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# HARDY-TYPE INEQUALITIES INVOLVING GENERALIZED FRACTIONAL INTEGRALS VIA SUPERQUADRATIC FUNCTIONS

SAJID IQBAL[1], JOSIP PEČARIĆ[2], MUHAMMAD SAMRAIZ[3]

Abstract. The aim of this article is to give Hardy-type inequalities involving linear differential operator, Widder's derivative and generalized fractional integral using superquadratic functions.

*Key words* : Inequality, Superquadratic function, Kernel, Green's function, Linear differential operator, Widder's derivative, Integral operator.
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## 1. Introduction

In 1920, G. H. Hardy stated and proved (see [6]) the integral inequality

$$\int_0^\infty \left( \frac{1}{x} \int_0^\infty f(t)dt \right)^p dx \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty f^p(x)dx, \tag{1}$$

where $p > 1$ and $f$ is a non-negative function such that $\int_0^\infty f^p(x)dx < \infty$. This is original form of Hardy's integral inequality, which later on has been extensively studied.

In last few decades, Hardy's inequality engrossed the interest of many mathematicians and they discover important and useful Hardy-type inequalities for convex functions as well as for superquadratic functions. Iqbal, Čižmešija,

[1]Department of Mathematics, University of Sargodha (Sub-Campus Bhakkar), Bhakkar, Pakistan. Email: sajid_uos2000@yahoo.com
[2]Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia. Email: pecaric@element.hr
[3]Department of Mathematics, University of Sargodha, Sargodha, Pakistan. Email: msamraiz@uos.edu.pk

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Krulić, Pečarić and Persson ([2], [3], [5], [7]) also give an inconceivable contribution in theory of inequalities. The extended form of (1) currently known as Hardy-type inequality

$$\left[\int_a^b u(x)\left(\frac{1}{x}\int_a^x f(t)dt\right)^q dx\right]^{\frac{1}{q}} \leq C_{p,q}\left(\int_a^b v(x)f^p(x)dx\right)^{\frac{1}{p}},$$

where $f$ is non-negative function, $u, v$ are given weight functions and the parameters $a, b, p$ and $q$ are such that $-\infty \leq a < b \leq \infty, 1 \leq p \leq \infty$ and $0 < q \leq \infty$.

In this paper, our particular interest is to give the applications of Hardy-type inequalities for linear differential operator, Widder's derivative and more generalized fractional integral operators involving superquadratic functions. The concept of superquadratic functions was first given by Abramovich, Jameson and Sinnamon in [1].

**Definition 1.** [1, Definition 2.1] *A function $\varphi : [0,\infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that*

$$\varphi(t) - \varphi(x) - \varphi(|t-x|) \geq C_x(t-x),$$

*for all $t \in \mathbb{R}$. We say that $\varphi$ is subquadratic if $-\varphi$ is superquadratic.*

**Definition 2.** [2, Definition 2] *A function $f : [0,\infty) \to \mathbb{R}$ is superadditive provided that for all $x$, $t \geq 0$, inequality*

$$f(x+t) \geq f(x) + f(t)$$

*holds true. If the reverse inequality holds, then $f$ is said to be subadditive.*

Let $(\Sigma_1, \Omega_1, \mu_1)$ and $(\Sigma_2, \Omega_2, \mu_2)$ be measure spaces with positive $\sigma$-finite measures. Let $U(f,k)$ denote the class of functions $g : \Omega_1 \to \mathbb{R}$ with the representation

$$g(x) = \int_{\Omega_2} k(x,t)f(t)d\mu_2(t),$$

and $A_k$ be an integral operator defined by

$$A_kf(x) := \frac{1}{K(x)}\int_{\Omega_2} k(x,t)f(t)d\mu_2(t), \tag{2}$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative kernel, $f : \Omega_2 \to \mathbb{R}$ is measurable function and

$$0 < K(x) := \int_{\Omega_2} k(x,t)d\mu_2(t), \qquad x \in \Omega_1. \tag{3}$$

The upcoming result is given in [2].

**Theorem 1.** *Let* $(\Sigma_1, \Omega_1, \mu_1)$ *and* $(\Sigma_2, \Omega_2, \mu_2)$ *be measure spaces with positive* $\sigma$*-finite measures,* $u$ *be a weight function and* $k(x,t) \geq 0$*. Assume that* $x \mapsto \frac{k(x,t)}{K(x)} u(x)$ *is integrable on* $\Omega_1$ *for each fixed* $t \in \Omega_2$*. Define* $v$ *on* $\Omega_2$ *by*

$$v(t) := \int_{\Omega_1} \frac{k(x,t)}{K(x)} u(x) d\mu_1(x) < \infty. \tag{4}$$

*Suppose* $I = [0, c), c \leq \infty$ *and* $\varphi : I \to \mathbb{R}$ *is a superquadratic function, then the inequality*

$$\int_{\Omega_1} \varphi(A_k f(x)) u(x) d\mu_1(x) + \int_{\Omega_2} \int_{\Omega_1} \varphi\left(|f(t) - A_k f(x)|\right) \frac{u(x) k(x,t)}{K(x)} d\mu_1(x)\, d\mu_2(t) \leq \int_{\Omega_2} \varphi(f(t)) v(t) d\mu_2(t) \tag{5}$$

*holds for all non-negative measurable functions* $f : \Omega_2 \to \mathbb{R}$*, such that* $Imf \subseteq I$*, where* $A_k$ *and* $K$ *are defined by* (2) *and* (3)*. If* $\varphi$ *is subquadratic, then the inequality sign in* (5) *is reversed.*

Let us define a linear functional as a difference between right-hand side and left-hand side of the inequality (5) as:

$$A(\varphi) = \int_{\Omega_2} \varphi(f(t)) v(t) d\mu_2(t) - \int_{\Omega_1} \varphi(A_k f(x)) u(x) d\mu_1(x) - \int_{\Omega_2} \int_{\Omega_1} \varphi\left(|f(t) - A_k f(x)|\right) \frac{u(x) k(x,t)}{K(x)} d\mu_1(x)\, d\mu_2(t). \tag{6}$$

It is clear that if $\varphi$ is superquadratic, then $A(\varphi) \geq 0$.

Next Lagrange's type mean value theorems are given in [5].

**Theorem 2.** *Let* $\varphi : [0, \infty) \to \mathbb{R}, \varphi(0) = 0$ *and let the assumptions of Theorem 1 be satisfied. Assume that* $A$ *is a strictly positive functional. If* $\frac{\varphi'(x)}{x} \in C^1(0, \infty)$*, then there exists* $\xi \in (0, \infty)$ *such that following equality holds:*

$$A(\varphi) = \frac{1}{3} \frac{\xi \varphi''(\xi) - \varphi'(\xi)}{\xi^2} \left( \int_{\Omega_2} f^3(t)\, v(t) d\mu_2(t) - \int_{\Omega_1} (A_k f(x))^3\, u(x) d\mu_1(x) - \int_{\Omega_2} \int_{\Omega_1} |f(t) - A_k f(x)|^3 \frac{u(x) k(x,t)}{K(x)} d\mu_1(x) d\mu_2(t) \right),$$

*where $A_k f(x)$ and $v(t)$ are defined by (2) and (4) respectively.*

**Theorem 3.** *Let $\varphi, \psi : [0, \infty) \to \mathbb{R}, \varphi(0) = \psi(0) = 0$ and let the assumptions of Theorem 1 be satisfied. If $\frac{\varphi'(x)}{x}, \frac{\psi'(x)}{x} \in C^1(0, \infty)$, then for a strictly positive functional $A$ there exists $\xi \in (0, \infty)$ such that*

$$\frac{A(\varphi)}{A(\psi)} = \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)},$$

*provided that denominators are not equal to zero.*

**Lemma 4.** *Consider the function $\varphi_p$ for $p > 0$ defined by*

$$\varphi_p(x) = \begin{cases} \frac{x^p}{p(p-2)}, & p \neq 2, \\ \frac{x^2}{2} \log x, & p = 2. \end{cases} \tag{7}$$

*Then with the convention $0 \log 0 = 0, \varphi_p$ is superquadratic.*

For linear functional $A$ defined by (6) we have $A(\varphi_p) \geq 0$ for all $p > 0$.

**Definition 3.** [8, page 373] *A function $\varphi : (a, b) \to \mathbb{R}$ is exponentially convex if it is continuous and*

$$\sum_{i,j=1}^{n} t_i t_j \varphi(\zeta_i + \zeta_j) \geq 0$$

*holds for every $n \in \mathbb{N}$ and for all sequences $t_n$ and $\zeta_n$ of all real numbers, such that $\zeta_i + \zeta_j \in (a, b), 1 \leq i, j \leq n$.*

The next result contains properties of the mapping $p \mapsto A(\varphi_p)$.

**Theorem 5.** [5] *For $A$ as in (6) and $\varphi_p$ as in (7) we have the followings:*

(i) *the mapping $p \mapsto A(\varphi_p)$ is continuous for $p > 0$,*

(ii) *for every $n \in \mathbb{N}$ and $p_i \in \mathbb{R}_+, p_{ij} = \frac{p_i + p_j}{2}$, $i, j = 1, 2, 3 \cdots, n$, the matrix $[A(\varphi_{p_{ij}})]_{i,j=1}^n$ is positive semi-definite, that is*

$$det[A(\varphi_{p_{ij}})]_{i,j=1}^n \geq 0,$$

(iii) *the mapping $p \mapsto A(\varphi_p)$ is exponentially convex,*

(iv) *the mapping $p \mapsto A(\varphi_p)$ is $\log -$convex,*

(v) *for $p_i \in \mathbb{R}, i = 1, 2, 3, p_1 < p_2 < p_3$,*

$$[A(\varphi_{p_2})]^{p_3 - p_1} \leq [A(\varphi_{p_1})]^{p_3 - p_2} [A(\varphi_{p_3})]^{p_2 - p_1}.$$

The paper is organized in the following way: after this Introduction, in Section 2, we prove new Hardy inequalities involving linear differential operator and for Widder's derivative using superquadratic functions. Section 3 consists of applications of Hardy-type inequalities for generalized fractional integrals. As special case we obtain the results for the Saigo, the Riemann-Liouville and the Erdélyi-Kober fractional integral operators. Finally, we give an integral operator involving generalized Mittage-Leffler function in its kernel as an application of results given in Section 1.

## 2. Hardy-type inequalities for linear differential operator and Widder's derivatives

Let $[a,b] \subset \mathbb{R}$, $a_i(x), i = 0,1,...,n-1$ $(n \in \mathbb{N})$, $h(x)$ be continuous functions on $[a,b]$. Let

$$L = D^n + a_{n-1}(x)D^{n-1} + ... + a_0(x), \ x \in (a,b),$$

be a fixed linear differential operator on $C^n[a,b]$. Let $y_1(x), y_2(x), \cdots, y_n(x)$ be a set of linearly independent solution to $Ly = 0$ and the associated Green's function for $L$ is

$$H(x,t) := \frac{\begin{vmatrix} y_1(t) & \cdot\ \cdot\ \cdot & y_n(t) \\ y_1'(t) & \cdot\ \cdot\ \cdot & y_n'(t) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ y_1^{(n-2)}(t) & \cdot & y_n^{(n-2)}(t) \\ y_1(x) & \cdot\ \cdot\ \cdot & y_n(x) \end{vmatrix}}{\begin{vmatrix} y_1(t) & \cdot\ \cdot\ \cdot & y_n(t) \\ y_1'(t) & \cdot\ \cdot\ \cdot & y_n'(t) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ y_1^{(n-2)}(t) & \cdot & y_n^{(n-2)}(t) \\ y_1^{(n-1)}(t) & \cdot\ \cdot\ \cdot & y_n^{(n-1)}(t) \end{vmatrix}},$$

which is continuous function on $[a,b]^2$, then

$$y(x) = \int_a^x H(x,t)h(t)dt, \ \text{for all } x \in [a,b],$$

is the unique solution to the initial value problem

$$Ly = h, \quad y^{(i)}(a) = 0, \quad i = 0,1,...,n-1.$$

Our first result is given in upcoming theorem.

**Theorem 6.** *Let $u$ be a weight function on $(a,b)$, $H(x,t)$ be a non-negative measurable Green function associated to linear differential operator $L$. Suppose $\widetilde{H}(x) > 0$ for all $x \in (a,b)$, the function $x \mapsto u(x)\frac{H(x,t)}{\widetilde{H}(x)}$ is integrable on $(a,b)$ and for each fixed $t \in (a,b)$, $\bar{v}$ is defined as:*

$$\bar{v}(t) := \int_t^b u(x)\frac{H(x,t)}{\widetilde{H}(x)}\,dx < \infty. \tag{8}$$

*Suppose $I = [0,c), c \leq \infty, \varphi : I \to \mathbb{R}$. If $\varphi$ is a superquadratic function on an interval $I$ then the inequality*

$$\begin{aligned} &\int_a^b \varphi\left(\frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)u(x)dx \\ &+ \int_a^b\int_a^x \varphi\left(\left|h(t) - \frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right|\right)\frac{u(x)H(x,t)}{\widetilde{H}(x)}dx\,dt \\ &\leq \int_a^b \varphi(h(t))\bar{v}(t)dt \end{aligned} \tag{9}$$

*holds for all measurable functions $h : (a,b) \to \mathbb{R}$, such that $h(t) \in I$ for all fixed $t \in (a,b)$, where $\widetilde{H}(x)$ defined as:*

$$0 < \widetilde{H}(x) := \int_a^x H(x,t)dt. \tag{10}$$

*Proof.* Applying Theorem 1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $k(x,t) = H(x,t)$, we get inequality (9). □

Let us define the linear functional given in (6) for linear differential operator which is in fact positive difference of (9) i.e.

$$\begin{aligned} A^{\diamond}(\varphi) = &\int_a^b \varphi(h(t))\bar{v}(t)dt - \int_a^b \varphi\left(\frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)u(x)dx \\ - &\int_a^b\int_a^x \varphi\left(\left|h(t) - \frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right|\right)\frac{u(x)H(x,t)}{\widetilde{H}(x)}dx\,dt \end{aligned} \tag{11}$$

The upcoming results represents mean value theorems of Lagrange-type for linear differential operator.

**Theorem 7.** *Let $\varphi : [0,\infty) \to \mathbb{R}, \varphi(0) = 0$ and let the assumptions of Theorem 6 be satisfied. Assume that $A^\diamond$ is a strictly positive functional. If $\frac{\varphi'(x)}{x} \in C^1(0,\infty)$, then there exists $\xi \in (0,\infty)$ such that following equality*

$$A^\diamond(\varphi) = \frac{1}{3}\frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi^2}\left(\int_a^b h^3(t)\,\bar{v}(t)dt - \int_a^b \left(\frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)^3 u(x)dx\right.$$
$$\left. - \int_a^b\int_a^x \left|h(t) - \frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right|^3 \frac{u(x)H(x,t)}{\widetilde{H}(x)}dxdt\right) \tag{12}$$

*holds for all measureable functions $h : (a,b) \to \mathbb{R}$, where $\bar{v}(t)$ and $\widetilde{H}(x)$ are defined by* (8) *and* (10) *respectively.*

*Proof.* Applying Theorem 2 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $k(x,t) = H(x,t)$ and $K(x) = \widetilde{H}(x)$, we get equality (12). □

**Theorem 8.** *Let $\varphi,\psi : [0,\infty) \to \mathbb{R}, \varphi(0) = \psi(0) = 0$ and let the assumptions of Theorem 7 be satisfied. Assume that $A^\diamond$ is strictly positive functional. If $\frac{\varphi'(x)}{x}, \frac{\psi'(x)}{x} \in C^1(0,\infty)$ then there exists $\xi \in (0,\infty)$ such that*

$$\frac{A^\diamond(\varphi)}{A^\diamond(\psi)} = \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)}, \tag{13}$$

*where the linear functional $A^\diamond(\varphi)$ is defined by* (11)*. It is provided that denominators are not equal to zero.*

*Proof.* Applying Theorem 3 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $k(x,t) = H(x,t)$, we get equation (13). □

**Theorem 9.** *Let $p > 2$, $\varphi_p$, $\bar{v}$ and $A^\diamond$ be given in (7), (8) and (11) respectively. Moreover*

$$A^\diamond(\varphi_p) = \frac{1}{p(p-2)}\left[\int_a^b h^p(t)\bar{v}(t)\,dt - \int_a^b \left(\frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)^p u(x)\,dx\right.$$
$$\left. - \int_a^b\int_a^x \left|h(t) - \frac{1}{\widetilde{H}(x)}\int_a^x H(x,t)h(t)dt\right|^p \frac{u(x)H(x,t)}{\widetilde{H}(x)}dx\,dt\right],$$

*then the map $p \to A^\diamond(\varphi_p)$ have the following properties:*

(i) *the mapping $p \mapsto A^\diamond(\varphi_p)$ is continuous for $p > 0$,*

(ii) *for every $n \in \mathbb{N}$ and $p_i \in \mathbb{R}_+, p_{ij} = \frac{p_i+p_j}{2}$, $i,j = 1,2,3\cdots,n$, the matrix $[A^\diamond(\varphi_{p_{ij}})]_{i,j=1}^n$ is positive semi-definite, i.e.,*

$$det[A^\diamond(\varphi_{p_{ij}})]_{i,j=1}^n \geq 0,$$

(iii) *the mapping* $p \mapsto A^{\diamond}(\varphi_p)$ *is exponentially convex,*

(iv) *the mapping* $p \mapsto A^{\diamond}(\varphi_p)$ *is log-convex,*

(v) *for* $p_i \in \mathbb{R}, i = 1, 2, 3,\ p_1 < p_2 < p_3,$

$$[A^{\diamond}(\varphi_{p_2})]^{p_3 - p_1} \leq [A^{\diamond}(\varphi_{p_1})]^{p_3 - p_2} [A^{\diamond}(\varphi_{p_3})]^{p_2 - p_1}.$$

*Proof.* Applying Theorem 5 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $k(x, t) = H(x, t)$, we get results $(i) - (v)$. □

Now we give the Hardy-type inequalities for Widder's derivative. First it is necessary to introduce some basic notations and facts about Widder's derivative (see[13]). Let $f, u_0, u_1, \ldots, u_n \in C^{n+1}[a, b], n \geq 0$, and the Wronskians

$$W_i(x) := W[u_0(x), u_1(x), \ldots, u_i(x)] = \begin{vmatrix} u_0(x) & \cdot & \cdot & \cdot & u_i(x) \\ u_0'(x) & \cdot & \cdot & \cdot & u_i'(x) \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ u_0^{(i)}(x) & \cdot & \cdot & \cdot & u_i^{(i)}(x) \end{vmatrix},$$

$i = 0, 1, \ldots, n$. Here $W_0(x) = u_0(x)$. Assume $W_i(x) > 0$ over $[a, b]$. For $i \geq 0$, the differential operator of order $i$ (Widder's derivative):

$$L_i f(x) := \frac{W[u_0(x), u_1(x), \ldots, u_{i-1}(x), f(x)]}{W_{i-1}(x)},$$

$i = 1, \ldots, n + 1; L_0 f(x) = f(x)$ for all $x \in [a, b]$. Consider also

$$g_i(x, t) := \frac{1}{W_i(t)} \begin{vmatrix} u_0(t) & \cdot & \cdot & \cdot & u_i(t) \\ u_0'(t) & \cdot & \cdot & \cdot & u_i'(t) \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ u_0^{(i-1)}(t) & \cdot & \cdot & \cdot & u_i^{(i-1)}(t) \\ u_0(x) & \cdot & \cdot & \cdot & u_i(x) \end{vmatrix},$$

$i = 1, 2, ..., n$; $g_0(x, t) := \frac{u_0(x)}{u_0(t)}$ for all $x, t \in [a, b]$.

**Example 1.** [13]. *Sets of the form* $\{u_0, u_1, u_2, \ldots, u_n\}$ *are* $\{1, x, x^2, \ldots, x^n\}$, $\{1, \sin x, -\cos x, -\sin 2x, \cos 2x, \ldots, (-1)^{n-1} \sin nx, (-1)^n \cos nx\}$, *etc. fulfill the above theory.*

We also mention the generalized Widder-Taylor's formula, see [13].

**Theorem 10.** *Let the functions* $f, u_0, u_1, \ldots, u_n \in C^{n+1}[a,b]$, *and let the Wronskians* $W_0(x), W_1(x), \ldots, W_n(x) > 0$ *on* $[a,b], x \in [a,b]$. *Then for* $t \in [a,b]$ *we have*

$$f(x) = f(t)\frac{u_0(x)}{u_0(t)} + L_1 f(t) g_1(x,t) + ... + L_n f(t) g_n(x,t) + R_n(x),$$

*where*

$$R_n(x) := \int_s^x g_n(x,s) L_{n+1} f(s) ds.$$

For example (see [13]) one could take $u_0(x) = c > 0$. If $u_i(x) = x^i, i = 0, 1, \ldots, n$, defined on [a,b], then

$$L_i f(t) = f^{(i)}(t) \ \ and \ \ g_i(x,t) = \frac{(x-t)^i}{i!}, \quad t \in [a,b].$$

**Corollary 11.** *By additionally assuming for fixed* $a$ *that* $L_i f(a) = 0, i = 0, 1, ..., n$, *we get that*

$$f(x) := \int_a^x g_n(x,t) L_{n+1} f(t) dt \quad for\ all\ x \in [a,b].$$

**Theorem 12.** *Let* $u$ *be a weight function on* $(a,b)$ *and* $g_n(x,t)$ *be a non-negative measurable kernel. Suppose* $\widetilde{g}_n(x) > 0$ *for all* $x \in (a,b)$ *the function* $x \mapsto u(x)\frac{g_n(x,t)}{\widetilde{g}_n(x)}$ *is integrable on* $(a,b)$ *for each fixed* $t \in (a,b), w$ *is defined on* $(a,b)$ *by*

$$w(t) := \int_t^b u(x)\frac{g_n(x,t)}{\widetilde{g}_n(x)}\, dx < \infty. \tag{14}$$

*Suppose* $I = [0,c), c \leq \infty, \varphi : I \to \mathbb{R}$. *If* $\varphi$ *is a superquadratic function on an interval* $I$ *then the inequality*

$$\begin{aligned} & \int_a^b \varphi\left(\frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t) L_{n+1} f(t) dt\right) u(x) dx \\ + & \int_a^b \int_a^x \varphi\left(\left|L_{n+1} f(t) - \frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t) L_{n+1} f(t) dt\right|\right) \frac{u(x) g_n(x,t)}{\widetilde{g}_n(x)} dx\, dt \\ \leq & \int_a^b \varphi(L_{n+1} f(t)) w(t) dt \end{aligned} \tag{15}$$

*holds for all measurable functions* $L_{n+1}f : (a,b) \to \mathbb{R}$*, such that* $L_{n+1}f(t) \in I$ *for all fixed* $t \in (a,b)$ *and* $\widetilde{g}_n(x)$ *is defined as*

$$0 < \widetilde{g}_n(x) := \int_a^x g_n(x,t)dt. \tag{16}$$

*Proof.* Applying Theorem 1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $k(x,t) = g_n(x,t)$, we get inequality (15). □

Now let us define the linear functional defined by (6) for Widder's derivative i.e.

$$A^*(\varphi) = \int_a^b \varphi(L_{n+1}f(t))w(t)dt - \int_a^b \varphi\left(\frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)u(x)dx$$
$$-\int_a^b\int_a^x \varphi\left(\left|L_{n+1}f(t) - \frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right|\right)\frac{u(x)g_n(x,t)}{\widetilde{g}_n(x)}dx\,dt. \tag{17}$$

Next mean value theorems for Widder's derivative are given in upcoming theorems.

**Theorem 13.** *Let* $\varphi : [0,\infty) \to \mathbb{R}, \varphi(0) = 0$ *and let the assumptions of the Theorem 12 be satisfied. Assume that* $A^*$ *is a strictly positive functional defined by* (17)*. If* $\frac{\varphi'(x)}{x} \in C^1(0,\infty)$*, then there exists* $\xi \in (0,\infty)$ *such that following equation*

$$A^*(\varphi) = \frac{1}{3}\frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi^2}\left(\int_a^b (L_{n+1}f(t))^3 w(t)dt\right.$$
$$- \int_a^b \left(\frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)^3 u(x)dx$$
$$\left.- \int_a^b\int_a^x \left|L_{n+1}f(t) - \frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)h(t)dt\right|^3 \frac{u(x)g_n(x,t)}{\widetilde{g}_n(x)}dx\,dt\right) \tag{18}$$

*holds for all measureable functions* $L_{n+1}f : (a,b) \to \mathbb{R}$ *, where* $w(t)$ *and* $\widetilde{g}_n(x)$ *are defined by* (14) *and* (16) *respectively.*

*Proof.* Applying Theorem 2 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $k(x,t) = g_n(x,t)$ and $K(x) = \widetilde{g}_n(x)$, we get equality (18). □

**Theorem 14.** *Let* $\varphi, \psi : [0,\infty) \to \mathbb{R}, \varphi(0) = \psi(0) = 0$ *and let the assumptions of Theorem 13 be satisfied. If* $\frac{\varphi'(x)}{x}, \frac{\psi'(x)}{x} \in C^1(0,\infty)$ *then there exists* $\xi \in$

$(0,\infty)$ *such that*

$$\frac{A^*(\varphi)}{A^*(\psi)} = \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)}, \tag{19}$$

*where the linear functional* $A^*(\varphi)$ *is defined by* (17) *provided that* $\xi\psi''(\xi) - \psi'(\xi) \neq 0$.

*Proof.* Applying Theorem 3 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $k(x,t) = g_n(x,t)$, we get equation (19). $\square$

Next result contains the properties of the mapping $p \to A^*(\varphi_p)$.

**Theorem 15.** *Let* $p > 2, \varphi_p, w(t)$ *and* $A^*$ *be given by (7), (14) and (17) respectively. If*

$$A^*(\varphi_p) = \frac{1}{p(p-2)}\left[\int_a^b (L_{n+1}f(t))^p\, w(t)\, dt - \int_a^b \left(\frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)^p u(x)\, dx \right.$$
$$\left. - \int_a^b\int_a^x \left|L_{n+1}f(t) - \frac{1}{\widetilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right|^p \frac{u(x)g_n(x,t)}{\widetilde{g}_n(x)} dx\, dt\right]$$

*then the following properties holds:*

(i) *the mapping* $p \mapsto A^*(\varphi_p)$ *is continuous for* $p > 0$,

(ii) *for every* $n \in \mathbb{N}$ *and* $p_i \in \mathbb{R}_+, p_{ij} = \frac{p_i+p_j}{2}$, $i,j = 1,2,3\cdots,n$, *the matrix* $[A^*(\varphi_{p_{ij}})]_{i,j=1}^n$ *is positive semi-definite, that is*

$$det[A^*(\varphi_{p_{ij}})]_{i,j=1}^n \geq 0,$$

(iii) *the mapping* $p \mapsto A^*(\varphi_p)$ *is exponentially convex,*

(iv) *the mapping* $p \mapsto A^*(\varphi_p)$ *is* $\log -$*convex,*

(v) *for* $p_i \in \mathbb{R}, i = 1,2,3, p_1 < p_2 < p_3$,

$$[A^*(\varphi_{p_2})]^{p_3-p_1} \leq [A^*(\varphi_{p_1})]^{p_3-p_2}[A^*(\varphi_{p_3})]^{p_2-p_1}.$$

*Proof.* Applying Theorem 5 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $k(x,t) = g_n(x,t)$, we get results $(i)-(v)$. $\square$

## 3. Hardy-type inequalities for generalized fractional integral operators

In this section we first give the definition of generalized fractional integral operator involving Gauss hypergeometric function in its kernel defined by Luis Curiel et al. in [4].

**Definition 4.** [4] *Let* $\alpha > 0, \mu > -1, \beta, \eta \in \mathbb{R}$. *Then the generalized fractional integral* $I_{a,x}^{\alpha,\beta,\eta,\mu}$ *of order* $\alpha$, *for a real-valued continuous function* $f$ *is defined by:*

$$I_{a,x}^{\alpha,\beta,\eta,\mu} f(x) = \frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} \int_a^x t^{\mu}(x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta+\mu, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)dt, \;\; x \in [a,b], \tag{20}$$

*where the function* ${}_2F_1(.\,,\,.\,,\,;\,.)$ *appearing in kernel for operator (20) is the Gaussian hypergeometric function defined by*

$${}_2F_1(a,b;c;t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} t^n,$$

*and* $(a)_n$ *is the Pochhammer symbol:* $(a)_n = a(a+1)...(a+n-1), (a)_0 = 1.$

The operator (20) includes the Saigo, the Riemann-Liouville and the Erdélyi-Kober fractional integral operators i.e.,

$$\begin{aligned} I_{a,x}^{\alpha,\beta,\eta} f(x) &= I_{a,x}^{\alpha,\beta,\eta,0} f(x) \\ &= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)dt, \; x \in [a,b]. \end{aligned}$$

$$R^{\alpha} f(x) = I_{a,x}^{\alpha,-\alpha,\eta} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \; x \in [a,b].$$

and

$$I^{\alpha,\eta} f(x) = I_{a,x}^{\alpha,0,\eta} f(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} t^{\eta} f(t)dt, \; x \in [a,b].$$

First we give our general result for generalized fractional integral of order $\alpha$, then as special cases we establish the inequalities for the Saigo, the Riemann-Liouville and the Erdélyi-Kober fractional integral operators.

**Theorem 16.** *Let* $\alpha > 0, \mu > -1, \beta, \eta \in \mathbb{R}$, $I_{a,x}^{\alpha,\beta,\eta,\mu}$ *denotes the generalized fractional integral of order* $\alpha$ *and* $u$ *be a weight function defined on* $(a,b)$. *Moreover for each fixed* $t \in (a,b)$ *define* $\hat{v}$ *by*

$$\hat{v}(t) = \frac{1}{\Gamma(\alpha)} \int_t^b u(x) \frac{x^{-\alpha-\beta-2\mu} {}_2F_1\left(\alpha+\beta+\mu, -\eta, \alpha; 1-\frac{t}{x}\right) t^{\mu}(x-t)^{\alpha-1}}{\hat{K}(x)} dx < \infty.$$

*If $I = [0, c), c \leq \infty$ and $\varphi : I \to \mathbb{R}$ is a superquadratic function, then the inequality*

$$\int_a^b \varphi\left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right) u(x)dx + \int_a^b \int_a^x \varphi\left(\left|f(t) - \frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right|\right) \frac{u(x)k(x,t)}{\hat{K}(x)} dx\, dt \leq \int_a^b \varphi(f(t))\hat{v}(t)dt \quad (21)$$

*holds for all non-negative measurable functions $f : (a, b) \to \mathbb{R}$, such that $Imf \subseteq I$, where*

$$\hat{K}(x) = \int_a^x \frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} {}_2F_1\left(\alpha+\beta+\mu, -\eta, \alpha; 1 - \frac{t}{x}\right) t^{\mu}(x-t)^{\alpha-1} dt. \quad (22)$$

*If $\varphi$ is subquadratic, then the inequality sign in* (21) *is reversed.*

*Proof.* Applying Theorem 1 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$

$$k(x,t) = \begin{cases} \frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} {}_2F_1\left(\alpha+\beta+\mu, -\eta, \alpha; 1 - \frac{t}{x}\right) t^{\mu}(x-t)^{\alpha-1}, & a \leq t \leq x; \\ 0, & x < t \leq b. \end{cases} \quad (23)$$

and

$$A_k f(x) = \frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)},$$

so inequality (21) follows. □

Now let us define the linear functional defined by (6) for generalized fractional integral i.e.

$$\tilde{A}(\varphi) = \int_a^b \varphi(f(t))\hat{v}(t)dt - \int_a^b \varphi\left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right) u(x)dx - \int_a^b \int_a^x \varphi\left(\left|f(t) - \frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right|\right) \frac{u(x)k(x,t)}{\hat{K}(x)} dx\, dt. \quad (24)$$

**Theorem 17.** *Let $\varphi : [0, \infty) \to \mathbb{R}, \varphi(0) = 0$ and let the assumptions of Theorem 16 be satisfied. Assume that $\widetilde{A}$ is a strictly positive functional. If*

$\frac{\varphi'(x)}{x} \in C^1(0,\infty)$, *then there exists* $\xi \in (0,\infty)$ *such that following equality*

$$\widetilde{A}(\varphi) = \frac{1}{3}\frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi^2}\left(\int_a^b f^3(t)\,\hat{v}(t)dt - \int_a^b \left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right)^3 u(x)dx \right.$$
$$\left. - \int_a^b\int_a^x \left|f(t) - \frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right|^3 \frac{u(x)k(x,t)}{\hat{K}(x)}dxdt\right) \tag{25}$$

*holds for all measurable functions* $f : (a,b) \to \mathbb{R}$*, where* $\hat{K}(x)$ *is defined by (22).*

*Proof.* Applying Theorem 2 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $k(x,t)$ given in (23), we obtain equality (25). □

**Theorem 18.** *Let* $\varphi, \psi : [0,\infty) \to \mathbb{R}, \varphi(0) = \psi(0) = 0$ *and let the assumptions of Theorem 16 be satisfied. Assume that* $\tilde{A}$ *is strictly positive functional. If* $\frac{\varphi'(x)}{x}, \frac{\psi'(x)}{x} \in C^1(0,\infty)$ *then there exists* $\xi \in (0,\infty)$ *such that*

$$\frac{\tilde{A}(\varphi)}{\tilde{A}(\psi)} = \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)}, \tag{26}$$

*where the linear functional* $\tilde{A}(\varphi)$ *is defined as* (24). *It is provided that* $\xi\psi''(\xi) - \psi'(\xi) \neq 0$.

*Proof.* Applying Theorem 3 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, we get equality (26). □

**Theorem 19.** *Let* $p > 2, \alpha > 0, \mu > -1, \beta, \eta \in \mathbb{R}$, $I_{a,x}^{\alpha,\beta,\eta,\mu}$ *denotes the generalized fractional integral of order* $\alpha$ *and* $\hat{v}$ *by* (16). *If*

$$\tilde{A}(\varphi_p) = \frac{1}{p(p-2)}\left[\int_a^b f^p(t)\hat{v}(t)\,dt - \int_a^b \left(\frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right)^p u(x)\,dx \right.$$
$$\left. - \int_a^b\int_a^x \left|f(t) - \frac{I_{a,x}^{\alpha,\beta,\eta,\mu} f(x)}{\hat{K}(x)}\right|^p \frac{u(x)k(x,t)}{\hat{K}(x)}dx\,dt\right],$$

*then the properties of the mapping* $p \mapsto \tilde{A}(\varphi_p)$ *are given as follows:*

(i) *the mapping* $p \mapsto \tilde{A}(\varphi_p)$ *is continuous for* $p > 0$,

(ii) *for every* $n \in \mathbb{N}$ *and* $p_i \in \mathbb{R}_+, p_{ij} = \frac{p_i+p_j}{2}$, $i,j = 1,2,3\cdots,n$, *the matrix* $[\tilde{A}(\varphi_{p_{ij}})]_{i,j=1}^n$ *is positive semi-definite, that is*

$$det[\tilde{A}(\varphi_{p_{ij}})]_{i,j=1}^n \geq 0,$$

(iii) *the mapping* $p \mapsto \tilde{A}(\varphi_p)$ *is exponentially convex,*

(iv) *the mapping* $p \mapsto \tilde{A}(\varphi_p)$ *is log-convex,*

(v) *for* $p_i \in \mathbb{R}, i = 1, 2, 3, p_1 < p_2 < p_3$,

$$[\tilde{A}(\varphi_{p_2})]^{p_3-p_1} \leq [\tilde{A}(\varphi_{p_1})]^{p_3-p_2}[\tilde{A}(\varphi_{p_3})]^{p_2-p_1}.$$

*Proof.* Applying Theorem 5 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $k(x, t)$ given in (23), then we get results $(i) - (v)$. □

**Remark 1.** *If we take* $\mu = 0$ *in the inequality* (21), *equations* (24), (25) *and in properties* $(i) - (v)$ *from Theorem 19, then we get results for the Saigo fractional integral.*

**Remark 2.** *If along* $\mu = 0$ *we take* $\beta = -\alpha$ *in the inequality* (21), *equations* (24), (25) *and in properties* $(i) - (v)$ *from Theorem 19, then we get results for the Riemann-Liouvill's fractional integral.*

**Remark 3.** *If we take* $\beta = 0$ *and* $\mu = 0$ *in the inequality* (21), *equations* (24), (25) *and in properties* $(i) - (v)$ *from Theorem 19, then we get results for the Erdélyi-Kober fractional integral operator.*

Now our purpose is to give the Hardy type inequalities for the fractional integral operator involving generalized Mittag-Leffler function appearing in the kernel (see [11]).

**Definition 5.** [11] *Let* $\alpha, \beta, \gamma, \delta \in \mathbb{C}; min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0; p, q > 0$ *and* $q < \Re\alpha + p$. *Then the integral operator defined by*

$$\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} g(x) = \int_a^x (x - t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x - t)^{\alpha})g(t)dt, \tag{27}$$

*which contains the generalized Mittage-Leffler function*

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_{pn}}, \tag{28}$$

*in its kernel is investigated and its boundedness is proved under certain conditions. Equation (28) represents all the previous generalizations of Mittage-Leffler function by setting*

- $\delta = p = q = 1$, *we get* $E^{\gamma}_{\alpha,\beta}(z) = \sum\limits_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}$ *classify by Prabhakar* [9].
- $p = q = 1$, *it reduces to* $E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum\limits_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{(\delta)_n}$ *defined by Salim in* [10].

- $\delta = p = 1$, *it represents* $E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}$ *which was introduced by H. M. Srivastava and Z. Tomovski in* [12].
- $\gamma = \delta = p = q = 1$, *it reduces to Wiman's function* [14], *moreover if* $\beta = 1$, *Mittage-Leffler function* $E_\alpha(z)$ *will be the result.*

**Theorem 20.** *Let* $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, *and* $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ *denotes the generalized Mittage-Leffler function. Moreover let* $u$ *be a weight function defined on* $(a,b)$ *and for each fixed* $t \in (a,b)$ *define* $\tilde{v}$ *by*

$$\tilde{v}(t) := \int_t^b u(x) \frac{(x-t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^\alpha)}{\tilde{K}(x)} dx. \tag{29}$$

*If* $I = [0,c), c \leq \infty$ *and* $\varphi : I \to \mathbb{R}$ *is a superquadratic function, then the inequality*

$$\int_a^b \varphi\left(\frac{\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} g(x)}{\tilde{K}(x)}\right) u(x)dx + \int_a^b \int_a^x \varphi\left(\left|g(t) - \frac{\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} g(x)}{\tilde{K}(x)}\right|\right)$$
$$\times \frac{u(x)(x-t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^\alpha)}{\tilde{K}(x)} dxdt \leq \int_a^b \varphi(g(t))\tilde{v}(t)dt \tag{30}$$

*holds for all non-negative measurable functions* $g : (a,b) \to \mathbb{R}$, *such that* $Img \subseteq I$, *where* $\tilde{K}$ *is defined as:*

$$\tilde{K}(x) = \int_a^x (x-t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^\alpha)dt, \tag{31}$$

*and* $\tilde{v}$ *as in (29).*

*If* $\varphi$ *is subquadratic, then the inequality sign in* (30) *is reversed.*

*Proof.* Applying Theorem 1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$,

$$\tilde{k}(x,t) = \begin{cases} (x-t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^\alpha), & a \leq t \leq x\,; \\ 0, & x < t \leq b\,, \end{cases} \tag{32}$$

so inequality (30) follows. □

Now let us define the linear functional defined by (6) for fractional integral operator with generalized Mittag-Leffler function appearing in the kernel.

$$\widehat{A}(\varphi) = \int_a^b \varphi(g(t))\tilde{v}(t)dt - \int_a^b \varphi\left(\frac{\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}g(x)}{\tilde{K}(x)}\right)u(x)dx$$
$$-\int_a^b\int_a^x \varphi\left(\left|g(t) - \frac{\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}g(x)}{\tilde{K}(x)}\right|\right)\frac{u(x)(x-t)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^{\alpha})}{\tilde{K}(x)}dx\,dt. \quad (33)$$

Next, results represents mean value theorems for the generalized fractional integral given in (27).

**Theorem 21.** *Let $\varphi : [0,\infty) \to \mathbb{R}, \varphi(0) = 0$ and let the assumptions of Theorem 20 be satisfied. Assume that $\widehat{A}$ is a strictly positive functional. If $\frac{\varphi'(x)}{x} \in C^1(0,\infty)$, then there exists $\xi \in (0,\infty)$ such that following equality*

$$\widehat{A}(\varphi) = \frac{1}{3}\frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi^2}\left(\int_a^b g^3(t)\,\tilde{v}(t)dt - \int_a^b \left(\frac{\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}g(x)}{\tilde{K}(x)}\right)^3 u(x)dx\right.$$
$$\left. - \int_a^b\int_a^x \left|g(t) - \frac{\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}g(x)}{\tilde{K}(x)}\right|^3 \frac{u(x)(x-t)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^{\alpha})}{\tilde{K}(x)}dxdt\right) \quad (34)$$

*holds for all measureable functions $g : (a,b) \to \mathbb{R}$, where $\tilde{K}(x)$ is given by (31).*

*Proof.* Applying Theorem 2 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $\tilde{K}(x)$ and $\tilde{k}(x,t)$ given in (31) and (32) respectively, we obtain equality (34). □

**Theorem 22.** *Let $\varphi, \psi : [0,\infty) \to \mathbb{R}, \varphi(0) = \psi(0) = 0$ and let the assumptions of Theorem 21 be satisfied. Assume that $\widehat{A}$ is strictly positive functional. If $\frac{\varphi'(x)}{x}, \frac{\psi'(x)}{x} \in C^1(0,\infty)$ then there exists $\xi \in (0,\infty)$ such that*

$$\frac{\widehat{A}(\varphi)}{\widehat{A}(\psi)} = \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)}, \quad (35)$$

*where the linear functional $\widehat{A}(\varphi)$ is defined by* (33).
*It is provided that $\xi\psi''(\xi) - \psi'(\xi) \neq 0$.*

*Proof.* Applying Theorem 3 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, we get equality (35). □

**Theorem 23.** *Let $p > 2, \alpha, \beta, \gamma, \delta \in \mathbb{C}$, and $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ denotes the generalized Mittage-Leffler function. Then the linear functional (6) for generalized integral* *operator (27) is given as:*

$$\widehat{A}(\varphi_p) = \frac{1}{p(p-1)} \left[ \int_a^b g^p(t)\tilde{v}(t)\,dt - \int_a^b \left( \frac{\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} g(x)}{\tilde{K}(x)} \right)^p u(x)\,dx \right.$$

$$\left. - \int_a^b \int_a^x \left| g(t) - \frac{\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} g(x)}{\tilde{K}(x)} \right|^p \frac{u(x)(x-t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^{\alpha})}{\tilde{K}(x)} dx\,dt \right],$$

*where $\varphi_p$ is given by (7) then the mapping $p \mapsto \widehat{A}(\varphi_p)$ admits the following properties:*

(i) *the mapping $p \mapsto \widehat{A}(\varphi_p)$ is continuous for $p > 0$,*

(ii) *for every $n \in \mathbb{N}$ and $p_i \in \mathbb{R}_+, p_{ij} = \frac{p_i+p_j}{2}$, $i,j = 1,2,3\cdots,n$, the matrix $[\widehat{A}(\varphi_{p_{ij}})]^n_{i,j=1}$ is positive semi-definite, that is*

$$det[\widehat{A}(\varphi_{p_{ij}})]^n_{i,j=1} \geq 0,$$

(iii) *the mapping $p \mapsto \widehat{A}(\varphi_p)$ is exponentially convex,*

(iv) *the mapping $p \mapsto \widehat{A}(\varphi_p)$ is log-convex,*

(v) *for $p_i \in \mathbb{R}, i = 1,2,3, p_1 < p_2 < p_3$,*

$$[\widehat{A}(\varphi_{p_2})]^{p_3-p_1} \leq [\widehat{A}(\varphi_{p_1})]^{p_3-p_2}[\widehat{A}(\varphi_{p_3})]^{p_2-p_1}.$$

*Proof.* Applying Theorem 5 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $\tilde{k}(x,t)$ given in (32), we get results $(i) - (v)$. □

**Remark 4.** *If we substitute $p = q = 1$ in the inequality (30) equations (33), (34) and in properties $(i) - (v)$ from Theorem 23, then we get results for the Mittage-Leffler function defined by Salim in* [10].

**Remark 5.** *If we take $\delta = p = 1$ in the inequality (30) equations (33), (34) and in properties $(i) - (v)$ from Theorem 23, then we get results for the Mittage-Leffler function introduced by Tomovoski et al. in* [12].

**Remark 6.** *If we take $\delta = p = q = 1$ in the inequality (30) equations (33), (34) and in properties $(i) - (v)$ from Theorem 23, then we get results for the Mittage-Leffler function represented in* [9] *by Parabhakar.*

**Remark 7.** *If we take $\gamma = \delta = p = q = 1$ in the inequality (30) equations (33), (34) and in properties $(i) - (v)$ from Theorem 23, then we get results for the Wiman's function* [14].

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