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A NINTH-ORDER ITERATIVE METHOD FOR NONLINEAR EQUATIONS ALONG WITH POLYNOMIOGRAPHY

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Abstract. In this paper, we suggest a new ninth order predictor-corrector iterative method to solve nonlinear equations. It is also shown that this new iterative method has convergence of order nine and has efficiency index 1.7321. Moreover, some examples are given to check its validity and efficiency. Finally, we present polynomiographs for some complex polynomials via our new method.

Key words: nonlinear equation, iterative method, polynomiography.
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1. Introduction

The boundary value problems in kinetic theory of gases, elasticity, and other applied areas are mostly reduced in solving single variable nonlinear equations. Hence, the problem of approximating a solution of a nonlinear equation is important. The numerical methods to find solutions of such equations are called iterative methods [31]. Many such iterative methods for solving nonlinear equations are described in literature; see for detail [31, 29, 10, 23, 1, 32, 11, 26, 27, 5, 6, 7, 8, 9, 12, 21, 22, 2, 3, 16, 17, 24, 25]. There are two types of iterative methods, the methods that involve derivatives [29] and the methods that do not involve derivatives [10, 23, 1, 32, 11, 26, 27, 5, 6, 7, 8, 9, 12, 21, 22,

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2, 3, 16, 17, 24]. Presently, we are interested in finding higher order iterative method that involve derivatives.

In this paper, we suggest new predictor-corrector iterative method for solving nonlinear equations. It is shown that suggested method has convergence of order nine and efficiency index 1.7321.

The breakup of the paper is as follows: In the second section, we suggested a predictor-corrector iterative method. In third section, we proved that convergence order of this method is at least nine. In fourth section, we compared the efficiency index of this method with some other existing iterative methods. In fifth section, some numerical examples are solved to check the convergence speed of the presented method. In the sixth section, the polynomiography is presented via double Abbasbandy's method, and in the last section we make some conclusions.

2. New Iterative Method

Consider a nonlinear algebraic equation of the form

$$f(x) = 0. \tag{1}$$

We assume that α is a simple zero of Eq. (1), and γ is an initial guess, sufficiently close to α. Using the Taylors series, we have

$$f(\gamma) + (x-\gamma)f'\gamma) + \frac{1}{2!}(x-\gamma)^2 f''(\gamma) + ... = 0. \tag{2}$$

If $f'(\gamma) \neq 0$, we can evaluate (2) as $f(\gamma) + (x-\gamma)f'(\gamma) = 0$.

We now present our method following several steps:

Step I. For a given x_0, compute the approximate solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This is well known Newton's method (NM) for root-finding of nonlinear functions, which converges quadratically [31, 7]. Also from (2), we obtain

$$x = \gamma - \frac{2f(\gamma)f'(\gamma)}{2f'^2(\gamma) - f(\gamma)f''(\gamma)}$$

This formulation allows us to suggest the following iterative method for solving nonlinear equation (1).

Step II. For a given x_0, compute the approximate solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}$$

This is Halley's Method, which has cubic convergence [31, 10, 23, 5, 7].

Step III. For a given x_0, compute the approximate solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)f''(x_n)}{2f'^3(x_n)}$$

This is so-called Households method, which has convergence of order three [31, 7].

Abbasbandy [1] improved the Newton-Raphson method by modified Adomian decomposition method, and developed the following third order iterative method

Step IV. For a given x_0, compute the approximate solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)f''(x_n)}{2f'^3(x_n)} - \frac{f^3(x_n)f'''(x_n)}{6f'^4(x_n)}.$$

This is so-called Abbasbandy method for root-finding of nonlinear functions. Noor and Noor [28] suggested the following two-step method

Step V.
For a given x_0, compute the approximate solution x_{n+1} by the iterative scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = y_n - \frac{2f(y_n)f'(y_n)}{2f'^2(y_n) - f(y_n)f''(y_n)}$$

We suggest the following two-step method, using Househölder method (Step I), as predictor, and Abbasbandy method (Step II) as a corrector

Step VI.
For a given x_0, compute the approximate solution x_{n+1} by the following iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)f''(x_n)}{2f'^3(x_n)} \tag{3}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)} - \frac{f^2(y_n)f''(y_n)}{2f'^3(y_n)} - \frac{f^3(y_n)f'''(y_n)}{6f'^4(y_n)}. \tag{4}$$

We call it predictor-corrector iterative method (PCIM).

3. Convergence Analysis

In the following theorem, we will find convergence order of predictor-corrector iterative method (PCIM)

Theorem 1. *Suppose α is a root of a nonlinear equation $f(x) = 0$. If $f(x)$ is sufficiently smooth in the neighborhood of α, then the predictor-corrector iterative method (PCIM) has 9th order of convergence.*

Proof. Suppose that α is a root of the equation $f(x) = 0$, and e_n is the error at nth iteration. Then $e_n = x_n - \alpha$ then by using Taylor series expansion, we have
$f(x_n) = f'(x_n)e_n + \frac{1}{2!}f''(x_n)e_n^2 + \frac{1}{3!}f'''(x_n)e_n^3 + \frac{1}{4!}f^{(iv)}(x_n)e_n^4 + \frac{1}{5!}f^{(v)}(x_n)e_n^5 + \frac{1}{6!}f^{(vi)}(x_n)e_n^6 + O(e_n^7)$

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + O(e_n^8)] \quad (5)$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + O(e_n^7)] \quad (6)$$

$$\begin{aligned} f''(x_n) &= f'^2(\alpha)[2c_2 + 6c_3e + 12c_4e_n^2 + 20c_5e_n^3 + 30c_6e_n^4 + 42c_7e_n^5 + 56c_8e_n^6 \\ &\quad +72c_9e_n^7 + O(e_n^8)] \end{aligned} \quad (7)$$

Here

$$c_n = \frac{1}{n!}\frac{f^{(n)}(\alpha)}{f'(\alpha)}.$$

Using 5, 6, and 7 in 3, we have

$$\begin{aligned} y_n &= f'(\alpha)[\alpha + (2c_2^2 - c_3)e_n^3 + (12c_2c_3 - 9c_2^3 - 3c_4)e_n^4 + (-63c_3c_2^2 + 30c_2^4 \\ &+ 24c_2c_4 + 15c_3^2 - 6c_5)e_n^5 + (-10c_6 + 40c_2c_5 + 55c_4c_3 - 112c_4c_2^2 \\ &- 136c_2c_3^2 + 251c_3c_2^3 - 88c_2^5)e_n^6 + (-15c_7 + 60c_2c_6 + 87c_3c_5 + 753c_3^2c_2^2 \\ &- 864c_3c_2^4 + 48c_4^2 + 420c_4c_2^3 - 174c_5c_2^2 - 462c_2c_3c_4 - 93c_3^3 + 240c_2^6)e_n^7 \\ &+ (-21c_8 + 84c_2c_7 + 147c_4c_5 - 381c_2c_4^2 - 459c_4c_3^2 - 1392c_4c_2^4 + 627c_5c_2^3 \\ &+ 126c_6c_3 - 249c_6c_2^2 + 972c_2c_3^3 - 3294c_3^2c_2^3 + 2712c_3c_2^5 + 244c_4c_3c_2^2 \\ &- 696c_2c_5c_3 - 624c_2^7)e_n^8 + (-28c_9 + 112c_2c_8 + 110c_5^2 - 676c_5c_3^2 - 2024c_5c_2^4 \\ &- +172c_7c_3 - 337c_7c_2^2 + 208c_6c_4 + 872c_6c_2^3 - 739c_4^2c_3 + 1950c_4^2c_2^2 + 4256c_4c_2^5 \\ &- 6138c_3^3c_2^2 + 12500c_3^2c_2^4 - 7984c_3c_2^6 - 1124c_2c_5c_4 + 3579c_3c_5c_2^2 - 974c_2c_6c_3 \\ &+ 4638c_2c_4c_3^2 - 10400c_4c_3c_2^3 + 459c_3^4 + 1568c_2^8)e_n^9 + O(e_n^{10})] \end{aligned} \quad (8)$$

$$\begin{aligned}
f(y_n) &= f'(\alpha)[(2c_2^2 - c_3)e_n^3 + (12c_2c_3 - 9c_2^3 - 3c_4)e_n^4 + (-63c_3c_2^2 + 30c_2^4 \\
&+ 24c_2c_4 + 15c_3^2 - 6c_5)e_n^5 + (-10c_6 + 40c_2c_5 + 55c_4c_3 - 112c_4c_2^2 \\
&- 135c_2c_3^2 + 247c_3c_2^3 - 84c_2^5)e_n^6 + (-798c_3c_2^4 + 204c_2^6 + 408c_4c_2^3 \\
&+ 729c_3^2c_2^2 - 456c_2c_3c_4 - 15c_7 + 60c_2c_6 + 87c_3c_5 + 48c_4^2 - 174c_5c_2^2 \\
&- 93c_3^3)e_n^7 + (2184c_3c_2^5 - 423c_2^7 - 1242c_4c_2^4 - 2964c_3^2c_2^3 + 603c_5c_2^3 \\
&+ 2328c_4c_3c_2^2 + 942c_2c_3^3 - 684c_2c_5c_3 - 372c_2c_4^2 - 21c_8 + 84c_2c_7 \\
&+ 147c_4c_5 - 459c_4c_3^2 + 126c_6c_3 - 249c_6c_2^2)e_n^8 + (-28c_9 + 112c_2c_8 \\
&+ 110c_5^2 - 676c_5c_3^2 - 1756c_5c_2^4 + 172c_7c_3 - 337c_7c_2^2 + 208c_6c_4 \\
&+ 832c_6c_2^3 - 739c_4^2c_3 + 1806c - 4^2c_2^2 + 3196c_4c_2^5 - 5500c_3^3c_2^2 \\
&+ 9660c_3^2c_2^4 - 4942c_3c_2^6 - 1088c_2c_5c_4 + 3355c_3c_5c_2^2 - 954c_2c_6c_3 \\
&+ 4438c_2c_4c_3^2 - 9002c_4c_3c_2^3 + 458c_3^4 + 676c_2^8)e_n^9 + O(e_n^{10})] \qquad (9)
\end{aligned}$$

$$\begin{aligned}
f'(y_n) &= f'(\alpha)[1 + (4c_2^3 - 2c_2c_3)e_n^3 + (24c_3c_2^2 - 18c_2^4 - 6c_2c_4)e_n^4 \\
&- (126c_3c_2^3 + 60c_2^5 + 48c_4c_2^2 + 30c_2c_3^2 - 12c_2c_5)e_n^5 + (-20c_2c_6 \\
&+ 80c_5c_2^2 + 110c_2c_3c_4 - 224c_4c_2^3 - 284c_3^2c_2^2 + 514c_3c_2^4 - 176c_2^6 \\
&+ 3c_3^3)e_n^6 + (1704c_3^2c_2^3 - 1836c_3c_2^5 - 960c_4c_3c_2^2 - 258c_2c_3^3 \\
&+ 18c_4c_3^2 - 30c_2c_7 + 120c_6c_2^2 + 174c_2c_5c_3 + 96c_2c_4^2 + 840c_4c_2^4 \\
&- 348c_5c_2^3 + 480c_2^7)e_n^7 + (-8172c_3^2c_2^4 + 6027c_3c_2^6 + 5346c_4c_3c_2^3 \\
&+ 2934c_3^3c_2^2 - 1464c_5c_3c_2^2 - 1278c_2c_4c_3^2 - 90c_3^4 + 36c_5c_3^2 \\
&+ 27c_4^2c_3 - 42c_2c_8 + 168c_7c_2^2 + 294c_2c_5c_4 - 762c_4^2c_2^2 \\
&- 2784c_4c_2^5 + 1254c_5c_2^4 + 252c_2c_6c_3 - 498c_6c_2^3 - 1248c_2^8)e_n^8 \\
&+ (3136c_2^9 + 2814c_2c_3^4 - 56c_2c_9 + 224c_2^2c_8 + 220c_2c_5^2 \\
&- 4048c_5c_2^5 - 674c_7c_2^3 + 1744c_6c_2^4 + 3900c_4^2c_2^3 + 8544c_4c_2^6 \\
&+ 344c_2c_7c_3 - 2248c_5c_2^2c_4 + 416c_2c_6c_4 - 1910c_2c_3c_4^2 \\
&- 2068c_3c_6c_2^2 + 7962c_5c_3c_2^3 - 2024c_2c_5c_3^2 - 24028c_4c_3c_2^4 \\
&+ 108c_4c_3c_5 + 13494c_4c_3^2c_2^2 - 20760c_3^3c_2^3 + 34102c_3^2c_2^5 \\
&- 604c_4c_3^3 - 18644c_3c_2^7 + 60c_6c_3^2)e_n^9 + O(e_n^{10})] \qquad (10)
\end{aligned}$$

$$\begin{aligned}
f''(y_n) &= f'^2(\alpha)[2c_2 + (12c_3c_2^2 - 6c_3^2)e_n^3 + (72c_2c_3^2 - 54c_3c_2^3 - 18c_4c_3)e_n^4 \\
&- (378c_3^2c_2^2 + 180c_3c_2^4 + 144c_2c_3c_4 + 90c_3^3 - 36c_5c_3)e_n^5 + (-60c_3c_6 \\
&+ 240c_2c_5c_3 + 342c_4c_3^2 - 720c_4c_3c_2^2 - 816c_2c_3^3 + 1506c_3^2c_2^3 \\
&- 528c_3c_2^5 + 48c_4c_2^4)e_n^6 + (3312c_4c_3c_2^3 - 432c_4c_2^5 - 144c_4^2c_2^2 \\
&- 3060c_2c_4c_3^2 + 360c_4^2c_3 - 90c_7c_3 + 360c_2c_6c_3 + 522c_5c_3^2 + 4518c_3^3c_2^2
\end{aligned}$$

$$
\begin{aligned}
&- 5184c_3^2c_2^4 - 1044c_5c_3c_2^2 - 558c_3^4 + 1440c_3c_2^6)e_n^7 + (-14688c_4c_3c_2^4 \\
&+ +2412c_4c_2^6 + 1800c_4^2c_2^3 + 18648c_4c_3^2c_2^2 - 288c_5c_2^2c_4 - 3726c_2c_3c_4^2 \\
&- 3114c_4c_3^3 + 1026c_4c_3c_5 + 108c_4^3 - 126c_3c_8 + 504c_2c_7c_3 + 3762c_5c_3c_2^3 \\
&+ 756c_6c_3^2 - 1494c_3c_6c_2^2 + 5832c_2c_3^4 - 19764c_3^3c_2^3 + 16272c_3^2c_2^5 \\
&- 4176c_2c_5c_3^2 - 3744c_3c_2^7)e_n^8 + (-9432c_2c_5c_4c_3 + 21594c_5c_3^2c_2^2 \\
&+ 35412c_2c_4c_3^3 - 96336c_4c_3^2c_2^3 + 5232c_3c_6c_2^3 + 61944c_4c_3c_2^5 \\
&+ 3216c_4c_5c_2^3 + 1488c_4c_6c_3 - 2022c_7c_3c_2^2 + 672c_2c_3c_8 \\
&+ 28476c_4^2c_3c_2^2 - 480c_4c_6c_2^2 - 5844c_2c_6c_3^2 - 12384c_5c_3c_2^4 \\
&+ 2754c_3^5 - 12720c_4^2c_2^4 - 10704c_4c_2^7 - 6834c_4^2c_3^2 + 432c_4^2c_5 \\
&- 168c_3c_9 + 660c_3c_5^2 - 4076c_5c_3^3 + 1032c_7c_3^2 - 36828c_3^4c_2^2 + 75000c_3^3c_2^4 \\
&- 47904c_3^2c_2^6 + 9408c_3c_2^8 + 160c_5c_2^6 - 1728c_2c_4^3)e_n^9 + O(e_n^{10})] \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
f'''(y_n) &= f'^3(\alpha)[6c_3 + (48c_4c_2^2 - 24c_4c_3)e_n^3 + (288c_3c_2c_4 - 216c_4c_2^3 - 72c_4^2)e_n^4 \\
&+ (-1512c_4c_3c_2^2 + 720c_4c_2^4 + 576c_2c_4^2 + 360c_4c_3^2 - 144c_4c_5)e_n^5 \\
&+ (-240c_6c_4 + 960c_5c_2c_4 + 1320c_4^2c_3 - 2688c_4^2c_2^2 - 3264c_4c_2c_3^2 \\
&+ 6024c_4c_3c_2^3 - 2112c_4c_2^5 + 240c_5c_2^4 - 240c_5c_3c_2^2 + 60c_5c_3^2)e_n^6 \\
&+ (3960c_5c_3c_2^3 - 2160c_5c_2^5 - 4896c_5c_2^2c_4 - 1440c_5c_2c_3^2 + 2448c_4c_3c_5 \\
&- 360c_4c_7 + 1440c_2c_6c_4 + 18072c_4c_3^2c_2^2 - 20736c_4c_3c_2^4 + 1152c_4^3 \\
&+ 10080c_4^2c_2^3 - 11088c_3c_2c_4^2 - 2232c_4c_3^3 + 5760c_4c_2^6)e_n^7 + (-31680c_5c_3c_2^4 \\
&+ 12060c_5c_2^6 + 24048c_4c_5c_2^3 + 19800c_5c_3^2c_2^2 - 1440c_5^2c_2^2 - 23904c_5c_2c_4c_3 \\
&- 1800c_5c_3^3 + 720c_3c_5^2 + 4068c_4^2c_5 - 504c_4c_8 + 2016c_4c_2c_7 - 9144c_2c_4^3 \\
&- 11016c_4^2c_3^2 - 33408c_4^2c_2^4 + 3024c_6c_4c_3 - 5976c_6c_4c_2^2 + 23328c_4c_2c_3^3 \\
&- 79056c_4c_3^2c_2^3 + 65088c_4c_3c_2^5 + 58752c_4^2c_3c_2^2 - 14976c_4c_2^7)e_n^8 \\
&+ (16080c_5^2c_2^3 - 53520c_5c_2^7 + 4800c_4c_5^2 - 672c_4c_9 + 4992c_6c_4^2 \\
&- 17736c_4^3c_3 + 46800c_4^3c_2^2 + 102144c_4^2c_2^5 + 11016c_4c_3^4 + 37632c_4c_2^8 \\
&+ 960c_6c_2^6 - 120c_6c_3^3 - 23376c_2c_6c_4c_3 + 169776c_4c_3c_5c_2^2 - 2400c_5c_6c_2^2 \\
&- 112176c_4c_5c_2^4 - 191616c_4c_3c_2^6 - 249600c_4^2c_3c_2^3 - 1440c_6c_3c_2^4 \\
&- 147312c_4c_3^3c_2^2 - 35616c_5c_2c_4^2 + 720c_6c_3^2c_2^2 + 37920c_5c_2c_3^3 \\
&+ 300000c_4c_3^2c_2^4 + 1200c_5c_6c_3 + 20928c_6c_4c_2^3 + 111312c_2c_4^2c_3^2 \\
&- 169680c_5c_3^2c_2^3 + 182040c_5c_3c_2^5 - 28224c_4c_5c_3^2 + 2688c_4c_2c_8 \\
&- 13440c_5^2c_2c_3 + 4128c_4c_7c_3 - 8088c_4c_7c_2^2)e_n^9 + O(e_n^{10})] \qquad (12)
\end{aligned}
$$

Using $8, 9, 10, 11, 12$ in 4, we get

$$x_{n+1} = \alpha + (-14c_3^3c_2^2 + 36c_3^2c_2^4 - 40c_3c_2^6 + 2c_3^4 + 16c_2^8)e_n^9 + O(e_n^{10}).$$

This implies that

$$e_{n+1} = (-14c_3^3c_2^2 + 36c_3^2c_2^4 - 40c_3c_2^6 + 2c_3^4 + 16c_2^8)e_n^9 + O(e_n^{10}),$$

which shows that the predictor-corrector iterative method (PCIM) has ninth order convergence. □

4. Comparison of Efficiency Indices

The term efficiency index is used to compare the performance of different iterative methods. It depends upon the order of convergence and number of functional evaluations of the iterative method. If r denote the order of convergence and N_f denote the number of functional evaluations of an iterative method, then the efficiency index EI is defined as

$$EI = r^{\frac{1}{N_f}}.$$

On this basis, the Newton's method [31, 7] has an efficiency of $2^{\frac{1}{2}} \approx 1.4142$. House-Hölder method [31, 7] has order of convergence three and the number of functional evaluations required for this method is three, so its efficiency is $3^{\frac{1}{3}} \approx 1.4422$. The Abbasbandy method [1] has order of convergence three and number of functional evaluation required is four, so its efficiencies is $3^{\frac{1}{4}} \approx 1.3161$. Kuo [21] has developed several method and one each requires two function evaluations and two derivative evaluations. These methods achieve order of convergence six, so having efficiencies $6^{\frac{1}{4}} \approx 1.5651$.

Now we move to calculate the efficiency index of our predictor-corrector iterative method: The PCIM needs one evaluation of the function and three of its first, second, and third derivatives. So, the number of functional evaluations of this method is four. i.e., $N_f = 4$. Also, in the earlier section, we have proved that the order of convergence of PCIM is nine, i.e., $r = 9$. Thus, the efficiency index of this method is $E.I = 9^{\frac{1}{4}} \approx 1.7321$.

The efficiencies of the methods we have discussed are summarized in Table 1, and can see that the efficiency of the PCIM is higher than the efficiencies of other methods.

Table 1. Comparison of efficiencies of various methods

Method	Number of function or derivative evaluations	Efficiency index
Newton, quadratic	2	$2^{\frac{1}{2}} \approx 1.4142$
House-Hölder 3rd order	3	$3^{\frac{1}{3}} \approx 1.4422$
AM's 3rd order	4	$3^{1/4} \approx 1.3161$
Kou's 6th order	4	$6^{\frac{1}{4}} \approx 1.5651$
(PCIM) 9th order	4	$9^{\frac{1}{4}} \approx 1.7321$

It can be seen from the above comparison table that the efficiency of the developed iterative method is much higher as compare to other iterative methods.

5. Numerical Examples

We now present some examples to illustrate the efficiency of PCIM. We compare the Newton's method (NM), the Halley's method (HM), the Househölder's method (HHM), the Abbasbandy's method (AM), Noor and Noor's method (NNM), and predictor-corrector iterative method (PCIM). We used $\varepsilon = 10^{-15}$. The following stopping criteria is used for computer program:

(1) $|x_{n+1} - x_n| < \varepsilon$.
(2) $|f(x_{n+1})| < \varepsilon$.

The functions considered in the following tables are respectively $f_1 = x^3 - e^x - 3x + 3$, $f_2 = x^4 - 4x^3 + x^2 + 10$, $f_3 = x^3 - 2$, $f_4 = sinx - 10x + 1$, $f_5 = cos(x) - 2x + 5$, and $f_6 = x^3 - 4x^2 + x - 10$.

Table 2. Comparison of NM, HM, HHM, AM, NNM and PCIM

Method	N	N_f	$\|f(x_{n+1})\|$	x_{n+1}
$f_1, x_0 = 0.5$				
NM	3	6	$3.115876e-23$	
HM	2	6	$9.537532e-23$	0.493921988169693004893251498668
HHM	2	6	$1.600038e-22$	
AM	2	8	$1.536254e-21$	
NNM	2	6	$9.537532e-23$	
PCIM	1	4	$2.822358e-22$	

Table 3. Comparison of NM, HM, HHM, AM, NNM and PCIM

Method	N	N_f	$\|f(x_{n+1})\|$	x_{n+1}
$f_2, x_0 = 1.6$				
NM	4	8	$1.218133e-23$	
HM	3	9	$5.598413e-24$	1.834167902560702964341153927900
HHM	3	9	$4.791293e-23$	
AM	3	12	$9.634615e-20$	
NNM	3	9	$5.598413e-24$	
PCIM	2	8	$1.867061e-67$	

Table 4. Comparison of NM, HM, HHM, AM, NNM and PCIM

Method	N	N_f	$\|f(x_{n+1})\|$	x_{n+1}
$f_3, x_0 = 0.8$				
NM	6	12	$1.865087e-21$	
HM	4	12	$1.828577e-36$	1.2599210498948731647672106072780
HHM	6	18	$2.516988e-25$	
AM	4	16	$1.018848e-18$	
NNM	4	12	$1.828577e-36$	
PCIM	3	12	$2.824425e-59$	

Table 5. Comparison of NM, HM, HHM, AM, NNM and PCIM

Method	N	N_f	$\|f(x_{n+1})\|$	x_{n+1}
$f_4, x_0 = 0.9$				
NM	4	8	$9.417757e-29$	
HM	3	9	$1.262482e-26$	0.111085741533827042910504718797
HHM	3	9	$3.437559e-27$	
AM	3	12	$2.234972e-23$	
NNM	4	12	$1.262482e-26$	
PCIM	2	8	$9.585298e-81$	

Table 6. Comparison of NM, HM, HHM, AM, NNM and PCIM

Method	N	N_f	$\|f(x_{n+1})\|$	x_{n+1}
$f_5, x_0 = 0$				
NM	6	12	$1.760968e-29$	
HM	4	12	$7.099233e-25$	2.204096081050027306553912292370
HHM	5	15	$9.612143e-34$	
AM	5	20	$5.917080e-24$	
NNM	4	12	$7.099233e-25$	
PCIM	3	12	$2.760185e-80$	
$x_0 = 0.6$				
NM	5	10	$1.973932e-25$	
HM	4	12	$7.113915e-35$	2.204096081050027306553912292370
HHM	4	12	$3.597006e-29$	
AM	4	16	$5.524693e-20$	
NNM	4	12	$7.113915e-35$	
PCIM	2	6	$1.538421e-26$	

Table 7. Comparison of NM, HM, HH, AM, NNM and PCIM

Method	N	N_f	$\lvert f(x_{n+1})\rvert$	x_{n+1}
$f_6, x_0 = 5.3$				
NM	5	10	$2.024500e-15$	
HM	4	12	$5.276438e-45$	4.306913199721865187030462632430
HHM	4	12	$2.475287e-34$	
AM	4	16	$1.928654e-36$	
NNM	4	12	$5.276438e-45$	
PCIM	2	8	$4.714885e-35$	

6. Polynomiography

Polynomiography was introduced by Kalantari in [13] as a visual technique to find approximate roots of of complex polynomials. An individual image is called a polynomiograph, and are colored based on the number of iterations needed to obtain the approximated root with a given accuracy and a chosen iteration method. For details, see [13, 14, 20, 25, 18].

6.1. **Iteration.** Let $p(z)$ be a complex polynomial. Then

$$\begin{aligned} y_n &= z_n - \frac{p(z_n)}{p'(z_n)} - \frac{p^2(z_n)p''(z_n)}{2p'^3(z_n)}, n = 0, 1, 2, ..., \\ z_{n+1} &= y_n - \frac{p(y_n)}{p'(y_n)} - \frac{p^2(y_n)p''(y_n)}{2p'^3(y_n)} - \frac{p^3(y_n)p'''(y_n)}{6p'^4(y_n)}, \end{aligned} \tag{13}$$

where $z_o \in \mathbb{C}$ is a starting point, is the modified AM with Householder's method for solving nonlinear complex equations. The sequence $\{z_n\}_{n=0}^{\infty}$ is called the orbit of the point z_o converges to a root z^* of p. We say that z_o is attracted to z^*. The set of all such starting points for which $\{z_n\}_{n=0}^{\infty}$ converges to z^* is called the basin of attraction of z^*.

6.2. **Convergence Test.** In numerical algorithms, that are based on iterative processes, we need a stopping criterion, a test that tells us that the process will terminate after a finite number of steps. Usually, in iterative processes the standard convergence test has the form:

$$|z_{n+1} - z_n| < \varepsilon, \tag{7.1}$$

where z_{n+1} and z_n are two successive points in the iteration process, and $\varepsilon > 0$ is a given accuracy. In our case, we also use this stopping criterion.

6.3. **Applications.**

6.3.1. *Polynomiograph for* $z^2 - 1 = 0$. The polynomiograph of $z^2 - 1 = 0$ contains two distinct basins of attraction corresponding to its two roots.

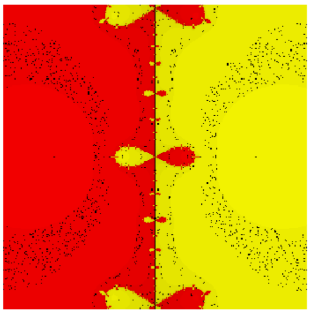

Figure. 1. Polynomiography for $z^2 - 1 = 0$.

6.3.2. *Polynomiograph for* $z^3 - 1 = 0$. Its polynomiograph is has three distinct basins of attraction, as it has three roots.

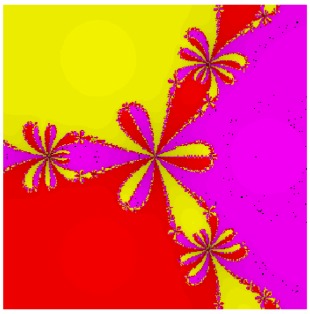

Figure. 2. Polynomiography for $z^3 - 1 = 0$.

6.3.3. *Polynomiograph for* $z^4 - 1 = 0$.

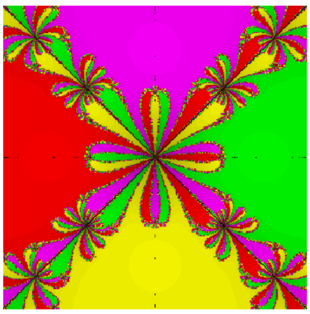

Fig. 3. Polynomiography for $z^4 - 1 = 0$.

6.3.4. *Polynomiograph for* $z^4 - z^3 + z^2 - z + 1 = 0$. The four roots of the equation $z^4 - z^3 + z^2 - z + 1 = 0$ are $-0.309017 - 0.951057I$, $-0.309017 + 0.951057I$, $0.809017 - 0.587785I$, and $0.809017 + 0.587785I$, and hence its polynomiograph has four distinct basins of attraction to the four roots.

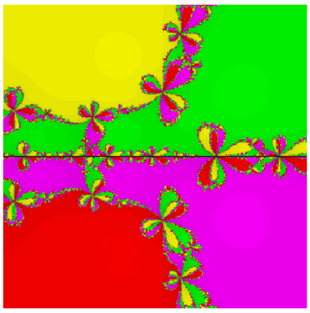

Fig. 4. Polynomiography for $z^4 - z^3 + z^2 - z + 1 = 0$.

6.3.5. *Polynomiograph for* $z(z^2 + 1)(z^2 + 4) = 0$.

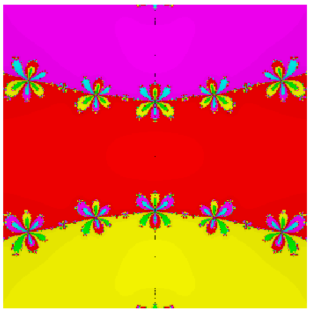

Fig. 5. Polynomiography for $z(z^2 + 1)(z^2 + 4) = 0$.

6.3.6. *Polynomiograph for* $z^5 - 1 = 0$.

Fig. 6. Polynomiography for $z^5 - 1 = 0$.

6.3.7. *Polynomiograph for* $z^{20} - 1 = 0$.

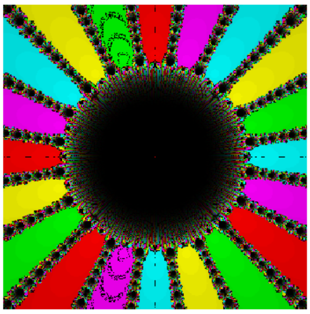

Fig. 7. Polynomiography for $z^{20} - 1 = 0$.

7. Conclusions

In this article we introduced the PCIM to solve nonlinear equations. We can concluded from tables $(1 - 7)$ that

(1) The efficiency index of two-step predictor-corrector iterative method is 1.7321.
(2) The convergence order of two-step predictor-corrector iterative method is 9.
(3) Its performance is better than some well-known methods.

We also gave examples of polynomiographs of some complex polynomials.

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