w_{3(2i)+2}^{F_{r,6(n-i)}}\right)\prod_{i=0}^{n-1}\left(w_{3(2i+1)+1}^{F_{r,6(n-i)-2}}\right)z_{-1}^{F_{r,6n+3}}},$$

$$z_{6n+3}=\frac{\prod_{i=0}^{n}\left(w_{3(2i)}^{F_{r,6(n-i)+3}}\right)\left(w_{3(2i)+2}^{F_{r,6(n-i)+1}}\right)\prod_{i=0}^{n-1}\left(w_{3(2i+1)+1}^{F_{r,6(n-i)-1}}\right)z_{-1}^{F_{r,6n+4}}}{\prod_{i=0}^{n}\left(w_{3(2i)+1}^{F_{r,6(n-i)+2}}\right)\left(w_{3(2i+1)}^{F_{r,6(n-i)}}\right)\prod_{i=0}^{n-1}\left(w_{3(2i+1)+2}^{F_{r,6(n-i)-2}}\right)z_{-2}^{rF_{r,6n+3}}},$$

$$z_{6n+4}=\frac{\prod_{i=0}^{n}\left(w_{3(2i)+1}^{F_{r,6(n-i)+3}}\right)\left(w_{3(2i+1)}^{F_{r,6(n-i)+1}}\right)\prod_{i=0}^{n-1}\left(w_{3(2i+1)+2}^{F_{r,6(n-i)-1}}\right)z_{-2}^{rF_{r,6n+4}}}{\prod_{i=0}^{n}\left(w_{3(2i)}^{F_{r,6(n-i)+4}}\right)\left(w_{3(2i)+2}^{F_{r,6(n-i)+2}}\right)\left(w_{3(2i+1)+1}^{F_{r,6(n-i)}}\right)z_{-1}^{F_{r,6n+5}}},$$

$$z_{6n+5}=\frac{\prod_{i=0}^{n}\left(w_{3(2i)}^{F_{r,6(n-i)+5}}\right)\left(w_{3(2i)+2}^{F_{r,6(n-i)+3}}\right)\left(w_{3(2i+1)+1}^{F_{r,6(n-i)+1}}\right)z_{-1}^{F_{r,6(n+1)}}}{\prod_{i=0}^{n}\left(w_{3(2i)+1}^{F_{r,6(n-i)+4}}\right)\left(w_{3(2i+1)}^{F_{r,6(n-i)+2}}\right)\left(w_{3(2i+1)+2}^{F_{r,6(n-i)}}\right)z_{-2}^{rF_{r,6n+5}}},$$

where the sequences $\{u_n\}_{n\in\mathbb{N}_0}$, $\{v_n\}_{n\in\mathbb{N}_0}$ *and* $\{v_n\}_{n\in\mathbb{N}_0}$ *are defined by the formulas*

$$u_{3n}=\left[\prod_{j=0}^{n-1}a_{3j+2}c_{3j+1}s_{3j}\right]\frac{x_{-2}^{p}}{x_0x_{-1}}+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}a_{3j+2}c_{3j+1}s_{3j}\right]\left(a_{3r+2}c_{3r+1}t_{3r}+a_{3r+2}d_{3r+1}+b_{3r+2}\right),$$

$$u_{3n+1}=\frac{\left[\prod_{j=0}^{n-1}a_{3(j+1)}c_{3j+2}s_{3j+1}\right]\left[a_0y_{-2}^{q}+b_0y_0y_{-1}\right]}{y_0y_{-1}}$$
$$+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}a_{3(j+1)}c_{3j+2}s_{3j+1}\right]\left(a_{3(r+1)}c_{3r+2}t_{3r+1}+a_{3(r+1)}d_{3r+2}+b_{3(r+1)}\right),$$

$$u_{3n+2}=\frac{\left[\prod_{j=0}^{n-1}a_{3j+4}c_{3(j+1)}s_{3j+2}\right]\left[a_1c_0z_{-2}^{r}+a_1d_0z_0z_{-1}+b_1z_0z_{-1}\right]}{z_0z_{-1}}$$
$$+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}a_{3j+4}c_{3(j+1)}s_{3j+2}\right]\left(a_{3r+4}c_{3(r+1)}t_{3r+2}+a_{3r+4}d_{3(r+1)}+b_{3r+4}\right),$$

$$v_{3n}=\frac{\left[\prod_{j=0}^{n-1}c_{3j+2}s_{3j+1}a_{3j}\right]y_{-2}^{q}}{y_0y_{-1}}+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}c_{3j+2}s_{3j+1}a_{3j}\right]\left(c_{3r+2}s_{3r+1}b_{3r}+c_{3r+2}t_{3r+1}+d_{3r+2}\right),$$

$$v_{3n+1}=\frac{\left[\prod_{j=0}^{n-1}c_{3(j+1)}s_{3j+2}a_{3j+1}\right]\left[c_0z_{-2}^{r}+d_0z_0z_{-1}\right]}{z_0z_{-1}}$$
$$+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}c_{3(j+1)}s_{3j+2}a_{3j+1}\right]\left(c_{3(r+1)}s_{3r+2}b_{3r+1}+c_{3(r+1)}t_{3r+2}+d_{3(r+1)}\right),$$

$$v_{3n+2} = \frac{\left[\prod_{j=0}^{n-1} c_{3j+4}s_{3(j+1)}a_{3j+2}\right]\left[c_1 s_0 x_{-2}^p + c_1 t_0 x_0 x_{-1} + d_1 x_0 x_{-1}\right]}{x_0 x_{-1}}$$
$$+ \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} c_{3j+4}s_{3(j+1)}a_{3j+2}\right]\left(c_{3r+4}s_{3(r+1)}b_{3r+2} + c_{3r+4}t_{3(r+1)} + d_{3r+4}\right),$$

and

$$w_{3n} = \frac{\left[\prod_{j=0}^{n-1} s_{3j+2}a_{3j+1}c_{3j}\right] z_{-2}^r}{z_0 z_{-1}} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} s_{3j+2}a_{3j+1}c_{3j}\right]\left(s_{3r+2}a_{3r+1}d_{3r} + s_{3r+2}b_{3r+1} + t_{3r+2}\right),$$

$$w_{3n+1} = \frac{\left[\prod_{j=0}^{n-1} s_{3(j+1)}a_{3j+2}c_{3j+1}\right]\left[s_0 x_{-2}^p + t_0 x_0 x_{-1}\right]}{x_0 x_{-1}}$$
$$+ \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} s_{3(j+1)}a_{3j+2}c_{3j+1}\right]\left(s_{3(r+1)}a_{3r+2}d_{3r+1} + s_{3(r+1)}b_{3r+2} + t_{3(r+1)}\right),$$

$$w_{3n+2} = \frac{\left[\prod_{j=0}^{n-1} s_{3j+4}a_{3(j+1)}c_{3j+2}\right]\left[say_{-2}^q + sby_0 y_{-1} + ty_0 y_{-1}\right]}{y_0 y_{-1}}$$
$$+ \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} s_{3j+4}a_{3(j+1)}c_{3j+2}\right]\left(s_{3r+4}a_{3(r+1)}d_{3r+2} + s_{3r+4}b_{3(r+1)} + t_{3r+4}\right).$$

in the case of variables coefficients, and formulas

$$u_{3n} = \begin{cases} \frac{x_{-2}^p}{x_0 x_{-1}} + (act + ad + b)n, & acs = 1, \\ \frac{(acs)^n x_{-2}^p}{x_0 x_{-1}} + \left(\frac{(acs)^n - 1}{acs - 1}\right)(act + ad + b), & otherwise, \end{cases}$$

$$u_{3n+1} = \begin{cases} \frac{ay_{-2}^q + by_0 y_{-1}}{y_0 y_{-1}} + (act + ad + b)n, & acs = 1, \\ \frac{(acs)^n (ay_{-2}^q + by_0 y_{-1})}{y_0 y_{-1}} + \left(\frac{(acs)^n - 1}{acs - 1}\right)(act + ad + b), & otherwise, \end{cases}$$

$$u_{3n+2} = \begin{cases} \frac{acz_{-2}^r + adz_0 z_{-1} + bz_0 z_{-1}}{z_0 z_{-1}} + (act + ad + b)n, & acs = 1, \\ (acs)^n \frac{acz_{-2}^r + adz_0 z_{-1} + bz_0 z_{-1}}{z_0 z_{-1}} + \left(\frac{(acs)^n - 1}{acs - 1}\right)(act + ad + b), & otherwise, \end{cases}$$

$$v_{3n} = \begin{cases} \frac{y_{-2}^q}{y_0 y_{-1}} + (csb + ct + d)n, & acs = 1, \\ \frac{(aa)^n y_{-2}^q}{y_0 y_{-1}} + \left(\frac{(acs)^n - 1}{acs - 1}\right)(csb + ct + d), & otherwise, \end{cases}$$

$$v_{3n+1} = \begin{cases} \dfrac{cz_{-2}^r + dz_0 z_{-1}}{z_0 z_{-1}} + (csb + ct + d)n, & acs = 1, \\ (csa)^n \dfrac{cz_{-2}^r + dz_0 z_{-1}}{z_0 z_{-1}} + \left(\dfrac{(csa)^n - 1}{csa - 1}\right)(csb + ct + d), & otherwise, \end{cases}$$

$$v_{3n+2} = \begin{cases} \dfrac{csx_{-2}^p + ctx_0 x_{-1} + dx_0 x_{-1}}{x_0 x_{-1}} + (csb + ct + d)n, & acs = 1, \\ (csa)^n \dfrac{csx_{-2}^p + ctx_0 x_{-1} + dx_0 x_{-1}}{x_0 x_{-1}} + \left(\dfrac{(csa)^n - 1}{csa - 1}\right)(csb + ct + d), & otherwise, \end{cases}$$

and

$$w_{3n} = \begin{cases} \dfrac{z_{-2}^r}{z_0 z_{-1}} + (sad + sb + t)n, & acs = 1, \\ (sac)^n \dfrac{z_{-2}^r}{z_0 z_{-1}} + \left(\dfrac{(sac)^n - 1}{sac - 1}\right)(sad + sb + t), & otherwise, \end{cases}$$

$$w_{3n+1} = \begin{cases} \dfrac{sx_{-2}^p + tx_0 x_{-1}}{x_0 x_{-1}} + (sad + sb + t)n, & acs = 1, \\ (sac)^n \dfrac{sx_{-2}^p + tx_0 x_{-1}}{x_0 x_{-1}} + \left(\dfrac{(sac)^n - 1}{sac - 1}\right)(sad + sb + t), & otherwise, \end{cases}$$

$$w_{3n+2} = \begin{cases} \dfrac{say_{-2}^q + sby_0 y_{-1} + ty_0 y_{-1}}{y_0 y_{-1}} + (sad + sb + t)n, & acs = 1, \\ (sac)^n \dfrac{say_{-2}^q + sby_0 y_{-1} + ty_0 y_{-1}}{y_0 y_{-1}} + \left(\dfrac{(sac)^n - 1}{sac - 1}\right)(sad + sb + t), & otherwise. \end{cases}$$

in the case of constant coefficients.

4. Conclusion

In this work we have solved in closed form a general system of difference equations of third order defined by one-to-one functions. The formulas of the solutions are expressed using the terms of a generalized Fibonacci sequence of second order but also in terms of the coefficients and the initial values. An application on a concrete system was provided. The present contribution, which is a part of the first author thesis [20], is a continuation of the work [21], which is also a part of [20]. For interested readers and as extension of the system studied here, we propose the following open problem.

Open problem: Solve in closed form the following system of k-difference equations defined for all $n \in \mathbb{N}_0$ by

$$x_{n+1}^1 = f_1^{-1}\left(\frac{f_2(x_n^2)f_2(x_{n-1}^2)(f_1(x_{n-1}^1))^{p_1}}{f_1(x_n^1)\left[a_n^1(f_2(x_{n-2}^2))^{p_2} + b_n^1 f_2(x_n^2)f_2(x_n^2)\right]}\right),$$

$$x_{n+1}^2 = f_2^{-1}\left(\frac{f_3(x_n^3)f_3(x_{n-1}^3)(f_2(x_{n-1}^2))^{p_2}}{f_2(x_n^2)\left[a_n^2(f_3(x_{n-2}^3))^{p_3} + b_n^2 f_3(x_n^3)f_3(x_{n-1}^3)\right]}\right),$$

$$\vdots$$

$$x_{n+1}^{k-1} = f_{k-1}^{-1}\left(\frac{f_k(x_n^k)f_k(x_{n-1}^k)(f_{k-1}(x_{n-1}^{k-1}))^{p_{k-1}}}{f_{k-1}(x_n^{k-1})\left[a_n^{k-1}(f_k(x_{n-2}^{k-1}))^{p_k} + b_n^{k-1} f_1(x_n^k)f_k(x_{n-1}^k)\right]}\right),$$

$$x_{n+1}^k = f_k^{-1}\left(\frac{f_1(x_n^1)f_1(x_{n-1}^1)(f_k(x_{n-1}^k))^{p_k}}{f_k(x_n^k)\left[a_n^k(f_1(x_{n-2}^1))^{p_1} + b_n^k f_1(x_n^1)f_1(x_{n-1}^1)\right]}\right)$$

where $k = 4, 5, \ldots, p_1, \ldots, p_k \in \mathbb{N}$, $f_1, \ldots, f_k : D \to \mathbb{R}$ are continuous one-to-one functions on $D \subseteq \mathbb{R}$, the sequences $(a_n^1)_{n\in\mathbb{N}_0}, \ldots, (a_n^k)_{n\in\mathbb{N}_0}$, $(b_n^1)_{n\in\mathbb{N}_0}, \ldots, (b_n^k)_{n\in\mathbb{N}_0}$ and the initial values x_{-2}^1, x_{-1}^1, x_0^1, ..., x_{-2}^k, x_{-1}^k, ..., x_0^k are nonzero real numbers.

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