

MAGNETOHYDRODYNAMICS OF ROTATING FRACTIONAL SECOND GRADE FLUID IN POROUS MEDIUM

AZHAR ALI ZAFAR^{1,2}, DUMITRU VIERU³, SHAHRAZ AKHTAR¹

ABSTRACT. Exact solution for the unsteady flow of a fractional second grade fluid through the porous medium under the influence of magnetic field in the direction normal to the flow has been investigated using the integral transforms. Expressions for dimensionless velocity have been obtained and are presented in terms of Fox's H-function. The influence of the fractional parameter on the fluid motion is studied and a comparison between velocity of the fractional and classical fluid is made.

Key words : Second grade fluid, fractional calculus, magnetohydrodynamics, velocity field.

AMS SUBJECT : (2010). 76A05, 76A10.

1. INTRODUCTION

Non-Newtonian fluids play an active and significant role in comparison with Newtonian fluids in a large number of industries and the branches of knowledge concerned with applied sciences. The dynamics of non-Newtonian fluids is much more complex and delicate enough to analyze [3]. The distinctive character of the response of non-Newtonian fluids cannot be studied by the classical Navier-Stokes equations but a huge count of models are in agreement to justify the rheological response of non-Newtonian fluids [8], [38]. A number of investigators in the field, propose/quotes a large number of applications of non-Newtonian fluids in rheological problems, geophysics, petroleum, chemical industries and biological sciences [50].

¹Abdus Salam School of Mathematical Sciences, GC University Lahore, Pakistan.
Email: azharalizafar@gmail.com, shahraz5768@gmail.com

²Department of Mathematics, GC University Lahore, Pakistan.

³Department of Theoretical Mechanics, Technical University of Iasi, Romania. Email: dumitru.vieru@yahoo.com.

Generally, the non-Newtonian fluids are categorized as differential, rate and integral type fluids. Amongst them, the fluids of differential type have received special attention [5], [6], [9], [11], [12], [13], [14], [15], [39], [45]. The second grade fluids form a subclass of non-Newtonian fluids and is the simplest subclass of differential type fluids which can show the normal stress effects. The flow of electrically conductive fluids in the presence of magnetic field is known as magnetohydrodynamics (MHD). The subject MHD seems to be initiated by H. Alfvén [1]. The principle of MHD is helpful in stabilizing a flow against the transition from laminar to turbulent. In recent years, the utility of MHD led the researchers to renew their investigation of hydrodynamics in the setting of MHD. The dynamics of non-Newtonian fluids with and without influence of magnetic field has a lot of applications e.g. boundary layer control in aerodynamics [28], the handling of molten metals, biological fluids, polymers and fossil fuels etc. For further studies regarding MHD we refer [32].

Over the last few years, fractional calculus has attracted much attention in the mathematical modeling of dynamical systems. It also has been employed with success in the constitutive modeling of certain non-Newtonian fluid models. The major reason is that, a fractional model has the ability to explain with simplicity the complex response of viscoelastic fluid. Few major contributions in the discussion of viscoelastic fluids flows with a fractional calculus approach can be seen in [16], [17], [18], [22], [24], [26], [30], [33], [37], [42], [43], [44], [46], [47], [48], [49]. Several researchers have discussed flows of second grade fluid in different configurations, and there are on hand few attempts which include the effects of rotation and MHD (for instance, see studies in [4], [7], [20], [21], [27], [29], [31], [34], [36], [40], [41] and references therein). Recently, Imran et al. [25] analyzed the unsteady MHD flow of a rotating second grade fluid in a porous medium over an oscillating plate.

To the best of authors knowledge, so far no study has been reported to analyze the unsteady MHD flow of a rotating fractional second grade fluid in a porous medium past an oscillating plate. Therefore, it is here proposed to make such an attempt. The main aim of the current study is to establish analytic solutions for the velocity field of a fractional second grade fluid which flows in the presence of an applied magnetic field. The fluid occupies a half porous space bounded by a rigid and nonconducting plate and, the whole system is in a rigid-rotating motion. The boundary plate also has a secondary motion, namely, a translation in its plane with the time dependent velocity of the form $V_o \sin(\omega t)$ or $V_o \cos(\omega t)$ or $V_o(a + be^{-\omega t})$. The closed-form of the complex velocity field is obtained by means of Laplace transform method and is expressed in an elegant form using the Fox functions [19]. The Fox function, also referred as the Fox H-function, generalizes the Mellin-Barnes function.

The importance of the Fox function lies in the fact that it includes nearly all special functions occurring in applied mathematics and statistics as special cases. From our general solutions the flow of an ordinary fluid as well as the hydrodynamic flow appear as limiting cases of the fractional model.

2. STATEMENT OF THE PROBLEM

Consider an incompressible rotating fractional second grade fluid bounded by a rigid plate at $z = 0$. The fluid is electrically conducting and fills the porous region $z > 0$. The z -axis is taken to be normal to the plate. Initially, the fluid and plate are at rest. At time $t = 0^+$, the fluid and plate start rotating around the z -axis as a solid body with a constant angular velocity Ω . The plate is also subject to translation motion in its plane along the x -axis. A uniform transverse magnetic field of strength B_o is applied parallel to the axis of rotation. It is assumed that the induced magnetic field and the electric field due to polarization of charges are negligible. The motion of the second grade fluid is governed by the following partial differential equation [20], [27].

$$\frac{\partial F}{\partial t} + (2i\Omega + \frac{\sigma B_o^2}{\rho})F = \nu \frac{\partial^2 F}{\partial z^2} + \frac{\alpha_1}{\rho} \frac{\partial^3 F}{\partial z^2 \partial t} - \frac{\nu \phi}{k} (1 + \frac{\alpha_1}{\mu} \frac{\partial}{\partial t})F, \quad (1)$$

where $F(z, t) = u(z, t) + iv(z, t)$ is the complex velocity of the fluid, u and v are the velocity components along x and y axis respectively, ρ is the fluid density, μ is the dynamic viscosity, α_1 the normal stress module, ν is the kinematic viscosity, σ is the electrical conductivity of the fluid, ϕ ($0 < \phi < 1$) is the porosity and $k > 0$ is the permeability of the porous medium.

The governing equation corresponding to such a motion of fractional second grade fluids is [46], [48]

$$\frac{\partial F}{\partial t} + (2i\Omega + \frac{\sigma B_o^2}{\rho})F = \nu \frac{\partial^2 F}{\partial z^2} + \eta_\beta D_t^\beta \frac{\partial^2 F}{\partial z^2} - \frac{\nu \phi}{k} (1 + \frac{\alpha_1}{\mu} \frac{\partial}{\partial t})F, \quad (2)$$

where $D_t^\beta f$ is the Caputo fractional derivative operator defined as [24],

$$D_t^\beta f(t) = \begin{cases} \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\beta} d\tau, & 0 < \beta < 1, \\ \frac{df(t)}{dt}, & \beta = 1. \end{cases}$$

Into Eq. (2), $\eta_\beta [m^2 s^{\beta-1}]$ is a material coefficient and $\eta_1 = \frac{\alpha_1}{\rho}$.

The appropriate initial and boundary conditions are

$$F(z, 0) = 0; \quad z \geq 0, \quad (3)$$

$$F(0, t) = V_o \sin(\omega t) \text{ or } V_o \cos(\omega t) \text{ or } V_o(a + be^{-\omega t}); \quad t > 0, \quad (4)$$

$$F(z, t) \rightarrow 0 \text{ as } z \rightarrow \infty, \quad (5)$$

where V_o and ω are constants, $\omega > 0$.

Introducing the dimensionless variables

$$z^* = z\sqrt{\frac{\omega}{\nu}}, t^* = \omega t, F^* = \frac{F}{V_o}, \Omega^* = \frac{\Omega}{\omega}, \eta_\beta^* = \frac{\eta_\beta \omega^\beta}{\nu} \quad (6)$$

into Eqs.(2)-(5) and dropping the asteriks we get,

$$\frac{\partial F}{\partial t} + \frac{1}{\omega}(2i\Omega + \frac{\sigma B_o^2}{\rho})F = \frac{\partial^2 F}{\partial z^2} + \eta_\beta D_t^\beta \frac{\partial^2 F}{\partial z^2} - \frac{\nu \phi}{k\omega}(1 + \frac{\alpha_1 \omega}{\mu} \frac{\partial}{\partial t})F, \quad (7)$$

or

$$\frac{\partial^2 F}{\partial z^2} + \eta_\beta D_t^\beta \frac{\partial^2 F}{\partial z^2} - a_o \frac{\partial F}{\partial t} - b_o F = 0, \quad (8)$$

subject to

$$F(z, 0) = 0; \quad z > 0, \quad (9)$$

$$F(0, t) = \sin t \text{ or } \cos t \text{ or } a + be^{-t}; \quad t > 0$$

$$F(z, t) \rightarrow 0 \text{ as } z \rightarrow \infty, \quad (10)$$

where $a_o = 1 + \frac{\alpha}{K}$, $\alpha = \frac{\alpha_1 \omega}{\rho \nu}$, $b_o = \frac{1}{K} + 2i\Omega + M^2$, $\frac{1}{K} = \frac{\phi \nu}{k\omega}$ and $M^2 = \frac{\sigma B_o^2}{\rho \omega}$

3. CALCULATION OF THE VELOCITY FIELD

Let us denote by $F_s(z, t)$, $F_c(z, t)$ and $F_e(z, t)$ the solutions of Eq. (8) related to the boundary conditions (10)₁, (10)₂ and (10)₃, respectively. In the following, in order to avoid repetition, we shall present calculi for $F_s(z, t)$ only. Applying the Laplace transform to Eq. (8), and having the initial conditions (9) in mind we find that

$$\frac{\partial^2 \bar{F}_s(z, q)}{\partial z^2} = \frac{a_o q + b_o}{1 + \eta_\beta q^\beta} \bar{F}_s(z, q) \quad (11)$$

with the boundary conditions

$$\bar{F}_s(0, q) = \frac{1}{1 + q^2}, \quad \bar{F}_s(z, q) \rightarrow 0 \text{ as } z \rightarrow \infty, \quad (12)$$

where $\bar{F}(z, q)$ is the Laplace transform of $F(z, t)$ and q is transform parameter. Solution of the equation (11) satisfying (12) is

$$\bar{F}_s(z, q) = \frac{1}{q^2 + 1} \cdot \exp\left(-z \sqrt{\frac{a_o q + b_o}{1 + \eta_\beta q^\beta}}\right) \quad (13)$$

Application of inverse Laplace transform to Eq. (13), leads to

$$F_s(z, t) = (F_1 * F_2)(t) = \int_0^t F_1(t-s) F_2(z, s) ds, \quad (14)$$

where $F_1(z, t)$ and $F_2(z, t)$ are inverse Laplace transforms of $\bar{F}_1(z, q) = \frac{q}{q^2+1}$ and

$\bar{F}_2(z, q) = \frac{1}{q} \exp \left(-z \sqrt{\frac{a_o q + b_o}{1 + \eta_\beta q^\beta}} \right)$ respectively. Now, we write

$$\begin{aligned} \bar{F}_2(z, q) &= \frac{1}{q} \exp \left(-z \sqrt{\frac{a_o q + b_o}{1 + \eta_\beta q^\beta}} \right) \\ &= \sum_{k=0}^{\infty} \frac{\left(-z a_o^{\frac{1}{2}} \eta_\beta^{\frac{-1}{2}} \right)^k}{k!} \sum_{m=0}^{\infty} \frac{\left(\frac{-b_o}{a_o} \right)^m}{m!} \sum_{n=0}^{\infty} \frac{-\eta_\beta^{-n}}{n!} \frac{\Gamma(n + \frac{k}{2}) \Gamma(m - \frac{k}{2})}{\Gamma(\frac{k}{2}) \Gamma(-\frac{k}{2})} \frac{1}{q^{(\beta-1)\frac{k}{2} + \beta n + m + 1}}. \end{aligned} \quad (15)$$

Applying the inverse Laplace transform, we obtain

$$\begin{aligned} F_2(z, t) &= \sum_{k=0}^{\infty} \frac{\left(-z a_o^{\frac{1}{2}} \eta_\beta^{\frac{-1}{2}} \right)^k}{k!} \sum_{m=0}^{\infty} \frac{\left(\frac{-b_o}{a_o} \right)^m}{m!} \times \\ &\times \sum_{n=0}^{\infty} \frac{\eta_\beta^{-n}}{n!} \frac{\Gamma(n + \frac{k}{2}) \Gamma(m - \frac{k}{2})}{\Gamma(\frac{k}{2}) \Gamma(-\frac{k}{2}) \Gamma((\beta-1)\frac{k}{2} + \beta n + m + 1)} t^{(\beta-1)\frac{k}{2} + \beta n + m} \end{aligned} \quad (16)$$

or

$$\begin{aligned} F_2(z, t) &= \sum_{k=0}^{\infty} \frac{\left(-z a_o^{\frac{1}{2}} \eta_\beta^{\frac{-1}{2}} \right)^k}{k!} \sum_{m=0}^{\infty} \frac{\left(\frac{-b_o}{a_o} \right)^m}{m!} t^{(\beta-1)\frac{k}{2} + m} \times \\ &\times H_2^{1 \frac{2}{4}} \left[\eta_\beta^{-1} t^\beta \right]_{(0,1), (1-\frac{k}{2}, 0), (1+\frac{k}{2}, 0), (\frac{(1-\beta)k}{2} - m, \beta)}^{(1-\frac{k}{2}, 1), (1-m+\frac{k}{2}, 0)} \end{aligned} \quad (17)$$

where $H_2^{1 \frac{2}{4}}$ is the Fox functions [19]. Finally, as $F_1(z, t) = \cos t$, it results

$$\begin{aligned} F_s(z, t) &= \int_0^t \cos(t-s) \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{k! m!} \left(z \sqrt{\frac{a_o s^{\beta-1}}{\eta_\beta}} \right)^k \left(\frac{b_o s}{a_o} \right)^m \times \\ &\times H_2^{1 \frac{2}{4}} \left[\eta_\beta^{-1} s^\beta \right]_{(0,1), (1-\frac{k}{2}, 0), (1+\frac{k}{2}, 0), (\frac{(1-\beta)k}{2} - m, \beta)}^{(1-\frac{k}{2}, 1), (1-m+\frac{k}{2}, 0)} ds \end{aligned} \quad (18)$$

To avoid repetition, we present the final form for $F_c(z, t)$ and $F_e(z, t)$

$$\begin{aligned}
F_c(z, t) = & \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{k!m!} (z\sqrt{\frac{a_o}{\eta_\beta}})^k \left(\frac{b_o}{a_o}\right)^m t^{(\beta-1)\frac{k}{2}+m} \times \\
& \times H_2^{1\frac{1}{2}} \left[\eta_\beta^{-1} t^\beta \right]_{(0,1), (1-\frac{k}{2}, 0), (1+\frac{k}{2}, 0), (\frac{(1-\beta)k}{2}-m, \beta)}^{(1-\frac{k}{2}, 1), (1-m+\frac{k}{2}, 0)} - \\
& - \int_0^t \sin(t-s) \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{k!m!} (z\sqrt{\frac{a_o s^{\beta-1}}{\eta_\beta}})^k \left(\frac{b_o s}{a_o}\right)^m \times \\
& \times H_2^{1\frac{1}{2}} \left[\eta_\beta^{-1} s^\beta \right]_{(0,1), (1-\frac{k}{2}, 0), (1+\frac{k}{2}, 0), (\frac{(1-\beta)k}{2}-m, \beta)}^{(1-\frac{k}{2}, 1), (1-m+\frac{k}{2}, 0)} ds \quad (19)
\end{aligned}$$

$$\begin{aligned}
F_e(z, t) = & (a+b) \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{k!m!} (z\sqrt{\frac{a_o}{\eta_\beta}})^k \left(\frac{b_o}{a_o}\right)^m t^{(\beta-1)\frac{k}{2}+m} \times \\
& \times H_2^{1\frac{1}{2}} \left[\eta_\beta^{-1} t^\beta \right]_{(0,1), (1-\frac{k}{2}, 0), (1+\frac{k}{2}, 0), (\frac{(1-\beta)k}{2}-m, \beta)}^{(1-\frac{k}{2}, 1), (1-m+\frac{k}{2}, 0)} - \\
& - b \int_0^t e^{-(t-s)} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{k!m!} (z\sqrt{\frac{a_o s^{\beta-1}}{\eta_\beta}})^k \left(\frac{b_o s}{a_o}\right)^m \times \\
& \times H_2^{1\frac{1}{2}} \left[\eta_\beta^{-1} s^\beta \right]_{(0,1), (1-\frac{k}{2}, 0), (1+\frac{k}{2}, 0), (\frac{(1-\beta)k}{2}-m, \beta)}^{(1-\frac{k}{2}, 1), (1-m+\frac{k}{2}, 0)} ds. \quad (20)
\end{aligned}$$

4. CONCLUSIONS AND DISCUSSION

In this paper the unsteady flow of a second grade fluid in a rotating frame is studied. By introducing a complex field of velocity, the flow problem is formulated in a non-dimensional, suitable form. Solutions of studied problems are determined by means of the Laplace transform method and are expressed by Fox functions. Some existing results can be obtained as particular cases from our general solutions. Also it is worth to note that, by customizing our solutions for Newtonian fluids we obtained solutions for many problems that could be formulated for these fluids. Some of the results are known and others are novel in literature. So, by making $\beta \rightarrow 1$ into Eqs. (18) and (19) we recover the solutions obtained by Imran et al. [[25], Eq(17) and (20)] as it results from Figs. 1 and 2. For $\Omega \rightarrow 0$ and $b \rightarrow 0$ into Eq. (20) we recover the similar solutions obtained in [[10], Eq. (26)]. By making $\Omega \rightarrow 0$ and $\beta \rightarrow 1$ in Eqs. (18) and (20) we recover the solutions obtained by Ali et al. [[2], Eq(15) and (16)]. Further by making $M = 0$ (hydrodynamic fluid), $\frac{1}{K} = 0$ (non porous

space) and $\Omega \rightarrow 0$ into Eqs. (18) and (19), we get

$$F_s(z, t) = \int_0^t \cos(t-s) \sum_{k=0}^{\infty} \frac{(-1)^k z^k \eta_{\beta}^{\frac{-k}{2}}}{k!} s^{(\beta-1)\frac{k}{2}} \times \\ \times H_2^{1 \frac{2}{4}} \left[\eta_{\beta}^{-1} s^{\beta} \left| \begin{matrix} (1-\frac{k}{2}, 1), (1+\frac{k}{2}, 0) \\ (0, 1), (1-\frac{k}{2}, 0), (1+\frac{k}{2}, 0), (\frac{(1-\beta)k}{2}, \beta) \end{matrix} \right. \right] ds \quad (21)$$

$$F_c(z, t) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k \eta_{\beta}^{\frac{-k}{2}}}{k!} t^{(\beta-1)\frac{k}{2}} H_2^{1 \frac{2}{4}} \left[\eta_{\beta}^{-1} t^{\beta} \left| \begin{matrix} (1-\frac{k}{2}, 1), (1+\frac{k}{2}, 0) \\ (0, 1), (1-\frac{k}{2}, 0), (1+\frac{k}{2}, 0), (\frac{(1-\beta)k}{2}, \beta) \end{matrix} \right. \right] - \\ - \int_0^t \sin(t-s) \sum_{k=0}^{\infty} \frac{(-1)^k z^k \eta_{\beta}^{\frac{-k}{2}}}{k!} s^{(\beta-1)\frac{k}{2}} H_2^{1 \frac{2}{4}} \left[\eta_{\beta}^{-1} s^{\beta} \left| \begin{matrix} (1-\frac{k}{2}, 1), (1+\frac{k}{2}, 0) \\ (0, 1), (1-\frac{k}{2}, 0), (1+\frac{k}{2}, 0), (\frac{(1-\beta)k}{2}, \beta) \end{matrix} \right. \right] ds \quad (22)$$

Now for $\beta \rightarrow 1$ in Eqs. (21) and (22) we get similar results obtained by Nazar et al. [[35], Eqs. (13) and (14)].

Making $\eta_{\beta} = 0$ into Eq. (13) we find the velocity field corresponding to Newtonian fluids. Firstly, $\bar{F}_{sN}(z, q) = \frac{1}{q^2+1} e^{-z\sqrt{a_o}\sqrt{q+c_o}}$, $c_o = b_o/a_o$, or, the equivalent form $\bar{F}_{sN}(z, q) = \frac{1}{2i} \left[\frac{1}{q-i} - \frac{1}{q+1} \right] e^{-z\sqrt{a_o}\sqrt{q+c_o}}$, with the original function [23],

$$F_{sN}(z, t) = i[A_1(z, t) - A_2(z, t)], \quad (23)$$

where

$$A_1(z, t) = \frac{e^{-it}}{4} \left[e^{-z\sqrt{b_o-ia_o}} \operatorname{erfc} \left(\frac{z\sqrt{a_o}}{2\sqrt{t}} - \sqrt{\frac{b_o-ia_o}{a_o}} t \right) + \right. \\ \left. + e^{z\sqrt{b_o-ia_o}} \operatorname{erfc} \left(\frac{z\sqrt{a_o}}{2\sqrt{t}} + \sqrt{\frac{b_o-ia_o}{a_o}} t \right) \right] \quad (24)$$

and

$$A_2(z, t) = \frac{e^{it}}{4} \left[e^{-z\sqrt{b_o+ia_o}} \operatorname{erfc} \left(\frac{z\sqrt{a_o}}{2\sqrt{t}} - \sqrt{\frac{b_o+ia_o}{a_o}} t \right) + \right. \\ \left. + e^{z\sqrt{b_o+ia_o}} \operatorname{erfc} \left(\frac{z\sqrt{a_o}}{2\sqrt{t}} + \sqrt{\frac{b_o+ia_o}{a_o}} t \right) \right]. \quad (25)$$

Similarly,

$$F_{cN}(z, t) = A_1(z, t) + A_2(z, t), \quad (26)$$

or

$$F_{eN}(z, t) = \frac{a}{2} \left[e^{-z\sqrt{b_o}} \operatorname{erfc} \left(\frac{z\sqrt{a_o}}{2\sqrt{t}} - \sqrt{\frac{b_o t}{a_o}} \right) + e^{z\sqrt{b_o}} \operatorname{erfc} \left(\frac{z\sqrt{a_o}}{2\sqrt{t}} + \sqrt{\frac{b_o t}{a_o}} \right) \right] + \frac{b}{2} e^{-t} \left[e^{-z\sqrt{b_o - a_o}} \operatorname{erfc} \left(\frac{z\sqrt{a_o}}{2\sqrt{t}} - \sqrt{\frac{b_o - a_o}{a_o} t} \right) + e^{z\sqrt{b_o - a_o}} \operatorname{erfc} \left(\frac{z\sqrt{a_o}}{2\sqrt{t}} + \sqrt{\frac{b_o - a_o}{a_o} t} \right) \right]. \quad (27)$$

It is worth pointing out that the above solutions contain as particular cases much problems regarding to Newtonian fluids. For example;

Classical first problem of Stokes, by setting $\frac{1}{K} = 0$, $M = 0$, $\Omega = 0$, $b = 0$, $a_o = 1$, $b_o = 0$ into Eq. (27)

$$F_{eN}(z, t) = \frac{a}{2} \left[\operatorname{erfc} \left(\frac{z}{2\sqrt{t}} \right) + \operatorname{erfc} \left(\frac{z}{2\sqrt{t}} \right) \right] = a \operatorname{erfc} \left(\frac{z}{2\sqrt{t}} \right) \quad (28)$$

and the classical second Stokes' problem, $(\sin(\omega t))$, by setting $\frac{1}{k} = 0$, $M = 0$, $\Omega = 0$, $a_o = 1$, $b_o = 0$ into Eq. (23)

$$F_{sN}(z, t) = \frac{ie^{-it}}{4} \left[e^{-z\sqrt{-i}} \operatorname{erfc} \left(\frac{z}{2\sqrt{t}} - \sqrt{-it} \right) + e^{z\sqrt{-i}} \operatorname{erfc} \left(\frac{z}{2\sqrt{t}} + \sqrt{-it} \right) \right] - \frac{ie^{it}}{4} \left[e^{z\sqrt{i}} \operatorname{erfc} \left(\frac{z}{2\sqrt{t}} - \sqrt{it} \right) + e^{z\sqrt{i}} \operatorname{erfc} \left(\frac{z}{2\sqrt{t}} + \sqrt{it} \right) \right], \quad (29)$$

etc.

In order to analyze the influence of the fractional parameter β on the velocity components, the figures 3 and 4 were plotted. From these figures it is observed that, for high values of the fractional parameter β , the fluid flows more slowly for cosine oscillations and faster for the sine oscillations of the plate. Influence of fractional parameter is significant in the vicinity of the plate and far from the plate the motion of the fluid is mitigated. There, the fluid velocity approaches to zero. In figures 5 and 6 are presented comparisons between the classical fluid ($\beta = 1$) and the fractional fluid ($\beta = 0.4$), for two different values of the strength of the magnetic field. For both types of the oscillations, the increasing of strength of the magnetic field leads to a decreasing of the absolute values of velocity.

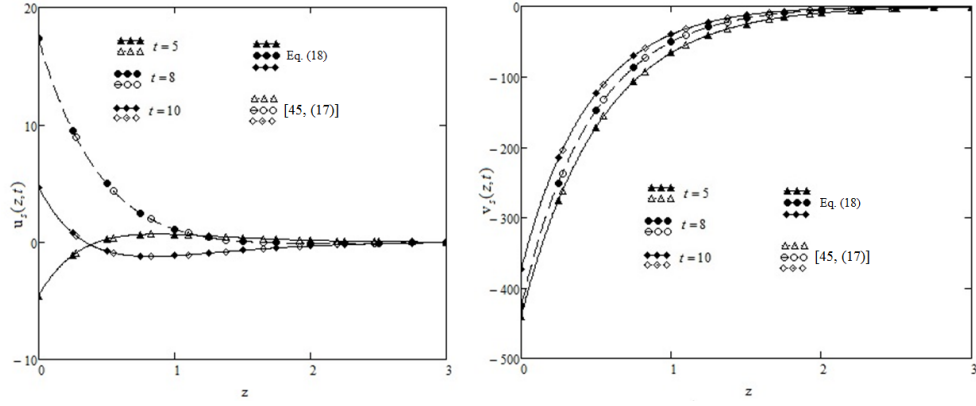


FIGURE 1. . Profile of the components $u(z,t)$ and $v(z,t)$ of the complex field $F_s(z,t)$ given by Eq. (18) and [45, (17)] with $M = 2.5$, $K = 2$, $\Omega = 0.05$, $\omega = 2$ and $\alpha = 1.2$ at different times.

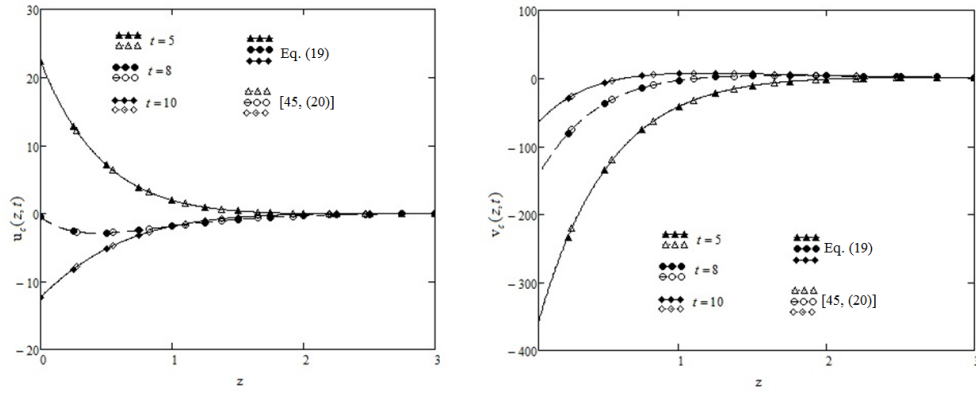


FIGURE 2. . Profile of the components $u(z,t)$ and $v(z,t)$ of the complex field $F_c(z,t)$ given by Eq. (19) and [45, (20)] with $M = 2.5$, $K = 2$, $\Omega = 0.05$, $\omega = 2$ and $\alpha = 1.2$ at different times.

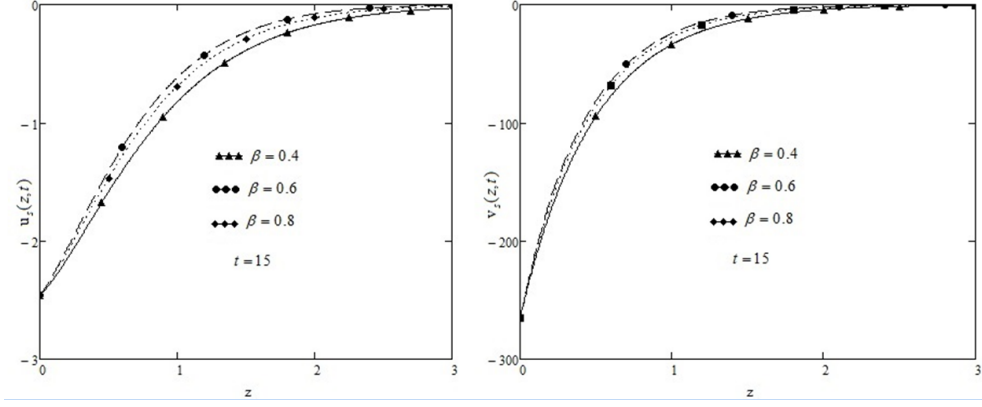


FIGURE 3. . Profile of the components $u(z, t)$ and $v(z, t)$ of the complex field $F_s(z, t)$ given by Eq. (18) with $M = 2.5$, $K = 2$, $\Omega = 0.05$, $\omega = 2$ and $\alpha = 1.2$ at $t = 15$ and different values of β .

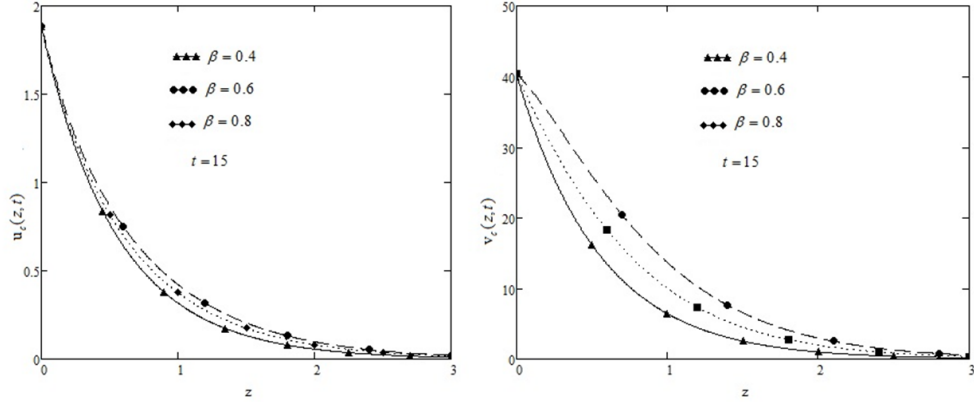


FIGURE 4. . Profile of the components $u(z, t)$ and $v(z, t)$ of the complex field $F_c(z, t)$ given by Eq. (19) with $M = 2.5$, $K = 2$, $\Omega = 0.05$, $\omega = 2$ and $\alpha = 1.2$ at $t = 15$ and different values of β .

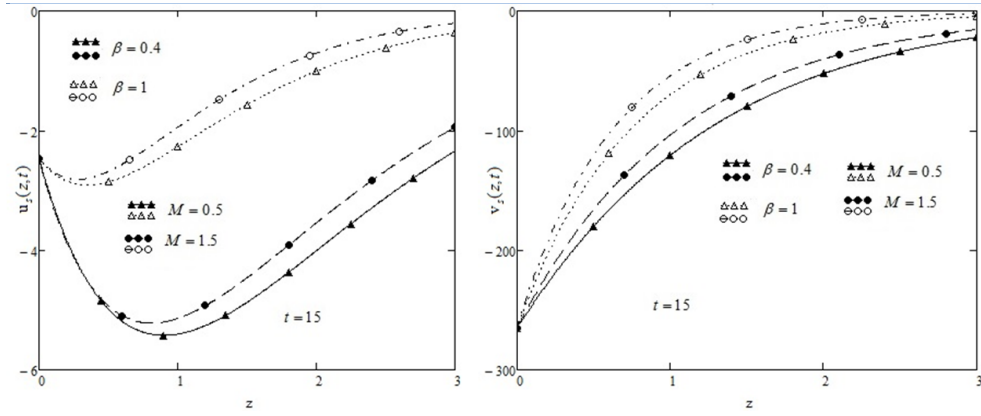


FIGURE 5. . Profile of the components $u(z, t)$ and $v(z, t)$ of the complex field $F_s(z, t)$ given by Eq. (18) with $K = 2$, $\Omega = 0.05$, $\omega = 2$ and $\alpha = 1.2$ at $t = 15$ and different values of β and M .

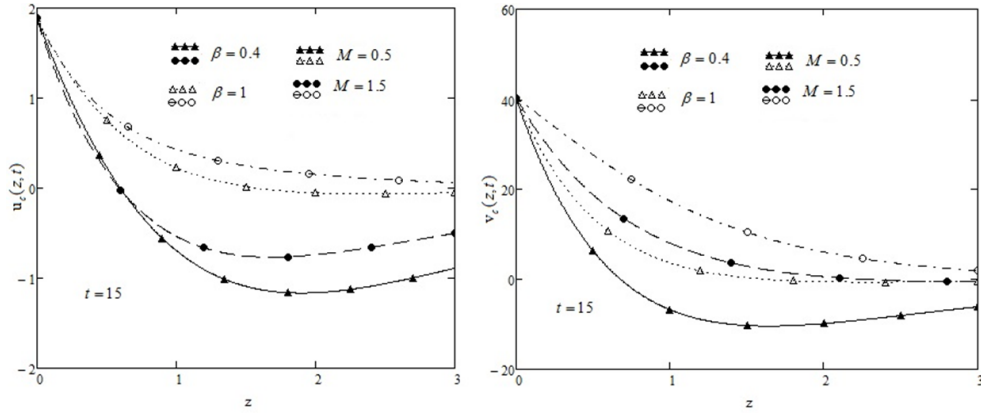


FIGURE 6. . Profile of the components $u(z, t)$ and $v(z, t)$ of the complex field $F_c(z, t)$ given by Eq. (19) with $K = 2$, $\Omega = 0.05$, $\omega = 2$ and $\alpha = 1.2$ at $t = 15$ and different values of β and M .

Acknowledgement: The authors are highly thankful and grateful to the Abdus Salam School of Mathematical Sciences, GC University Lahore, Pakistan and also Higher Education Commission of Pakistan for generous supporting and facilitating this research work.

REFERENCES

- [1] H. Alfven, Existence of electromagnetic-hyrdodynamic waves, *Nature*, 150, no.3805 (1942), 405–406.
- [2] F. Ali, M. Norzieha, S. Sharidan, I. Khan and T. Hayat, New exact solutions of Stokes second problem for an MHD second grade fluid in a porous space, *Int. J. Nonlinear Mech.*, doi: 10.1016/j.ijnonlinmec.2011.09.027 (2011).
- [3] P. D. Ariel, On exact solutions of flow problems of a second grade fluid through two parallel porous walls, *Internat. J. Engrg. Sci.* 40 (2002) 913–941.
- [4] S. Asghar, M. M. Gulzar and T. Hayat, Rotating flow of a third grade fluid by homotopy analysis method, *Appl. Math. Comput.*, 165, (2005) 213–221.
- [5] M. Ayub, A. Rasheed and T. Hayat, Exact flow of a third grade fluid past a porous plate using homotopy analysis method, *Int. J. Eng. Sci.*, 41, (2003) 2091–2103.
- [6] R. Bandelli, K. R. Rajagopal and G. P. Galdi, On some unsteady motions of fluids of second grade, *Arch. Mech.*, 47, (1995) 661–676.
- [7] S. Das, S. L. Maji, M. Guria and R. N. Janan, Unsteady MHD Couette flow in a rotating system, *Math and Comput. Model.*, 50, (2009) 1211–1217.
- [8] J. E. Dunn and K. R. Rajagopal, Fluid of differential type: critical review and thermodynamic analysis, *Internat. J. Engrg. Sci.* 33 (1995) 689–729.
- [9] J. E. Dunn and K. R. Rajagopal, Fluids of differential type: critical review and thermodynamic analysis, *Int. J. Eng. Sci.*, 33, (1995) 689–729.
- [10] M. El-Shahed, On the Impulsive Motion of Flat Plate in a Generalized Second Grade Fluid, *Z. Naturforsch.* 59a, (2004) 829–837.
- [11] M. E. Erdogan, On unsteady motions of a second-order fluid over a plane wall, *Int. J. Non-Linear. Mech.*, 38, (2003) 1045–1051.
- [12] K. . Fakhar, Z. Xu and C. Yi, Exact solutions of a third grade fluid flow on a porous plate, *Appl. Math. and Comput.*, 202, (2008) 376–382.
- [13] C. Fetecau and Corina Fetecau, Starting solutions for some unsteady unidirectional flows of a second grade fluid, *Int. J. Eng. Sci.*, 43, (2005) 781–789.
- [14] C. Fetecau, T. Hayat, Corina Fetecau and N. Ali, Unsteady flow of a second grade fluid between two side walls perpendicular to a plate, *Nonlinear Analysis: Real World Applications*, 9, (2008) 1236–1252.
- [15] C. Fetecau, Corina Fetecau and M. Rana, General solutions for the unsteady flow of second grade fluid over an infinite plate that applies arbitrary shear to the fluid, *Z. Naturforsch.* 66a, (2011) 753–759.
- [16] C. Fetecau, M. Athar and C. Fetecau, Unsteady flow of a generalized Maxwell fluid with fractional derivative due to a constantly accelerating plate, *Comput. Math. Appl.* 57 (2009) 596–603.
- [17] C. Fetecau, C. Fetecau, M. Jamil and A. Mahmood, Flow of fractional Maxwell fluid between coaxial cylinders, *Arch. Appl. Mech.* 81 (2011) 1153–1163. doi:10.1007/s00419-011-0536-x.
- [18] C. Fetecau, A. Mahmood and M. Jamil, Exact solutions for the flow of a viscoelastic fluid induced by a circular cylinder subject to a time dependent shear stress, *Commun. Nonlinear Sci. Numer. Simul.* 15 (2010) 3931–3938.

- [19] C. Fox, The G and H functions as symmetrical Fourier kernels, Transactions of the American Mathematical Society 98 (1961) 395-429.
- [20] T. Hayat, C. Fetecau and M. Sajid, Analytic solution for MHD transient rotating flow of a second grade fluid in a porous space., Non-Linear Anal. Real World Appl., 9, (2008) 1619-1627.
- [21] T. Hayat, C. Fetecau and M. Sajid, On MHD transient flow of Maxwell fluid in a porous medium and rotating frame., Phys. Lett. A, 372, (2008) 1639-1644.
- [22] A. Hernandez-Jimenez, J. Hernandez-Santiago, A. Macias-Garcia and J. Sanchez- Gonzalez, Relaxation modulus in PMMA and PTFE fitting by fractional Maxwell model, Polym. Test 21 (2002) 325-331.
- [23] R. B. Hetnarski, An algorithm for generating some inverse Laplace transforms of exponential form, Z. Angew. Math. Phys. 26, (1975) 249-253.
- [24] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific Press, Singapore, (2000).
- [25] M. A. Imran, M. Imran and C. Fetecau, MHD oscillating flows of a rotating second grade fluid in a porous medium, Communications in Numerical Analysis, volume 2014(2014), 1-12, <http://dx.doi.org/10.5899/2014/cna-00196>.
- [26] M. Khan, S. H. Ali and H. T. Qi, On accelerated flows of a viscoelastic fluid with the fractional Burgers model, Nonlinear Anal. Real World Appl. (2009) 2286-2296.
- [27] I. Khan, A. Farhad, S. Sharidan and M. Norzieha, Exact solutions for accelerated flows of a rotating second grade fluid in a porous medium, World Appl. Sci. J. 9, (2010) 55-68.
- [28] Y. J. Kim, Unsteady MHD convective heat transfer past a semi infinite vertical porous moving plate with variable suction, Internat. J. Engrg. Sci. 38 (2000) 833-845.
- [29] K. Kumar and C. L. Varshney, Viscous flow through a porous medium past an oscillating plate in a rotating system, Indian J. Pure Appl. Math., 15, (1984) 1014-1019.
- [30] A. Mahmood, C. Fetecau, N. A. Khan and M. Jamil, Some exact solutions of the oscillatory motion of a generalized second grade fluid in an annular region of two cylinders, Acta Mech. Sin. 26 (2010) 541-550.
- [31] G. Manna, S. N. Maji, M. Guria and R. N. Jana, Unsteady viscous flow past a flat plate in a rotating system., J. Phys. Sci., 11, (2008) 29-42.
- [32] R. Moreau, Magnetohydrodynamics, Kluwer Academic Publishers, Dordrecht, 1990.
- [33] M. Nazar, Q. Sultan, M. Athar and M. Kamran, Unsteady longitudinal flow of a generalized Oldroyd-B fluid in cylindrical domains, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 2737-2744.
- [34] R. Nazar, N. Amin and I. Pop, Unsteady boundary layer flow due to a stretching surface in a rotating fluid. Mech. Res. Comm., 31, (2004) 121-128.
- [35] M. Nazar, C. Fetecau, D. Vieru and C. Fetecau, New exact solutions corresponding to the second problem of Stokes for second grade fluids, Nonlinear Analysis: Real World Applications 11, (2010) 584-591.
- [36] I. Pop and V. M. Soundalgekar, Effects of Hall current on hydromagnetic flow near a porous plate., Acta Mech., 20, (1974) 315-318.
- [37] H. T. Qi and H. Jin, Unsteady rotating flows of viscoelastic fluid with the fractional Maxwell model between coaxial cylinders, Acta Mech. Sin. 22 (2006) 301-305. doi: 10.1007/s10409-006-0013-x
- [38] K. R. Rajagopal, Mechanics of non-Newtonian fluids, in: Recent Developments in Theoretical Fluid Mechanics, G.P. Galdi and J. Necas (Eds.) Pitman Research Notes in Math. Ser., 291, Longman Scientific and Technical, Essex, (1993) 129-162.
- [39] K. R. Rajagopal and A. S. Gupta, An exact solution for the flow of a non-newtonian fluid past an infinite porous plate, Meccanica, 19, (1984) 158-160.

- [40] V. Rajeswari and G. Nath, Unsteady flow over a stretching surface in a rotating fluid, *Int. J. Eng. Sci.*, 30, (1992) 347–756.
- [41] M. Sajid, T. Javed and T. Hayat, MHD rotating flow of a viscous fluid over a shrinking surface, *Nonlinear Dyn.*, 51, (2008) 259–265.
- [42] S. H. A. M. Shah and H. T. Qi, Starting solutions for a viscoelastic fluid with fractional Burgers model in an annular pipe, *Nonlinear Anal. Real World Appl.* 11 (2010) 547–554.
- [43] I. Siddique and Z. Sajid, Exact solutions for the unsteady axial flow of non-Newtonian fluids through a circular cylinder, *Commun. Nonlinear Sci. Numer. Simul.* 16 (2011) 226–238.
- [44] I. Siddique and D. Vieru, Exact solution for the rotational flow of a generalized second grade fluid in a circular cylinder, *Acta Mech. Sin.* 25 (2009) 777–785. doi: 10.1007/s10409-009-0277-z.
- [45] W. C. Tan and Takashi Masuoka, Stokes first problem for a second grade fluid in a porous half-space with heated boundary, *Int. J. Non-Linear. Mech.*, 40, (2005) 515–522.
- [46] W. Tan, W. Pan and M. Xu, A note on unsteady flows of a viscoelastic fluid with the fractional Maxwell model between two parallel plates, *Internat. J. Non-Linear Mech.* 38 (2003) 645–650.
- [47] D. Tong and L. T. Shan, Exact solution for generalized Burgers fluid in an annular pipe, *Meccanica*. 44 (2009) 427–431.
- [48] S. Wang and M. Xu, Exact solution on unsteady Couette flow of generalized Maxwell fluid with fractional derivative, *Acta Mech.* 187 (2006) 1–4.
- [49] S. Wang and M. Xu, Axial Couette flow of two kinds of fractional viscoelastic fluids in an annulus, *Nonlinear Anal. Real World Appl.* 10 (2009) 1087–1096.
- [50] Z. S. Yu and J.Z. Lin, Numerical research on the coherent structure in the viscoelastic second-order mixing layers, *Appl. Math. Mech.* 8 (1998) 717–723.