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RECIPROCAL PRODUCT DEGREE DISTANCE OF STRONG PRODUCT OF GRAPHS

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Abstract. In this paper, the exact formula for the reciprocal product degree distance of strong product of a connected graph and the complete multipartite graph with partite sets of sizes m_0, m_1, $\ldots$, m_{r-1} is obtained. Using the results obtained here, the formula for the reciprocal degree distance of the closed fence graph is computed.

Key words : degree distance,reciprocal product degree distance, strong product.
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1. Introduction

All the graphs considered in this paper are simple and connected. For vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G and let $d_G(v)$ be the degree of a vertex $v \in V(G)$. The *strong product* of graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) : u \in V(G), v \in V(H)\}$ and $(u, x)(v, y)$ is an edge whenever (i) $u = v$ and $xy \in E(H)$, or (ii) $uv \in E(G)$ and $x = y$, or (iii) $uv \in E(G)$ and $xy \in E(H)$.

A *topological index* of a graph is a real number related to the graph; it does not depend on labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [6]. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index.

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Let G be a connected graph. Then *Wiener index* of G is defined as $W(G) = \frac{1}{2} \sum\limits_{u,\, v \in V(G)} d_G(u,v)$ with the summation going over all pairs of distinct vertices of G. Dobrynin and Kochetova [2] and Gutman [5] independently proposed a vertex-degree-weighted version of Wiener index called *degree distance* or *Schultz molecular* topological index, which is defined for a connected graph G as $DD(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} (d_G(u) + d_G(v))d_G(u,v)$, where $d_G(u)$ is the degree of the vertex u in G. Note that the degree distance is a degree-weight version of the Wiener index. In the literature, many results on the degree distance $DD(G)$ have been put forward in past decades and they mainly deal with extreme properties of $DD(G)$. Tomescu[18] showed that the star is the unique graph with minimum degree distance within the class on n-vertex connected graphs. Tomescu[19] deduced properties of graphs with minimum degree distance in the class of n-vertex connected graphs with $m \geq n-1$ edges. For other related results along this line, see [3, 8, 11].

Additively weighted Harary index(H_A) or reciprocal degree distance(RDD) is defined in [1] as $H_A(G) = RDD(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} \frac{(d_G(u)+d_G(v))}{d_G(u,v)}$. In [7], Hamzeh et. al recently introduced generalized degree distance of graphs. Hua and Zhang [9] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants including the degree distance, Harary index, the first Zagreb index, the first Zagreb coindex, pendent vertices, independence number, chromatic number and vertex-, and edge-connectivity. Pattabiraman and Vijayaragavan [13, 14] have obtained the reciprocal degree distance of join, tensor product, strong product and wreath product of two connected graphs in terms of other graph invariants. The chemical applications and mathematical properties of the reciprocal degree distance are well studied in [1, 12, 16].

Similarly, the modified Schultz molecular topological index or Gutman index is defined as $DD_*(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} d_G(u)d_G(v)d_G(u,v)$. In Su et al.[17] introduce the multiplicatively weighted Harary indices or reciprocal product-degree distance of graphs, which can be seen as a product -degree-weight version of Harary index $RDD_*(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} \frac{d_G(u)d_G(v)}{d_G(u,v)}$. Hua and Zhang [9] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants including the degree distance, Harary index, the first Zagreb index, the first Zagreb coindex, pendent vertices, independence number, chromatic number and vertex and edge-connectivity. Reciprocal product-degree distance of join, tensor and strong product of two connected graphs are discussed by Pattabiraman [15].

It is well known that many graphs arise from simpler graphs via various graph operations. Hence it is important to understand how certain invariants of such product graphs are related to the corresponding invariants of the original graphs. In this paper, the exact formula for the reciprocal product degree distance of strong product $G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$, where $K_{m_0, m_1, \ldots, m_{r-1}}$ is the complete multipartite graph with partite sets of sizes m_0, m_1, $\ldots$, m_{r-1} is obtained.

The *first Zagreb index* is defined as $M_1(G) = \sum\limits_{u \in V(G)} d_G(u)^2$. In fact, one can rewrite the first Zagreb index as $M_1(G) = \sum\limits_{uv \in E(G)} (d_G(u) + d_G(v))$. The Zagreb indices are found to have applications in QSPR and QSAR studies as well, see [4].

If $m_0 = m_1 = \ldots = m_{r-1} = s$ in $K_{m_0, m_1, \ldots, m_{r-1}}$ (the complete multipartite graph with partite sets of sizes m_0, m_1, $\ldots$, m_{r-1}), then we denote it by $K_{r(s)}$. For $S \subseteq V(G)$, $\langle S \rangle$ denotes the subgraph of G induced by S. For two subsets $S, T \subset V(G)$, not necessarily disjoint, by $d_G(S,T)$, we mean the sum of the distances in G from each vertex of S to every vertex of T, that is, $d_G(S,T) = \sum\limits_{s \in S, t \in T} d_G(s,t)$.

2. Reciprocal product degree distance of strong product of graphs

In this section, we obtain the reciprocal product degree distance of $G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$. Let G be a simple connected graph with $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and let $K_{m_0, m_1, \ldots, m_{r-1}}$, $r \geq 2$, be the complete multipartite graph with partite sets V_0, V_1, $\ldots$, V_{r-1} and let $|V_i| = m_i$, $0 \leq i \leq r-1$. In the graph $G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$, let $B_{ij} = v_i \times V_j, v_i \in V(G)$ and $0 \leq j \leq r-1$. For our convenience, the vertex set of $G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$ is written as $V(G) \times V(K_{m_0, m_1, \ldots, m_{r-1}}) = \bigcup\limits_{\substack{i=0 \\ j=0}}^{\substack{r-1 \\ n-1}} B_{ij}$. Let $\mathscr{B} = \{B_{ij}\}_{\substack{i=0,1,\ldots,n-1 \\ j=0,1,\ldots,r-1}}$. Let $X_i = \bigcup\limits_{j=0}^{r-1} B_{ij}$ and $Y_j = \bigcup\limits_{i=0}^{n-1} B_{ij}$; we call X_i and Y_j as *layer* and *column* of $G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$, respectively. If we denote $V(B_{ij}) = \{x_{i1}, x_{i2}, \ldots, x_{im_j}\}$ and $V(B_{kp}) =$
$\{x_{k1}, x_{k2}, \ldots, x_{k\,m_p}\}$, then $x_{i\ell}$ and $x_{k\ell}, 1 \leq \ell \leq j$, are called the *corresponding vertices* of B_{ij} and B_{kp}. Further, if $v_i v_k \in E(G)$, then the induced subgraph $\langle B_{ij} \bigcup B_{kp} \rangle$ of $G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$ is isomorphic to $K_{|V_j||V_p|}$ or, m_p independent edges joining the corresponding vertices of B_{ij} and B_{kj} according as $j \neq p$ or $j = p$, respectively.

The proof of the following lemma follows easily from the properties and structure of $G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$.

Lemma 1. *Let G be a connected graph and let $B_{ij},\ B_{kp} \in \mathscr{B}$ of the graph $G' = G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$, where $r \geq 2$.*
(i) If $v_iv_k \in E(G)$ and $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kj}$, then

$$d_{G'}(x_{it}, x_{k\ell}) = \begin{cases} 1, & \text{if } t = \ell, \\ 2, & \text{if } t \neq \ell, \end{cases}$$

and if $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kp},\ j \neq p$, then $d_{G'}(x_{it}, x_{k\ell}) = 1$.
(ii) If $v_iv_k \notin E(G)$, then for any two vertices $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kp}$, $d_{G'}(x_{it}, x_{k\ell}) = d_G(v_i, v_k)$.
(iii) For any two distinct vertices in B_{ij}, their distance is 2.

The proof of the following lemma follows easily from Lemma 1. The lemma is used in the proof of the main theorems of this section.

Lemma 2. *Let G be a connected graph and let $B_{ij},\ B_{kp} \in \mathscr{B}$ of the graph $G' = G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$, where $r \geq 2$.*
(i) If $v_iv_k \in E(G)$, then

$$d^H_{G'}(B_{ij}, B_{kp}) = \begin{cases} m_jm_p, & \text{if } j \neq p, \\ \frac{m_j(m_j+1)}{2}, & \text{if } j = p, \end{cases}$$

(ii) If $v_iv_k \notin E(G)$, then $d^H_{G'}(B_{ij}, B_{kp}) = \begin{cases} \frac{m_jm_p}{d_G(v_i,v_k)}, & if\ j \neq p, \\ \frac{m_j^2}{d_G(v_i,v_k)}, & if\ j = p. \end{cases}$

(iii) $d^H_{G'}(B_{ij}, B_{ip}) = \begin{cases} m_jm_p, & if\ j \neq p, \\ \frac{m_j(m_j-1)}{2}, & if\ j = p. \end{cases}$

Lemma 3. *Let G be a connected graph and let B_{ij} in $G' = G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$. Then the degree of a vertex $(v_i, u_j) \in B_{ij}$ in G' is $d_{G'}((v_i, u_j)) = d_G(v_i) + (n_0 - m_j) + d_G(v_i)(n_0 - m_j)$, where $n_0 = \sum\limits_{j=0}^{r-1} m_j$.*

Remark 1. *The sums $\sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} m_jm_p = 2q$, $\sum\limits_{j=0}^{r-1} m_j^2 = n_0^2 - 2q$, $\sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} m_j^2 m_p = n_0^3 - 2n_0q - \sum\limits_{j=0}^{r-1} m_j^3 = \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} m_jm_p^2$ and $\sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} m_j^3 m_p = n_0 \sum\limits_{j=0}^{r-1} m_j^3 - \sum\limits_{j=0}^{r-1} m_j^4 = \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} m_jm_p^3$, where $n_0 = \sum\limits_{j=0}^{r-1} m_j$ and q is the number of edges of $K_{m_0, m_1, \ldots, m_{r-1}}$.*

Theorem 4. *Let G be a connected graph with n vertices and m edges. Then $RDD_*(G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}) = RDD_*(G)\Big(2qn_0^2 + 4n_0q + n_0^2 + 4q^2 + \sum\limits_{j=0}^{r-1} m_j^3\Big) +$*

$\frac{M_1(G)}{4}\Big(6qn_0^2 + 20n_0q - 3n_0^4 + 2q - 5n_0^3 + n_0^2 - n_0 + 8q^2 + (6n_0 + 5)\sum\limits_{j=0}^{r-1} m_j^3 - 3\sum\limits_{j=0}^{r-1} m_j^4\Big) + \frac{m}{2}\Big(6qn_0^2 + 8n_0q - 3n_0^4 - n_0^3 + 16q^2 - 4q + (6n_0+1)\sum\limits_{j=0}^{r-1} m_j^3 - 3\sum\limits_{j=0}^{r-1} m_j^4\Big) + \frac{n}{4}\Big(2qn_0^2 - 4n_0q - n_0^4 + n_0^3 + 8q^2 + (2n_0 - 2)\sum\limits_{j=0}^{r-1} m_j^3 - \sum\limits_{j=0}^{r-1} m_j^4\Big) + 4q^2H(G) + RDD(G)\Big(2qn_0^2 + 4n_0q - n_0^3 + 4q^2 + \sum\limits_{j=0}^{r-1} m_j^3\Big) + \frac{M_2(G)}{2}\Big(2qn_0^2 + 8n_0q - n_0^4 + 4q - 3n_0^3 + n_0 - 2n_0\sum\limits_{j=0}^{r-1} m_j^3 - \sum\limits_{j=0}^{r-1} m_j^4\Big), r \geq 2.$

Proof. Let $G' = G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$. Clearly,

$$\begin{aligned}
RDD_*(G') &= \frac{1}{2}\sum_{B_{ij},\, B_{kp} \in \mathscr{B}} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{kp})\Big) d_{G'}(B_{ij}, B_{kp}) \\
&= \frac{1}{2}\Bigg(\sum_{i=0}^{n-1}\sum_{\substack{j,p=0\\ j\neq p}}^{r-1} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{ip})\Big) d_{G'}(B_{ij}, B_{ip}) \\
&\quad + \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{j=0}^{r-1} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{kj})\Big) d_{G'}(B_{ij}, B_{kj}) \\
&\quad + \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,p=0\\ j\neq p}}^{r-1} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{kp})\Big) d_{G'}(B_{ij}, B_{kp}) \\
&\quad + \sum_{i=0}^{n-1}\sum_{j=0}^{r-1} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{ij})\Big) d_{G'}(B_{ij}, B_{ij})\Bigg) \\
&= \frac{1}{2}\{A_1 + A_2 + A_3 + A_4\}, \qquad (1)
\end{aligned}$$

where A_1, A_2, A_3 and A_4 are the sums of the terms of the above expression, in order.
We shall obtain A_1 to A_4 of (1), separately. By Lemmas 2 and 3, we have

$$A_1 = \sum_{i=0}^{n-1}\sum_{\substack{j,p=0\\ j\neq p}}^{r-1} \Big(d_G(v_i) + (n_0 - m_j) + d_G(v_i)(n_0 - m_j)\Big)$$
$$\Big(d_G(v_i) + (n_0 - m_p) + d_G(v_i)(n_0 - m_p)\Big) m_j m_p$$

$$= \sum_{i=0}^{n-1} \sum_{\substack{j,\,p\,=\,0\\ j\,\neq\,p}}^{r-1} \Bigg\{ \Big((n_0^2 + 2n_0 + 1) - (n_0+1)m_p - (n_0+1)m_j + m_j m_p \Big)$$

$$+ d_G(v_i)\Big(2n_0^2 + 2n_0 - (2n_0+1)m_p - (2n_0+1)m_j + 2m_j m_p \Big)$$

$$+ \Big(n_0^2 - n_0 m_p - n_0 m_j + m_j m_p \Big) \Bigg\} m_j m_p$$

$$= \sum_{\substack{j,\,p\,=\,0\\ j\,\neq\,p}}^{r-1} \Bigg\{ M_1(G)\Big((n_0^2 + 2n_0 + 1)m_j m_p - 2(n_0+1)m_j m_p^2 - (n_0+1)m_j^2 m_p$$

$$+ \ m_j^2 m_p^2 \Big) + 2m\Big((2n_0^2 + 2n_0)m_j m_p - (2n_0+1)m_j m_p^2 - (2n_0+1)m_j^2 m_p$$

$$+ \ 2m_j^2 m_p^2 \Big) + n\Big(n_0^2 m_j m_p - n_0 m_j m_p^2 - n_0 m_j^2 m_p + m_j^2 m_p^2 \Big) \Bigg\}$$

$$= M_1(G)\Big((n_0^2 + 2n_0 + 1)(2q) - 2(n_0+1)(n_0^3 - 2qn_0 - \sum_{j=0}^{r-1} m_j^3) + (n_0^2 - 2q)^2$$

$$- \sum_{j=0}^{r-1} m_j^4 \Big) + 2m\Big(2n_0(n_0+1)(2q) - 2(2n_0+1)(n_0^3 - 2qn_0 - \sum_{j=0}^{r-1} m_j^3) + 2(n_0^2 - 2q)^2$$

$$- \ 2\sum_{j=0}^{r-1} m_j^4 \Big) + n\Big(2n_0^2 q - 2n_0(n_0^3 - 2qn_0 - \sum_{j=0}^{r-1} m_j^3) + (n_0^2 - 2q)^2 - \sum_{j=0}^{r-1} m_j^4 \Big),$$

by Remark 1 and the definition of first Zagreb index

$$= M_1(G)\Big(2n_0^2 q - n_0^4 + 8n_0 q + 2q - 2n_0^3 + 4q^2 + 2(n_0+1)\sum_{j=0}^{r-1} m_j^3 - \sum_{j=0}^{r-1} m_j^4 \Big)$$

$$+ 2m\Big(4qn_0^2 - 2n_0^4 + 8n_0 q - 2n_0^3 + 8q^2 + 2(2n_0+1)\sum_{j=0}^{r-1} m_j^3 - 2\sum_{j=0}^{r-1} m_j^4 \Big)$$

$$+ n\Big(2qn_0^2 - n_0^4 + 4q^2 + 2n_0 \sum_{j=0}^{r-1} m_j^3 - \sum_{j=0}^{r-1} m_j^4 \Big). \qquad (2)$$

Next we compute A_2.

$$A_2 = \sum_{j\,=\,0}^{r-1} \Bigg(\sum_{\substack{i,\,k\,=\,0\\ i\,\neq\,k\\ v_i v_k \in E(G)}}^{n-1} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{kj}) \Big) d_{G'}(B_{ij}, B_{kj})$$

$$+ \sum_{\substack{i,\,k\,=\,0\\ i\,\neq\,k\\ v_i v_k \notin E(G)}}^{n-1} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{kj}) \Big) d_{G'}(B_{ij}, B_{kj}) \Bigg)$$

$$
\begin{aligned}
&= \sum_{j=0}^{r-1} \sum_{\substack{i,k=0 \\ i \neq k \\ v_i v_k \in E(G)}}^{n-1} \Big(d_G(v_i) + (n_0 - m_j) + (n_0 - m_j) d_G(v_i)\Big) \\
&\quad \Big(d_G(v_k) + (n_0 - m_j) + (n_0 - m_j) d_G(v_k)\Big) \frac{m_j(m_j+1)}{2} \\
&\quad + \sum_{j=0}^{r-1} \sum_{\substack{i,k=0 \\ i \neq k \\ v_i v_k \notin E(G)}}^{n-1} \Big(d_G(v_i) + (n_0 - m_j) + (n_0 - m_j) d_G(v_i)\Big) \\
&\quad \Big(d_G(v_k) + (n_0 - m_j) + (n_0 - m_j) d_G(v_k)\Big) \frac{m_j^2}{d_G(v_i, v_k)}, \text{ by Lemmas 2 and 3} \\
&= \sum_{j=0}^{r-1} \sum_{\substack{i,k=0 \\ i \neq k \\ v_i v_k \in E(G)}}^{n-1} \Big(d_G(v_i) + (n_0 - m_j) + (n_0 - m_j) d_G(v_i)\Big) \\
&\quad \Big(d_G(v_k) + (n_0 - m_j) + (n_0 - m_j) d_G(v_k)\Big) \Big(\frac{m_j(m_j+1)}{2} + m_j^2 - m_j^2\Big) \\
&\quad + \sum_{j=0}^{r-1} \sum_{\substack{i,k=0 \\ i \neq k \\ v_i v_k \notin E(G)}}^{n-1} \Big(d_G(v_i) + (n_0 - m_j) + (n_0 - m_j) d_G(v_i)\Big) \\
&\quad \Big(d_G(v_k) + (n_0 - m_j) + (n_0 - m_j) d_G(v_k)\Big) \frac{m_j^2}{d_G(v_i, v_k)} \\
&= \sum_{j=0}^{r-1} \sum_{\substack{i,k=0 \\ i \neq k \\ v_i v_k \in E(G)}}^{n-1} \Big(d_G(v_i) + (n_0 - m_j) + (n_0 - m_j) d_G(v_i)\Big) \\
&\quad \Big(d_G(v_k) + (n_0 - m_j) + (n_0 - m_j) d_G(v_k)\Big) \frac{m_j - m_j^2}{2} \\
&\quad + \sum_{j=0}^{r-1} \sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \Big(d_G(v_i) + (n_0 - m_j) + (n_0 - m_j) d_G(v_i)\Big) \\
&\quad \Big(d_G(v_k) + (n_0 - m_j) + (n_0 - m_j) d_G(v_k)\Big) \frac{m_j^2}{d_G(v_i, v_k)} \\
&= S_1 + S_2, \qquad (3)
\end{aligned}
$$

where S_1 and S_2 are the sums of the terms of the above expression, in order. We shall obtain the sums S_1 and S_2 separately.

$$\begin{aligned}
S_1 &= \sum_{j=0}^{r-1} \sum_{\substack{i,\,k=0 \\ i\neq k \\ v_iv_k\in E(G)}}^{n-1} \Big(d_G(v_i)+(n_0-m_j)+(n_0-m_j)d_G(v_i)\Big) \\
&\quad \Big(d_G(v_k)+(n_0-m_j)+(n_0-m_j)d_G(v_k)\Big)\frac{m_j-m_j^2}{2} \\
&= \sum_{j=0}^{r-1} \sum_{\substack{i,\,k=0 \\ i\neq k \\ v_iv_k\in E(G)}}^{n-1} \Bigg(\Big(n_0^2+m_j^2-2n_0m_j\Big)+(d_G(v_i)+d_G(v_k))\Big(n_0^2+n_0-(2n_0+1)m_j \\
&\quad +m_j^2\Big)+d_G(v_i)d_G(v_k)\Big(n_0^2+n_0+1-(2n_0+1)m_j+m_j^2\Big)\Bigg)\frac{m_j-m_j^2}{2} \\
&= \frac{1}{2}\sum_{j=0}^{r-1} \sum_{\substack{i,\,k=0 \\ i\neq k \\ v_iv_k\in E(G)}}^{n-1} \Bigg(\Big(n_0^2m_j-(n_0^2+n_0)m_j^2+(2n_0+1)m_j^3-m_j^4\Big) \\
&\quad +(d_G(v_i)+d_G(v_k))\Big((n_0^2+n_0)m_j-(n_0^2+3n_0+1)m_j^2+(2n_0+2)m_j^3-m_j^4\Big) \\
&\quad +d_G(v_i)d_G(v_k)\Big((n_0^2+n_0+1)m_j-(n_0^2+4n_0+2)m_j^2-2n_0m_j^3\Big)\Bigg) \\
&= \frac{1}{2}\Bigg(2m\Big(2qn_0^2+4n_0q-n_0^3-n_0^4+(2n_0+1)\sum_{j=0}^{r-1}m_j^3-\sum_{j=0}^{r-1}m_j^4\Big) \\
&\quad +2M_1(G)\Big(2qn_0^2+6n_0q-2n_0^3-n_0^4+2q+(2n_0+2)\sum_{j=0}^{r-1}m_j^3-\sum_{j=0}^{r-1}m_j^4\Big) \\
&\quad +2M_2(G)\Big(2qn_0^2+8n_0q-3n_0^3-n_0^4+n_0+4q-2n_0\sum_{j=0}^{r-1}m_j^3-\sum_{j=0}^{r-1}m_j^4\Big)\Bigg), \qquad (4)
\end{aligned}$$

by Remark 1 and the definitions of first and second Zagreb index.

$$\begin{aligned}
S_2 &= \sum_{j=0}^{r-1} \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \Big(d_G(v_i)+(n_0-m_j)+(n_0-m_j)d_G(v_i)\Big) \\
&\quad \Big(d_G(v_k)+(n_0-m_j)+(n_0-m_j)d_G(v_k)\Big)\frac{m_j^2}{d_G(v_i,v_k)} \\
&= \sum_{j=0}^{r-1} \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \Bigg(\Big(n_0^2+m_j^2-2n_0m_j\Big)+(d_G(v_i)+d_G(v_k))\Big(n_0^2+n_0-(2n_0+1)m_j
\end{aligned}$$

$$+m_j^2\Big) + d_G(v_i)d_G(v_k)\Big(n_0^2 + n_0 + 1 - (2n_0+1)m_j + m_j^2\Big)\Big)\frac{m_j^2}{d_G(v_i,v_k)}$$

$$= \sum_{j=0}^{r-1}\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\Bigg(\Big(n_0^2m_j^2 - 2n_0m_j^3 + m_j^4\Big)\frac{1}{d_G(v_i,v_k)}$$

$$+\frac{(d_G(v_i)+d_G(v_k))}{d_G(v_i,v_k)}\Big((n_0^2+n_0)m_j^2 - (2n_0+1)m_j^3 + m_j^4\Big)$$

$$+\frac{d_G(v_i)d_G(v_k)}{d_G(v_i,v_k)}\Big((n_0^2+n_0+1)m_j^2 - (2n_0+1)m_j^3 + m_j^4\Big)\Bigg)$$

$$= \Bigg(2H(G)\Big(n_0^4 - 2n_0^2q - 2n_0\sum_{j=0}^{r-1}m_j^3 + \sum_{j=0}^{r-1}m_j^4\Big)$$

$$+2RDD(G)\Big(n_0^4 + n_0^3 - 2qn_0^2 - 2n_0q - (2n_0+1)\sum_{j=0}^{r-1}m_j^3 + \sum_{j=0}^{r-1}m_j^4\Big)$$

$$+2RDD_*(G)\Big(n_0^4 + 2n_0^3 - 2qn_0^2 - 4n_0q + n_0^2 - 2q - (2n_0+1)\sum_{j=0}^{r-1}m_j^3 + \sum_{j=0}^{r-1}m_j^4\Big)\Bigg) \quad (5)$$

by Remark 1 and the definitions of Harary index, reciprocal degree distance and reciprocal product degree distance.

Now we obtain A_3. By Lemmas 2 and 3, we have

$$A_3 = \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,p=0,\\ j\neq p}}^{r-1}\Big(d_{G'}(B_{ij}) + d_{G'}(B_{kp}\Big)d_{G'}(B_{ij},B_{kp})$$

$$= \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,p=0,\\ j\neq p}}^{r-1}\Big(d_G(v_i) + (n_0-m_j) + d_G(v_i)(n_0-m_j)\Big)$$

$$\Big(d_G(v_k) + (n_0-m_p) + d_G(v_k)(n_0-m_p)\Big)\frac{m_jm_p}{d_G(v_i,v_k)}$$

$$= \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,p=0,\\ j\neq p}}^{r-1}\Bigg(\frac{d_G(v_i)d_G(v_k)}{d_G(v_i,v_k)}\Big((n_0^2+n_0+1)m_jm_p - (n_0+1)m_jm_p^2$$

$$- \quad (n_0+1)m_j^2m_p + m_j^2m_p^2\Big) + \frac{d_G(v_i)+d_G(v_k)}{d_G(v_i,v_k)}\Big(n_0(n_0+1)m_jm_p - (n_0+1)m_jm_p^2$$

$$- \quad n_0m_j^2m_p + m_j^2m_p^2\Big) + \frac{1}{d_G(v_i,v_k)}\Big(n_0^2m_jm_p - n_0m_jm_p^2 - n_0m_j^2m_p + m_j^2m_p^2\Big)\Bigg)$$

$$= 2RDD_*(G)\Big(4n_0^2q+8qn_0+2q-2n_0^3-n_0^4+4q^2+2(n_0+1)\sum_{j=0}^{r-1}m_j^3-\sum_{j=0}^{r-1}m_j^4\Big)$$

$$+2RDD(G)\Big(4n_0^2q+6qn_0-2n_0^3-n_0^4+4q^2+2(n_0+1)\sum_{j=0}^{r-1}m_j^3-\sum_{j=0}^{r-1}m_j^4\Big)$$

$$+2H(G)\Big(2n_0^2q-n_0^4+4q^2+2n_0\sum_{j=0}^{r-1}m_j^3-\sum_{j=0}^{r-1}m_j^4\Big), \qquad (6)$$

by Remark 1.

Finally, we obtain A_4. By Lemmas 2 and 3, we have

$$A_4 = \sum_{i=0}^{n-1}\left(\sum_{j=0}^{r-1}\Big(d_{G'}(B_{ij})+d_{G'}(B_{ij})\Big)d_{G'}(B_{ij},B_{ij})\right)$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{r-1}\Big(d_G(v_i)+(n_0-m_j)+d_G(v_i)(n_0-m_j)\Big)^2\ \frac{m_j(m_j-1)}{2},$$

by Lemmas 2 and 3

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{r-1}\left(\Big(n_0^2+m_j^2-2n_0m_j\Big)+\Big(n_0^2+2n_0+1+m_j^2-2(n_0+1)m_j\Big)(d_G(v_i))^2\right.$$

$$\left.+2\Big(n_0^2+n_0-(2n_0+1)m_j+m_j^2\Big)d_G(v_i)\right)\frac{m_j(m_j-1)}{2}$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{r-1}\left(\Big((n_0^2+n_0)m_j^2-n_0^2m_j-(2n_0+1)m_j^3+m_j^4\Big)\right.$$

$$+(d_G(v_i))^2\Big((n_0^2+4n_0+3)m_j^2-(n_0^2+2n_0+1)m_j-(2n_0+3)m_j^3+m_j^4\Big)$$

$$\left.+2d_G(v_i)\Big((n_0^2+3n_0+1)m_j^2-(n_0^2+n_0)m_j-(2n_0+2)m_j^3+(2n_0+1)m_j^4\Big)\right)$$

$$= \frac{n}{2}\Big(n_0^4-2qn_0^2+n_0^3-4qn_0-2(n_0+1)\sum_{j=0}^{r-1}m_j^3+\sum_{j=0}^{r-1}m_j^4\Big)$$

$$+\frac{M_1(G)}{2}\Big(n_0^4-2qn_0^2+3n_0^3-8qn_0-6q-n_0-(2n_0+3)\sum_{j=0}^{r-1}m_j^3+\sum_{j=0}^{r-1}m_j^4\Big)$$

$$+2m\Big(n_0^4-2qn_0^2+2n_0^3-6qn_0-2q-2(n_0+1)\sum_{j=0}^{r-1}m_j^3+\sum_{j=0}^{r-1}m_j^4\Big),$$

by Remark 1 (7)

Using (2), (4), (5), (6) and (7) in (1), we have

$$RDD_*(G') = RDD_*(G)\Big(2qn_0^2 + 4n_0q + n_0^2 + 4q^2 + \sum_{j=0}^{r-1} m_j^3\Big)$$
$$+\frac{M_1(G)}{4}\Big(6qn_0^2 + 20n_0q - 3n_0^4 + 2q - 5n_0^3 + n_0^2 - n_0 + 8q^2 + (6n_0+5)\sum_{j=0}^{r-1} m_j^3$$
$$-3\sum_{j=0}^{r-1} m_j^4\Big) + \frac{m}{2}\Big(6qn_0^2 + 8n_0q - 3n_0^4 - n_0^3 + 16q^2 - 4q + (6n_0+1)\sum_{j=0}^{r-1} m_j^3$$
$$-3\sum_{j=0}^{r-1} m_j^4\Big) + \frac{n}{4}\Big(2qn_0^2 - 4n_0q - n_0^4 + n_0^3 + 8q^2 + (2n_0-2)\sum_{j=0}^{r-1} m_j^3 - \sum_{j=0}^{r-1} m_j^4\Big)$$
$$+4q^2H(G) + RDD(G)\Big(2qn_0^2 + 4n_0q - n_0^3 + 4q^2 + \sum_{j=0}^{r-1} m_j^3\Big)$$
$$+\frac{M_2(G)}{2}\Big(2qn_0^2 + 8n_0q - n_0^4 + 4q - 3n_0^3 + n_0 - 2n_0\sum_{j=0}^{r-1} m_j^3 - \sum_{j=0}^{r-1} m_j^4\Big).$$

If $m_i = s,\ 0 \le i \le r-1$, in above theorem, we have the following corollary.

Corollary 5. *Let G be a connected graph with n vertices and m edges. Then* $RDD_*(G \boxtimes K_{r(s)}) = RDD_*(G \boxtimes K_{r(s)}) = RDD_*(G)\Big(r^4s^4 - r^3s^4 + 2r^3s^3 - 2r^2s^3 + r^2s^2 + rs^3 + r^2s^4(r-1)^2\Big) + \frac{M_1(G)}{4}\Big(5r^2s^2 - 3r^2s^3 - 10rs^2 + 6rs - 5s - 1 + 6rs^3 + 5s^2 - 3s^3\Big) + \frac{m}{2}\Big(3r^3s^3 - 3r^3s^4 - 4r^2s^3 - 2r^2s^2 + 2rs^2 + 4r62s^4(r-1)^2 + (6rs+1)rs^3 - 3rs^4\Big) + \frac{n}{4}\Big(2r^2s^4(r-1)^2 - r^3s^4 - r^3s^3 - 2r^2s^3 + 2r^2s^4 - 2rs^3 - rs^4\Big) + \Big(r^2s^4(r-1)^2\Big)H(G) + RDD(G)\Big(r^4s^4 - r^3s^4 + r^3s^3 - 2r^2s^3 + r^2(r-1)^2 + rs^3\Big) + \frac{M_2(G)}{2}\Big(r^3s^3 + rs - r^3s^4 - 4r^2s^3 + 2r^2s^2 - 2rs^2 - 2r^2s^4 - rs^4\Big),\ r \ge 2.$

As $K_r = K_{r(1)}$, the above corollary gives the following corollary which is the one of the main theorem in [15].

Corollary 6. [15] *Let G be a connected graph with n vertices and m edges. Then* $RDD_*(G \boxtimes K_r) = r^2\Big(r^2RDD_*(G) + r(r-1)RDD(G) + (r-1)^2H(G)\Big) + r(r-1)\Big(\frac{r^2M_1(G)}{2} + \frac{n(r-1)^2}{2} + 2mr(r-1)\Big),\ r \ge 2.$

Using above corollary, we obtain the following corollary.

Corollary 7. $RDD_*(C_n \boxtimes K_r) = (9r^2 - 6r + 1)\Big(r^2H(C_n) + \frac{nr(r-1)}{2}\Big).$

As an application we present formula for reciprocal product degree distance of closed fence graph, $C_n \boxtimes K_2$.

Example 1. *By Corolarry 7, we have*

$$RDD_*(C_n \boxtimes K_2) = \begin{cases} 25\Big(n + n\sum\limits_{i=1}^{\frac{n}{2}} \frac{1}{i} - 1\Big), & \text{if } n \text{ is even} \\ 25n\Big(1 + \sum\limits_{i=1}^{\frac{n-1}{2}} \frac{1}{i}\Big), & \text{if } n \text{ is odd.} \end{cases}$$

References

[1] Y. Alizadeh, A. Iranmanesh, T. Doslic, *Additively weighted Harary index of some composite graphs*, Discrete Math. 313 (2013) 26-34.

[2] A.A. Dobrynin, A.A. Kochetova,*Degree distance of a graph: a degree analogue of the Wiener index*, J. Chem. Inf. Comput. Sci. 34 (1994) 1082-1086.

[3] S. Chen , W. Liu ,*Extremal modified Schultz index of bicyclic graphs* MATCH Commun Math Comput Chem 64(2010)767-782.

[4] J. Devillers, A.T. Balaban, Eds.,*Topological indices and related descriptors in QSAR and QSPR*, Gordon and Breach, Amsterdam, The Netherlands, 1999.

[5] I. Gutman, *Selected properties of the Schultz molecular topological index*, J. Chem. Inf. Comput. Sci. 34 (1994) 1087-1089.

[6] I. Gutman, O.E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer-Verlag, Berlin, 1986.

[7] A. Hamzeh, A. Iranmanesh, S. Hossein-Zadeh , M.V. Diudea,*Generalized degree distance of trees, unicyclic and bicyclic graphs*, Studia Ubb Chemia, LVII, 4(2012) 73-85.

[8] B. Horoldagva , I. Gutman , *On some vertex-degree- based graph invariants*. MATCH Commun Math Comput Chem 65(2011)723-730.

[9] H. Hua, S. Zhang,*On the reciprocal degree distance of graphs*, Discrete Appl. Math. 160 (2012) 1152-1163.

[10] O. Ivanciuc, T.S. Balaban,*Reciprocal distance matrix, related local vertex invariants and topological indices.* J. Math. Chem.12(1993)309-318.

[11] O. Khormali , A. Iranmanesh , I. Gutman , A. Ahamedi ,*Generalized Schultz index and its edge versions*. MATCH Commun Math Comput Chem 64(2010)783-798.

[12] S.C. Li, X. Meng, *Four edge-grafting theorems on the reciprocal degree distance of graphs and their applications*, J. Comb. Optim. (2013) DOI 10.1007/s 10878-013-9649-1.

[13] K. Pattabiraman, M. Vijayaragavan,*Reciprocal degree distance of some graph operations, Transactions on Combinatorics*, 2(2013) 13-24.

[14] K. Pattabiraman, M. Vijayaragavan, *Reciprocal degree distance of product graphs*, Accepted in Discrete Appl. Math. (2014).

[15] K. Pattabiraman,*Multiplicatively Harary indices of some graph operations*, Accepted to publication in J. Appl. Math. Inf. (2014).

[16] G. F. Su, L.M. Xiong, X.F. Su, X.L. Chen, *Some results on the reciprocal sum-degree distance of graphs*, J. Comb. Optim (2013)DOI 10.1007/s 10878-013-9645-5.

[17] G. Su, I. Gutman, L. Xiong, L. Xv,*Reciprocal product degree distance of graphs*, Manuscript.

[18] I. Tomescu,*Some extremal properties of the degree distance of a graph.* Discrete Appl Math 98(1999)159-163.

[19] I. Tomescu, *Properties of connected graphs having minimum degree distance*. Discrete Math 309(2008)2745-2748.