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ON SOME PARAMETERS RELATED TO FIXING SETS IN GRAPHS

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ABSTRACT. The fixing number of a graph G is the smallest cardinality of a set of vertices $F \subseteq V(G)$ such that only the trivial automorphism of G fixes every vertex in F. In this paper, we introduce and study three new fixing parameters: fixing share, fixing polynomial and fixing value.

Key words : fixing number, fixing share, fixing polynomial, fixing value.
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1. INTRODUCTION

Unless otherwise specified, all the graphs G considered in this paper are simple, non-trivial and connected with vertex set $V(G)$ and edge set $E(G)$. We write $u \sim^e v$ if two vertices u and v form an edge in G and $u \not\sim^e v$ if u and v do not form an edge in G. The subgraph induced by a set S of vertices of G is denoted by $\langle S \rangle$. The *neighborhood* of a vertex v of G is the set $N(v) = \{u \in V(G) \ : \ u \sim^e v\}$. The number of elements in $N(v)$ is the *degree* of v, denoted by $d(v)$. The maximum degree of G is denoted by $\Delta(G)$. A vertex v with $d(v) = 0$ is an *isolated vertex*. If two distinct vertices u and v of G have the property that $N(u) - \{v\} = N(v) - \{u\}$, then u and v are called *twin vertices* (or simply twins) in G. If for a vertex u of G, there exists a vertex $v \neq u$ in G such that u, v are twins in G, then u is said to be a *twin* in G. A set $T \subseteq V(G)$ is said to be a *twin-set* in G if every two elements of T are twin vertices of G. The *complement* of G, denoted by $\overline{G}$, has the same vertex set as G and $x \sim^e y$ in $\overline{G}$ if and only if $x \not\sim^e y$ in G.

An *automorphism* of G is a bijective mapping $\phi : V(G) \to V(G)$ such that $(u)\phi \sim^e (v)\phi$ if and only if $u \sim^e v$. Thus, each automorphism of G is a

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permutation on the vertex set $V(G)$, which preserves adjacencies and non-adjacencies. The *automorphism group* of a graph G, denoted by $\Gamma(G)$, is the set of all automorphisms of a graph G. The *stabilizer* of a vertex v of a graph G, denoted by $\Gamma_v(G)$, is the set $\{\phi \in \Gamma(G) : (v)\phi = v\}$. The stabilizer of a subset $F \subseteq V(G)$ is $\Gamma_F(G) = \{\phi \in \Gamma(G) : (v)\phi = v \ \forall \ v \in F\}$. Note that $\Gamma_F(G) = \bigcap\limits_{v \in F} \Gamma_v(G)$. The *orbit* of a vertex v of a graph G, denoted by $\mathcal{O}(v)$, is the set $\{u \in V(G) : (v)\alpha = u \text{ for some } \alpha \in \Gamma(G)\}$. Two vertices u and v are said to be *similar* if they belong to the same orbit. The number $d(u, v)$ denotes the *distance* between two vertices u and v of G, which is the number of edges in a shortest $u-v$ path in G. We note a well established fact that every automorphism is also an *isometry*, that is, for any $\psi \in \Gamma(G)$ and $u, v \in V(G)$, $d(u, v) = d((u)\psi, (v)\psi)$ [4].

A vertex v of a graph G is said to be *fixed* by a group element $\phi \in \Gamma(G)$ if $\phi \in \Gamma_v(G)$. A subset $F \subseteq V(G)$ is called a *fixing set* of G if $\Gamma_F(G)$ is trivial. In this case, we say that F *fixes* G. The *fixing number* of G, $fix(G)$, is the minimum cardinality of a fixing set of G [8]. Each graph has a fixing set. Trivially, the set of vertices of G itself is a fixing set. It is also clear that any set containing all but one vertex is a fixing set. In [10], it was shown that the only connected graph with $fix(G) = n - 1$ is the complete graph on $n \geq 2$ vertices. Also, it has been noted that $fix(\overline{K_n}) = n - 1$. On the other hand, a graph G has $fix(G) = 0$ if and only if $\Gamma(G)$ is trivial. Thus, for a graph G on $n \geq 1$ vertices, $0 \leq fix(G) \leq n - 1$ [3]. Unless otherwise specified, all the graphs considered in this paper have non-trivial automorphisms group.

The fixing number of a graph G was first defined by Erwin and Harary in 2006 [8]. Boutin introduced the concept of determining set and defined it as follows: A subset D of the vertices in a graph G is called a *determining set* if whenever $g, h \in \Gamma(G)$ with the property that $(u)g = (u)h$ for all $u \in D$, then $(v)g = (v)h$ for all $v \in V(G)$ [5]. The minimum cardinality of a determining set is called the *determining number*. In [10], it was shown that fixing set and determining set are equivalent. A considerable literature has been developed in this field (see [2, 3, 6, 8, 12]). The concept of the fixing number originates from the idea of breaking symmetries in graphs which have applications in the problem of programming a robot to manipulate objects [13].

A finite sequence of real numbers $(x_0, x_1, \ldots, x_n)$ is said to be *unimodal* if there is some $k \in \{0, 1, \ldots, n\}$, called the *mode* of the sequence, such that $x_0 \leq x_1 \leq \ldots \leq x_{k-1} \leq x_k \geq x_{k+1} \geq \ldots \geq x_{n-1} \geq x_n$. The mode is unique if $x_{k-1} < x_k > x_{k+1}$. A polynomial is called *unimodal* if the sequence of its coefficients is unimodal.

The rest of the paper is organized as follows: In section 2, we quantify the participation of each vertex v of a graph G to fix any pair of vertices of G in order to break the symmetry of the graph, and we call this quantity the fixing share of v. We also investigate some useful results related to fixing share in this section. In section 3, a new graph polynomial, called the fixing polynomial, as well as a new fixing parameter, called the fixing value, are defined and studied. Some basic properties and useful results related to these newly defined parameters are also derived in this section. Moreover, fixing polynomials and fixing values in some well-known families of graphs such as cycles, complete multipartite graphs and lexicographic product of cycle with m isolated vertices are found. Also, we discuss the unimodality of the fixing polynomial of cycles.

2. Fixing Share

In this section, we define the concept of fixing share and investigate some basic results. We begin with the following useful preliminaries: Let G be a connected graph. A vertex v of G is said to be *fixed* if $(v)\psi = v$ for all $\psi \in \Gamma(G)$, that is, $\Gamma_v(G) = \Gamma(G)$. A vertex v of G is said to be *locally fixed* by an automorphism ϕ of G if $\phi \in \Gamma_v(G)$ and $\Gamma_v(G) \neq \Gamma(G)$. In order to avoid confusion of terms fixed and locally fixed, we shall use the term 'globally fixed' instead of just 'fixed'. For instance, in the graph G_1 of Figure 1, the vertex v_2 is locally fixed by the automorphism $(v_5\ v_6)$, whereas the vertex v_3 is globally fixed.

Figure 1. The graph G_1

From the definitions of globally fixed and locally fixed vertex, we have the following remark:

Remark 1. (1) *For a locally fixed vertex u and for a globally fixed vertex v of a graph G, there is no automorphism ψ of G such that $(u)\psi = v$ or $(v)\psi = u$.*
(2) *If v is a globally fixed vertex and u is a locally fixed vertex in a graph G, then $\mathcal{O}(v) = \{v\}$ and $|\mathcal{O}(u)| \geq 2$.*

Let $V_s(G) = \{(u, v) :\ u, v \in V(G)$ and u, v are distinct similar vertices$\}$. A vertex v of G is said to *locally fix a pair* $(x, y) \in V_s(G)$, if $(x)\psi \neq y$ and $(y)\psi \neq x$, for all $\psi \in \Gamma_v(G)$. We shall say that locally fixing v destroys all the automorphisms in which x is mapped onto y and y is mapped onto x. For instance, in the graph G_1, v_6 does not locally fix the pair (v_1, v_2), because there is an automorphism $\psi = (v_1\ v_2)$ in $\Gamma_{v_6}(G_1)$ such that $(v_1)\psi = v_2$ and

$(v_2)\psi = v_1$. However, the vertex v_6 locally fixes the pair (v_5, v_6) because there is no automorphism ϕ in $\Gamma_{v_6}(G_1)$ such that $(v_5)\phi = v_6$ and $(v_6)\phi = v_5$.

For a pair (u, v) of distinct vertices of G, the *fixing neighborhood* of (u, v) is denoted by $F(u, v)$ and is defined as: $F(u, v) = \{x \in V(G) : (u)\psi \neq v \wedge (v)\psi \neq u,\ \forall\ \psi \in \Gamma_x(G)\}$. From this definition, we observe that, the fixing neighborhood of $(u, v) \in V_s(G)$ contains both the vertices u and v.
From the definition of $F(u, v)$ and Remark 1, we have the following remark:

Remark 2. *If v is a globally fixed vertex of G, then $F(v, u) = V(G)$ for all $u \in V(G) - \{v\}$. Moreover, $v \notin F(x, y)$ for any distinct $x, y \in V(G)$.*

Definition 1. *(Fixing share) Let G be a connected graph. For any pair (u, v) of distinct vertices of G and for any $w \in V(G)$, the quantity*

$$f_w(u, v) = \begin{cases} 0 & \text{when } w \notin F(u, v), \\ \frac{1}{|F(u,v)|} & \text{when } w \in F(u, v), \end{cases}$$

is called the fixing share of w for the pair (u, v).

For example, in the graph G_1 of Figure 1, $F(v_1, v_2) = \{v_1, v_2\}$, and thus

$$f_w(v_1, v_2) = \begin{cases} \frac{1}{2} & \text{when } w \in\ F(v_1, v_2), \\ 0 & \text{otherwise.} \end{cases}$$

From the definition of fixing share, we observe that, $f_u(u, v) \neq 0 \neq f_v(u, v)$ for every two locally fixed vertices u and v.

In view of Remarks 1 and 2, from now onwards, each pair of vertices of a graph G considered for computing its fixing share is from the set $V_s(G) \subseteq V_p$, where V_p denotes the collection of all $\binom{n}{2}$ pairs of the vertices of G.

Properties 1. (1) *The fixing neighborhood of a pair $(u, v) \in V_s(G)$ is the class of all those vertices of G whose fixing share for the pair (u, v) is the same.*
(2) *For $w \in V(G)$ and $(u, v) \in V_s(G)$, $0 \leq f_w(u, v) \leq \frac{1}{2}$. The sharpness of the upper bound in this inequality follows if and only if u and v are twin vertices and $w \in \{u, v\}$.*
(3) *A twin in G is a locally fixed vertex. Because, for a twin x in G, there exists a vertex $y \neq x$ in G such that x and y are twin vertices, and hence there is an automorphism $\psi = (x\ y)$ of G with the property that $(x)\psi = y$ and $(y)\psi = x$, and can be destroyed only by fixing either x or y.*
(4) *Let $J = \{v_1, v_2, \ldots, v_k\} \subseteq V(G)$, $(k \geq 2)$. If for every two elements $u, v \in J$, $f_w(u, v) = \frac{1}{2}$, then at least $k - 1$ elements of J must belong to any fixing set F for G.*
(5) *For a pair $(u, v) \in V_s(G)$, if $f_w(u, v) = \frac{1}{|G|}$, then there is no globally fixed vertex in G. Since if there is a globally fixed vertex x in G, then $F(u, v) \neq V(G)$ because for a globally fixed vertex x, $\Gamma_x(G) = \Gamma(G)$ which implies that $f_w(u, v) \neq \frac{1}{|G|}$, a contradiction.*

(6) *For two distinct locally fixed vertices u and v of a graph G, $F(u,v) = \{u,v\}$ if and only if there is an automorphism of G which is a transposition $(u\ v)$ on $V(G)$ and can be destroyed by fixing u and v only.*
(7) *For any two distinct locally fixed vertices u and v of a graph G, $(u\ v)$ is a transposition if u and v are twins.*

Let D_i denotes the class of all the vertices of a connected graph G having degree i for $1 \leq i \leq \Delta(G)$, and is called the *degree class* in G. A degree class may be empty. Note that, all the non-empty degree classes in G form a partition of $V(G)$, called the *degree partition* of $V(G)$. Thus, we have the following straightforward lemma:

Lemma 1. *Let G be a connected graph and $\{U_i\ ;\ 1 \leq i \leq \Delta(G)\}$ be the degree partition of $V(G)$. Then for $u \in U_i$ and $v \in U_{j\neq i}$, $f_w(u,v) \neq 0$ for all $w \in V(G)$.*

Theorem 2. *Let G be a connected graph of order $n \geq 2$. Let J be the set of $p \geq 1$ globally fixed vertices of G and $\{U_i\ ;\ 1 \leq i \leq k\}$ $(k \leq \Delta(G))$ be the degree partition of $V(G)$. Let $S_i = U_i - J$ for $1 \leq i \leq k$. Then the number of pairs (u,v) in V_p for which $f_w(u,v) \neq 0$ for all $w \in V(G)$ is bounded below by*

$$\frac{p}{2}(2n-p-1) + \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} |S_i||S_j|.$$

Moreover, this bound is sharp.

Proof. Since there are $p \geq 1$ globally fixed vertices in G, so we have $\binom{p}{2} + p(n-p)$ pairs (u,v) in $V_p - V_s(G)$ for which $f_w(u,v) \neq 0$ for all $w \in V(G)$. Further, since for each $u \in S_i$ and $v \in S_{j\neq i}$, $d(u) \neq d(v)$, so Lemma 1 yields that there are at least

$$\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} |S_i||S_j|$$

pairs (u,v) in $V_s(G)$ for which $f_w(u,v) \neq 0$ for all $w \in V(G)$. It completes the proof of first part.

For sharpness, consider a complete graph K_t $(t \geq 2)$ and a star graph $K_{1,r}$ $(r \geq 2)$ with center, say c . Make a graph G of order $n = t + r + 1$ by joining the vertex c of $K_{1,r}$ by any edge with a vertex, say v, of K_t. One can see that for the pairs (c,x) and (v,y) with $x \in (V(K_t) \cup V(K_{1,r}) - \{c\})$ and $y \in (V(K_t) \cup V(K_{1,r}) - \{c,v\})$, $f_w(c,x) = f_w(v,y) \neq 0$ for all $w \in V(G)$, and there are $2(r+t) - 1$ such pairs in V_p. Also, for each pair (x,y) with $x \in V(K_t) - \{v\}$ and $y \in V(K_{1,r}) - \{c\}$, $f_w(x,y) \neq 0$ for all $w \in V(G)$, and there are $r(t-1)$ such pairs in V_p. Therefore, there are exactly

$$r(t+1)+2t-1=\frac{p}{2}(2n-p-1)+\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}|S_i||S_j|$$

pairs (u,v) in V_p for which $f_w(u,v)\neq 0$ for all $w\in V(G)$. Because, in this graph G, $J=\{c,v\}$ and we have four classes in the degree partition of $V(G)$, namely $U_1=D_1$ with $|U_1|=r$, $U_2=D_{r+1}=\{c\}$ with $|U_2|=1$, $U_3=D_t=\{v\}$ with $|U_3|=1$, and $U_4=D_{t-1}$ with $|U_4|=t-1$. Note that $S_1=U_1, S_4=U_4$ and $S_2=S_3=\emptyset$. □

3. Fixing polynomials and Fixing values

One of the most general approaches to graph polynomials was proposed by Farrell [9] in his theory of F-polynomials of a graph. According to Farrell, any such polynomial corresponds to a strictly prescribed family of connected subgraphs of the respective graph. For the domination polynomial of G [1], this family corresponds to all the dominating sets of G; for the chromatic polynomial of G [7], this family corresponds to all the color classes of G; for the matching polynomial of a graph G [9], this family corresponds to all the edges of G; for the independence polynomial of G [11], this family corresponds to all the stable (independent) sets of G; for the resolving polynomial of G [14], this family corresponds to all the resolving sets of G.

In this section, we introduce the fixing polynomial of G, this family includes all the fixing sets of G. In fact, various aspects of combinatorial information concerning a graph is stored in the coefficients of a specific graph polynomial.

For a graph G of order n with fixing number $fix(G)$, the fixing polynomial $fix(G,x)$ is a generating polynomial for the fixing sequence $(f_{fix(G)}, f_{fix(G)+1}, \ldots, f_n)$ which helps in counting all the fixing sets of cardinality i; $fix(G)\leq i\leq n$, for G. The fixing polynomial of a graph is a good representative of the fixing structure of the graph. This polynomial is defined as follows: Let G be a graph. An i-set is a subset of $V(G)$ of cardinality i. Let $\mathcal{F}(G,i)$ denotes the family of all the fixing sets of G which are i-sets and let $f_i=|\mathcal{F}(G,i)|$. Then the *fixing polynomial* of G, denoted by $fix(G,x)$, is defined as

$$fix(G,x)=\sum_{i=fix(G)}^{n} f_i x^i,$$

where $fix(G)$ is the fixing number of G. It is worth mentioning that $f_i=0$ if and only if $i=fix(G)=0$ or $i<fix(G)$.

Example 1. *Let G be the co-eiffeltower graph. Then $fix(G)=1$, because $\mathcal{O}(v_3)=|\Gamma(G)|$ [8]. The fixing sequence for G is $(4,18,34,35,21,7,1)$ and due*

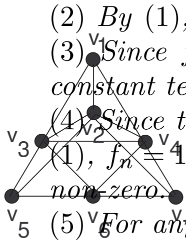

FIGURE 2. co-eiffeltower graph

to this sequence, we have the fixing polynomial of G *as* $x^7+7x^6+21x^5+35x^4+34x^3+18x^2+4x$.

In a polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$, the coefficient a_n is called the *leading coefficient* of $P(x)$. If $a_n = 1$, then the polynomial $P(x)$ is called *monic*. Followings are some properties of fixing polynomial $fix(G, x)$ of a graph G of order n.

Properties 2. (1) *Since the only fixing set of cardinality* n *is the set* $V(G)$ *and a fixing set for* G *of cardinality* $n-1$ *can be chosen in* n *possible different ways, so* $f_n = 1$ *and* $f_{n-1} = n$.
(2) *By* (1), *it follows that* $fix(G, x)$ *is monic.*
(3) *Since* $f_i = 0$ *for* $i = fix(G) = 0$ *or* $i < fix(G)$, *so* $fix(G, x)$ *has no constant term.*
(4) *Since there exists at least one fixing set of cardinality* $fix(G) \neq 0$ *and by* (1), $f_n = 1$. *So, each term of the fixing sequence* $(f_{fix(G)}, f_{fix(G)+1}, \ldots, f_n)$ *is non-zero.*
(5) *For any* $a, b \in [0, \infty)$ *such that* $a < b$, $fix(G, a) < fix(G, b)$. *It concludes that* $fix(G, x)$ *is strictly increasing function on* $[0, \infty)$.
(6) *If* H *is any subgraph of* G, *then* $deg(fix(G, x)) \geq deg(fix(H, x))$.

It is easy to see that if a graph G has n components $G_1, G_2, \ldots, G_n$, then a fixing set for a graph G can be obtained by taking the union of fixing sets for $G_1, G_2, \ldots, G_n$. Thus, $fix(G) = fix(G_1) + fix(G_2) + \ldots + fix(G_n)$. Therefore, we have the following remark:

Remark 3. (1) *If* G *is a graph with* n *components* $G_1, G_2, \ldots, G_n$, *then the fixing polynomial of* G *is* $fix(G, x) = \prod_{i=1}^{n} fix(G_i, x)$.
(2) *If* G *is the union of* $n \geq 2$ *isolated vertices, then* $fix(G, x) = x^n + nx^{n-1}$.
(3) *If* G *is a graph with* $r \geq 2$ *isolated vertices and* H *be a graph induced by the set* $V(G) - Y$, *where* Y *is the set of* r *isolated vertices, then* $fix(H, x) = \frac{fix(G,x)}{x^r + rx^{r-1}}$.

The number of isolated vertices in a graph G can be obtained if the fixing polynomials of G and its subgraph H are known, as we show in the following lemma.

Lemma 3. *Let G be a graph of order $n \geq 4$ with $r \geq 2$ isolated vertices and H be its subgraph induced by the set $V(G)-Y$, where Y is the set of r isolated vertices. If $fix(G,x) = \sum\limits_{i=fix(G)}^{n} f_i x^i$ and $fix(H,x) = \sum\limits_{j=fix(H)}^{n-r} f_j x^j$ are the fixing polynomials of G and H, respectively, then $r = \frac{f_{fix(G)}}{f_{fix(H)}}$.*

Proof. Let K be a graph induced by Y, then $fix(K,x) = x^r + rx^{r-1}$. Since H is a graph induced by $V(G)-Y$, so $fix(H,x) = \sum\limits_{i=fix(H)}^{m} f_i x^i$, where $m = |V(G)-Y|$. Since G is a graph consisting of two components K and H, so $fix(G,x) = fix(K,x)fix(H,x)$. This implies that

$$\sum_{i=fix(G)}^{n} f_i x^i = \left(x^r + rx^{r-1}\right)\left(\sum_{i=fix(H)}^{m} f_i x^i\right).$$

It follows that

$$f_{fix(G)} x^{fix(G)} + \sum_{i=fix(G)+1}^{m+r} f_i x^i = r f_{fix(H)} x^{fix(H)+r-1} + \ldots + f_{m+r} x^{m+r}.$$

Since $fix(G) = fix(H) + fix(K) = fix(H) + r - 1$. Therefore, by comparing the coefficients of $x^{fix(G)}$ and $x^{fix(H)+r-1}$, we have the required result. $\square$

In a graph G, we call a fixing set of G of cardinality $fix(G)$, the *fix-set* of G, and we denote the total number of the fix-sets of G by $T(G)$. It follows from the definition of fixing polynomial that $T(G) = f_{fix(G)} = |\mathcal{F}(G, fix(G))|$. Now, we define the fixing value of each vertex of G as follows: For each vertex $v \in V(G)$, the *fixing value* of v, denoted by $FV_G(v)$, is the number of fix-sets of G for which v belongs. We simply write $FV(v)$ instead of $FV_G(v)$ if G is clear from the context. Since, the fixing number of the graph G of Figure 2 is 1, so $T(G) = f_1 = 4$. This implies that $FV(v_3) = 1$.

The following straightforward assertions hold in the context of fixing value.

Proposition 4. *Let G be a graph, then*
(1) $\sum\limits_{v \in V(G)} FV(v) = T(G)fix(G)$.
(2) *If u and v are similar vertices in G, then $FV(v) = FV(u)$.*

(3) *If G has $n \geq 2$ components $G_1, G_2, \ldots, G_n$, then $T(G) = \prod_{i=1}^{n} T(G_i)$. Furthermore, for $v \in V(G)$, $FV(v) = FV_{G_i}(v) \prod_{\substack{j=1 \\ j \neq i}}^{n} T(G_j)$.*

According to the definition of twin vertices and twin-set, we have the followings:

Proposition 5. *Suppose that u, v are twins in a connected graph G and F is a fixing set of G. Then either u or v is in F. Moreover, if $u \in F$ and $v \notin F$, then $(F - \{u\}) \cup \{v\}$ is a fixing set of G.*

Proposition 6. *For each pair (u, v) of twin vertices of a graph G $|F(u, v)| = 2$ and $F(u, v) = \{u, v\}$.*

Remark 4. *Let T be a twin-set of order $m \geq 2$ in a connected graph G. Then every fixing set F of G contains at least $m - 1$ vertices of T.*

Proposition 7. *For each pair $(u, v) \in V_p$, we have*

$$T(G) \leq \sum_{v_0 \in F(u,v)} FV(v_0) \leq T(G) fix(G).$$

Proof. The upper bound follows from the Proposition 4(1). For the lower bound, note that any fixing set F of G must contain a vertex from the fixing neighborhood $F(u, v)$, otherwise it is not a fixing set of G. □

3.1. **Fixing polynomials and fixing values in some well-known families of graphs.** In this section, we consider cycles, complete multipartite graphs and lexicographic product of cycles with m isolated vertices in the context of fixing polynomial and fixing value. Also, we discuss the unimodality of the fixing polynomial of cycles.

Two vertices u and v in a connected graph G are said to be *antipodal* if $d(u, v) = diam(G)$. Otherwise, u and v are non-antipodal.

Theorem 8. *Let G be a cycle C_n with $n \geq 3$. Then*

$$fix(G, x) = \begin{cases} \frac{1}{2} n(n-2)x^2 + \sum_{i=3}^{n} \binom{n}{i} x^i & if\ n\ is\ even, \\ \sum_{i=2}^{n} \binom{n}{i} x^i & if\ n\ is\ odd. \end{cases}$$

Further, this polynomial is unimodal. Moreover, for each vertex v of G,

$$FV(v) = \begin{cases} n - 1 & if\ n\ is\ odd, \\ n - 2 & if\ n\ is\ even. \end{cases}$$

Proof. The fixing number of C_n, $n \geq 3$ is 2 [8]. Thus, we have to find the coefficients of the fixing polynomial $fix(G,x) = \sum_{i=2}^{n} f_i x^i$.

Case 1. (n is even) (a) For $i = 2$. Let $F \subseteq V(G)$ with $|F| = 2$ such that $F \nsubseteq \mathcal{F}(G,2)$. Then there are $\frac{n}{2}$ such F since the only 2-element subsets of $V(G)$ which can not belong to $\mathcal{F}(G,2)$ are those which consist of antipodal vertices. Therefore, $f_2 = \binom{n}{2} - \frac{n}{2}$.
(b) For $3 \leq i \leq n$, $f_i = \binom{n}{i}$ since choosing a fixing set of cardinality i from $V(G)$ is equivalent to selecting i vertices out of n vertices of G to destroy the symmetry of G.
Case 2. (n is odd) By the same argument as in part (b) of Case 1, $f_i = \binom{n}{i}$ for all $2 \leq i \leq n$.

Note that whenever n is even then $f_2 = |\mathcal{F}(G,2)| = \binom{n}{2} - \frac{n}{2}$, $f_i = \binom{n}{i}$ for all $3 \leq i \leq n$, and $f_i = \binom{n}{i}$ for all $2 \leq i \leq n$ when n is odd, so there exists a mode $k \in \{\frac{n}{2} - 1$ (n is even), $\frac{n+1}{2} - 1$ (n is odd)$\}$ such that, by using the property $\binom{n}{i} = \binom{n}{n-i}$, we have $f_2 \leq \ldots \leq f_{k-1} < f_k > f_{k+1} \geq \ldots \geq f_n$, which shows that $fix(G,x)$ is unimodal.

In the first part of the proof, we note that any two non-antipodal vertices of G form a fix-set of G. So, for each $v \in V(G)$, $FV(v) = n-2$ for even n, and $FV(v) = n-1$ for odd n. □

Theorem 9. *For $t \geq 2$, let G be a complete multipartite graph $K_{n_1,n_2,\ldots,n_t}$ with $n_i \geq 2$ for each i, and $n_1 + n_2 + \ldots + n_t = n$. Then*

$$fix(G,x) = x^n + \sum_{j=1}^{t} [\sum_{1=i_1<i_2<\ldots<i_j}^{t} n_{i_1} n_{i_2} \ldots n_{i_j} x^{n-j}].$$

Moreover, if V_j, $1 \leq j \leq t$ be the partite sets of G of cardinality n_j, then for each $v \in V_j$,

$$FV(v) = \prod_{\substack{i=1 \\ i \neq j}}^{t} n_i (n_j - 1).$$

Proof. It was shown in [6] that $fix(G) = n - t$. Therefore, we find the fixing sequence $(f_{n-t}, f_{n-t+1}, \ldots, f_n)$ to derive the fixing polynomial $fix(G,x) = \sum_{i=n-t}^{n} f_i x^i$. In fact, we have to compute each coefficient f_{n-j} for $2 \leq j \leq t$, where as, the coefficients $f_{n-1} = n$ and $f_n = 1$.

Note that, to make a fixing set F of cardinality $n-j$, we need to choose all the vertices of G except j vertices with one vertex from each partite set, and this can be done in $\sum_{1=i_1<i_2<\ldots<i_j}^{t} n_{i_1} n_{i_2} \ldots n_{i_j}$ different ways, and it completes the proof for the fixing polynomial of G.

Consider a locally fixed vertex v in V_j, then out of remaining $n_j - 1$ vertices of V_j, $n_j - 2$ vertices can be chosen in $n_j - 1$ different ways. Also, from each partite set V_i $(i \neq j)$, $n_i - 1$ vertices out of n_i vertices can be chosen in n_i different ways, where $i = 1, 2, \ldots, t$ $(i \neq j)$. Hence, $FV(v) = \prod_{\substack{i=1 \\ i \neq j}}^{t} n_i(n_j - 1)$. □

The *lexicographic product* of a graph G with a graph H, denoted by $G[H]$, is the graph having vertex set $V(G) \times V(H) = \{(u, v) \mid u \in V(G),\ v \in V(H)\}$ and for two distinct vertices $(u, v), (u', v')$ of $G[H]$, $(u, v) \sim^e (u', v')$ whenever $u = u'$ and $v \sim^e v'$ or $u \sim^e u'$.

The following result gives the fixing number of $C_n[\overline{K_m}]$.

Theorem 10. *Let G be the graph $C_n[\overline{K_m}]$ with $n \geq 3$ $(n \neq 4)$ and $m \geq 2$. Then $fix(G) = n(m - 1)$.*

Proof. Let $V(G) = \{(u_i, v_j);\ 1 \leq i \leq n,\ 1 \leq j \leq m\}$, where $u_i \in V(C_n)$ and $v_j \in V(\overline{K_m})$. Then there are n twin-sets of cardinality m in G, and hence at least $m - 1$ elements from each twin-set belong to any fixing set of G, so $fix(G) \geq n(m - 1)$. Further, note that, the set $F = \{(u_i, v_j);\ 1 \leq i \leq n, 1 \leq j \leq m - 1\}$ is a fixing set of G of cardinality $n(m - 1)$, which implies that $fix(G) = n(m - 1)$. □

Theorem 11. *Let G be the graph $C_n[\overline{K_m}]$ with $n \geq 3$ $(n \neq 4)$ and $m \geq 2$. Then*

$$fix(G, x) = \sum_{i=0}^{n} \binom{n}{i} m^{n-i} x^{n(m-1)+i}.$$

Further, for every vertex v of G, $FV(v) = (m - 1)m^{n-1}$.

Proof. There are n twin-sets of cardinality m in G. Out of these n twin-sets, we can choose r twin-sets from which we will choose all the m elements, and this can be done in $\binom{n}{r}$ different ways. Further, amongst the remaining $n - r$ twin-sets, we can choose $m - 1$ elements out of m elements, which can be done in m^{n-r} different ways. It yields that $fix(G, x) = \sum_{i=0}^{n} \binom{n}{i} m^{n-i} x^{n(m-1)+i}$.

To make a fix-set of G, note that, out of m elements of a twin-set, we must choose $m - 1$ elements and for a locally fixed vertex v in G, $m - 2$ elements from the twin-set containing v can be chosen in $m - 1$ different ways. For a twin-set, not containing v, $m - 1$ elements out of m elements can be chosen in m different ways. Hence, $FV(v) = (m - 1)m^{n-1}$. □

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