

FORGOTTEN INDEX OF GENERALIZED F-SUM GRAPHS

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ABSTRACT. Liu et al. [IEEE Access; 7(2019); 105479-105488] defined the concept of the generalized subdivided operations on graphs and obtained the generalized F-sum graphs. They also calculated the 1st and 2nd Zagreb indices of the generalized F-sum graphs. In the continuation of this work, we study the forgotten index (F-index) of the generalized F-sum graphs in terms of different topological indices (TI's) of their base graphs. In the end, the results of F-index on the generalized F-sum graphs acquired by the particular classes of alkane are also included.

Key words : Generalized operations; Zagreb indices; F-index.

AMS Classification: 05C12; 05C90; 05C35.

1. INTRODUCTION

A topological index (TI) is considered as a function $\phi : \Sigma \rightarrow R$ (set of real numbers) that links every element of the Σ to a unique real number, where Σ is collection of graphs. TI's forecast the bio-nature activities, chemical reactivities and physical attributes of the molecular graphs like heat of formation, heat of evaporation, viscosity, freezing point, boiling point, melting point, surface tension, stability, temperature, density, weight, polarizability, connectivity, and solubility. Several drugs, crystalline, and nano-materials that are utilized in various industries are analyzed with the help of TI's, see [14, 15, 10, 9, 8, 12, 22, 27]. These are also considered a useful tool to examine the quantitative structural properties relationships and quantitative structural activities relationships which connect molecular structures to their different molecular properties. For more detail, we refer to [12, 16, 17].

To study the paraffin's boiling point, Wiener first time used the distance-based TI [7]. Gutman and Trinajstić [19] calculated π -electrons total energy of a molecule through a TI called as the first Zagreb index. After that, many

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impressive TI's are introduced in chemical graph theory [20, 18] but the degree based TI's are well known than any other outcome, see [21].

In molecular graph theory, the different operations on a graph perform a fundamental role in the formation of different new classes of graphs, see [11]. Yan et al. [12] introduced the four operations S_1, R_1, Q_1, T_1 on graphs and obtained the Wiener index of these resultant graphs. For $F_1 \in \{S_1, R_1, Q_1, T_1\}$, Taeri and Eliasi [13] defined the F_1 -sum graphs $(H_1 +_{F_1} H_2)$ using the cartesian product on graphs $F(H_1)$ and H_2 , where H_1 & H_2 are connected. They also studied the Wiener index of these F_1 -sum graphs. Furthermore, Deng et al. [24], Shehnaz et al. [25] and Liu et al. [2] calculated the 1st & 2nd Zagreb indices, F-index and the 1st general Zagreb index of F_1 -sum graph.

Recently, Liu et al. [1] introduced the concept of generalized operations on graphs and obtained the generalized F-sum (F_k -sum) graphs. They also calculated the 1st & 2nd Zagreb TI's of F_k -sum graphs. In the present article, we calculate the F-index for the generalized F-sum graphs (F_k -sum), where k presents a counting number. This article is arranged as; the section II contains some primary terminologies, and definitions. The section III covers the key outcomes, and finally in the section IV closing remarks are included with the application of the achieved outcomes.

2. PRELIMINARIES

Let $H = (V(H), E(H))$ be a connected & simple graph having the node-set $V(H)$ as well as the edge-set $E(H) \subseteq V(H) \times V(H)$. Every node $u \in V(H)$ is named as atom in chemical graph theory and connection with in the two atoms is denoted by an edge. So the cardinalities of node-set and edge-set are known as order and size. The strength of edges which are incident on any node $v \in V(H)$ is known as its degree ($d_H(v)$). Here, we defined few topological indices that given:

Definition 2.1. For a simple graph H , the 1st Zagreb index ($M_1(H)$) and 2nd Zagreb index ($M_2(H)$) are $M_1(H) = \sum_{v \in V(H)} [d_H(v)]^2 = \sum_{uv \in E(H)} [d_H(u) + d_H(v)]$ and $M_2(H) = \sum_{uv \in E(H)} [d_H(u) \times d_H(v)]$.

In 1972, Trinajsti and Gutman [19] explained the above TI's that are later utilized to calculate the structure based properties of (molecular) graphs like energy, connectivity, complexity, branching, chirality and hetero-system, see [16, 17]. However, there was another formula which was not studied by the mathematicians for many decades. In 2015, Gutman and Furtula [3] set a new name for this index as forgotten index and they have also got some meaningful

outcomes in the same paper. For the graph H , the F-index is given as follows:

$$F(H) = \sum_{(v) \in V(H)} [d_H(v)]^3 = \sum_{uv \in E(H)} [[d_H(u)]^2 + [d_H(v)]^2].$$

The four generalized operations related to the subdivision of graphs defined in [1] are given as follows:

- The graph $S_k(H)$ is gained by the addition of k nodes in each edge of H , where $k \geq 1$ is an integral value.
- In $R_k(H)$, we join the vertices of $S_k(H)$ that were adjacent in H .
- $Q_k(H)$ is gained by joining the new added nodes of the edges having one vertex common in basic graph H .
- Finally, $T_k(H)$ is gained by both the operations R_k and Q_k on the graph $S_k(H)$. For more explanation, see Figure 1.

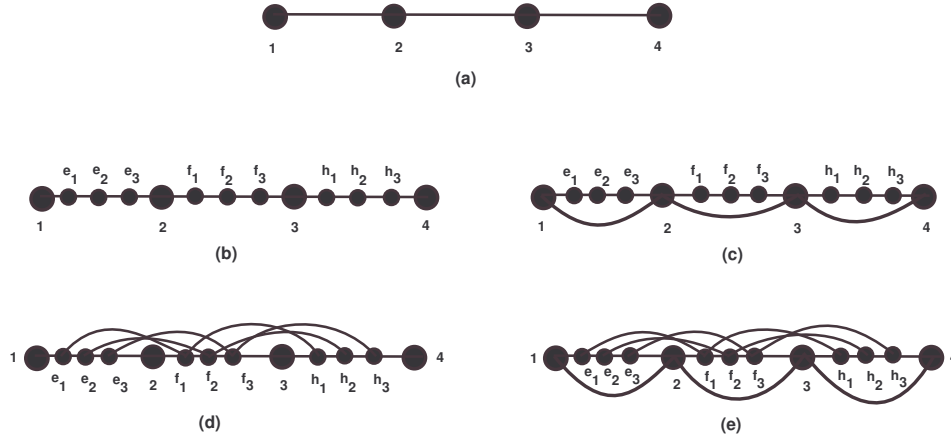


FIGURE 1. (a) $H \cong P_4$, (b) $S_3(H)$, (c) $R_3(H)$, (d) $Q_3(H)$ and (e) $T_3(H)$

Definition 2.2. Assume that H_1 & H_2 are two simple, undirected and connected graphs, $F_k \in \{S_k, R_k, Q_k, T_k\}$ is an operation and $F_k(H_1)$ is obtained after applying F_k on H_1 having edge-set $E(F_k(H_1))$ and node-set $V(F_k(H_1))$. The generalized F-sum graph $(H_1 +_{F_k} H_2)$ is a graph having node-set

$$V(H_1 +_{F_k} H_2) = V(F_k(H_1)) \times V(H_2) = (V(H_1) \cup E(H_1)) \times V(H_2)$$

such that two nodes (u_1, v_1) & (u_2, v_2) of $V(H_1 +_{F_k} H_2)$ are adjacent iff $[u_1 = u_2 \in V(H_1) \text{ \& } (v_1, v_2) \in E(H_2)]$ or $[v_1 = v_2 \in V(H_2) \text{ \& } (u_1, u_2) \in E(F_k(H_1))]$, where $k \geq 1$ is a positive number.

We noticed that the generalized F-sum graphs $(H_1 +_{F_k} H_2)$ contain $|V(H_2)|$ copies of graphs $F_k(H_1)$ that are labeled with the nodes of H_2 . For more explanation FIGURE 2 and FIGURE 3 are given below.

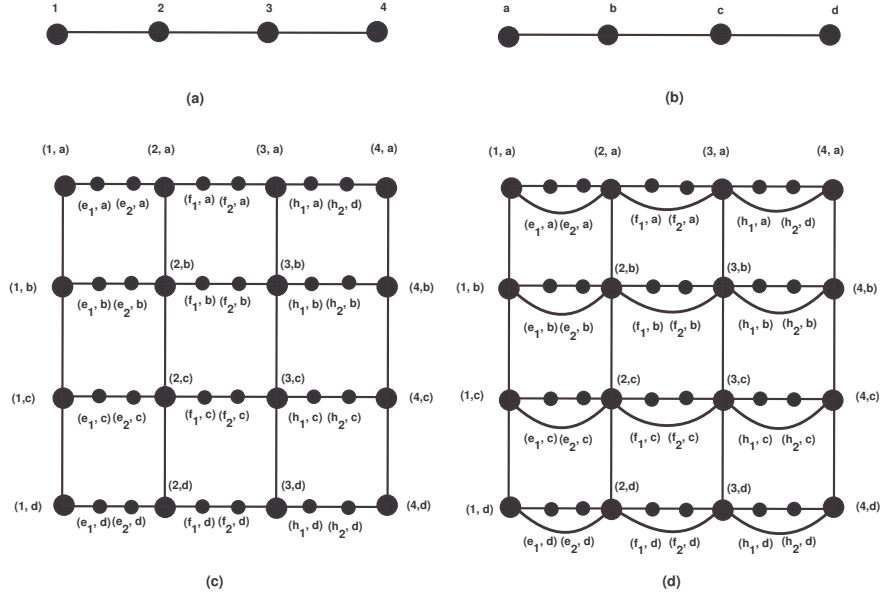


FIGURE 2. (a) $H_1 \cong P_4$, (b) $H_2 \cong P_4$, (c) $H_1 +_{S_2} H_2$, (d) $H_1 +_{R_2} H_2$

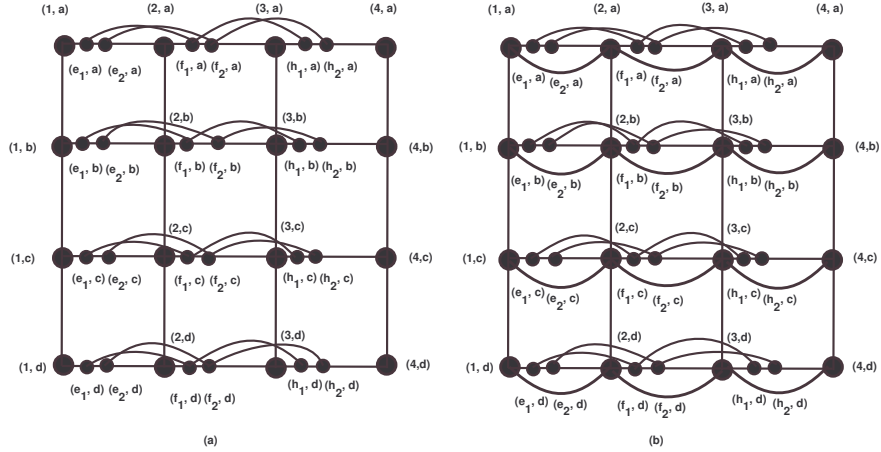


FIGURE 3. (a) $H_1 +_{Q_2} H_2$, (b) $H_1 +_{T_2} H_2$

3. MAIN RESULTS

Now, we prove the key results of F-index for miscellaneous classes of graphs which are described in previous section. Before the main results, we assume that H_1 and H_2 be two connected & simple graphs with order $|V(H_1)|$ and $|V(H_2)|$ respectively.

Theorem 3.1.

For $|V(H_1)|, |V(H_2)| \geq 4$ and $k \geq 1$, the F-index of $F(H_1 +_{S_k} H_2)$ is

$$|V(H_1)| F(H_2) + |V(H_2)| F(H_1) + 2|E(H_2)| M_1(H_1) + 6|E(H_1)| M_1(H_2) + 8|E(H_2)||E(H_1)| + 8|E(H_1)||V(H_2)| + 8|V(H_2)||E(H_1)|(k-1).$$

Proof. Let $d(s, t) = d_{H_1 +_{S_k} H_2}(s, t)$ for $(s, t) \in H_1 +_{S_k} H_2$.

$$F(H_1 +_{S_k} H_2) =$$

$$\begin{aligned} & \sum_{(s,t) \in V(H_1 +_{S_k} H_2)} d(s, t)^3 = \sum_{(s_1, t_1)(s_2, t_2) \in E(H_1 +_{S_k} H_2)} [d(s_1, t_1)^2 + d(s_2, t_2)^2] \\ &= \sum_{s \in V(H_1)} \sum_{t_1 t_2 \in E(H_2)} [d(s, t_1)^2 + d(s, t_2)^2] + \sum_{t \in V(H_2)} \sum_{s_1 s_2 \in E(S_k(H_1))} [d(s_1, t)^2 + d(s_2, t)^2] \\ &= \sum_{s \in V(H_1)} \sum_{t_1 t_2 \in E(H_2)} [d(s, t_1)^2 + d(s, t_2)^2] + \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(S_k(H_1)) \\ s_1 \in V(H_1), s_2 \in V(S_k(H_1) - H_1)}} [d(s_1, t)^2 + d(s_2, t)^2] \\ & \quad + \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(S_k(H_1)) \\ s_1, s_2 \in V(S_k(H_1) - H_1)}} [d(s_1, t)^2 + d(s_2, t)^2] \\ &= \sum 1 + \sum 2 + \sum 3. \end{aligned}$$

Consider,

$$\begin{aligned} \sum 1 &= \sum_{s \in V(H_1)} \sum_{t_1 t_2 \in E(H_2)} [(d_{H_1}(s) + d_{H_2}(t_1))^2 + (d_{H_1}(s) + d_{H_2}(t_2))^2] \\ &= \sum_{s \in V(H_1)} \sum_{t_1 t_2 \in E(H_2)} [2d_{H_1}(s)^2 + (d_{H_2}(t_1)^2 + d_{H_2}(t_2)^2) + 2d_{H_1}(s)(d_{H_2}(t_1) + d_{H_2}(t_2))] \\ &= \sum_{s \in V(H_1)} [|E(H_2)| \times 2d_{H_1}(s)^2 + F(H_2) + 2M_1(H_2)d_{H_1}(s)] \\ &= 2|E(H_2)| M_1(H_1) + |V(H_1)| F(H_2) + 4|E(H_1)| M_1(H_2), \end{aligned}$$

$$\sum 2 =$$

$$\begin{aligned}
& \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(S_k(H_1)) \\ s_1 \in V(H_1), s_2 \in V(S_k(H_1) - H_1)}} [(d_{S_k(H_1)}(s_1) + d_{H_2}(t))^2 + d_{S_k(H_1)}(s_2)^2] \\
&= \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(S_k(H_1)) \\ s_1 \in V(H_1), s_2 \in V(S_k(H_1) - H_1)}} [d_{S_k(H_1)}(s_1)^2 + d_{H_2}(t)^2 + 2d_{S_k(H_1)}(s_1)d_{H_2}(t) \\
&+ d_{S_k(H_1)}(s_2)^2]
\end{aligned}$$

Since $s_1 \in V(H_1)$ and $s_2 \in V(S_k(H_1) - H_1)$, therefore

$$\begin{aligned}
&= \sum_{t \in V(H_2)} [F(S_1(H_1)) + 2 \mid E(H_1) \mid d_{H_2}(t)^2 + 2 \times 2 \mid E(H_1) \mid d_{H_2}(t)] \\
&= \mid V(H_2) \mid F(S_1(H_1)) + 2 \mid E(H_1) \mid M_1(H_2) + 8 \mid E(H_2) \parallel E(H_1) \mid, \\
&\text{and}
\end{aligned}$$

$$\begin{aligned}
&\sum 3 = \\
&(k-1) \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(S_k(H_1)) \\ s_1, s_2 \in V(S_k(H_1) - H_1)}} [4+4]
\end{aligned}$$

Since in this case $|E(S_k(H_1))| = |E(H_1)|(k-1)$, so

$$= |E(H_1)|(k-1) \sum_{t \in V(H_2)} (8) = 8(k-1) \mid V(H_2) \parallel E(H_1).$$

Consequently,

$$\begin{aligned}
F(H_1 +_{S_k} H_2) &= \mid V(H_1) \mid F(H_2) + \mid V(H_2) \mid F(S_1(H_1)) + 2 \mid E(H_2) \mid M_1(H_1) \\
&+ 6 \mid E(H_1) \mid M_1(H_2) + 8 \mid E(H_2) \parallel E(H_1) \mid + 8 \mid V(H_2) \parallel E(H_1) \mid (k-1) \\
\text{Moreover, for } F(S_1(H_1)) &= F(H_1) + 8 \mid E(H_1) \mid, \text{ we have} \\
F(H_1 +_{S_k} H_2) &= \mid V(H_1) \mid F(H_2) + \mid V(H_2) \mid F(H_1) + 2 \mid E(H_2) \mid M_1(H_1) + 6 \\
&\mid E(H_1) \mid M_1(H_2) + 8 \mid E(H_2) \parallel E(H_1) \mid + 8 \mid E(H_1) \parallel V(H_2) \mid + 8 \mid V(H_2) \mid \\
&\mid E(H_1) \mid (k-1).
\end{aligned}$$

Theorem 3.2.

For $\mid V(H_1) \mid, \mid V(H_2) \mid \geq 4$ and $k \geq 1$, the F-index of $F(H_1 +_{R_k} H_2)$ is

$$\begin{aligned}
&8 \mid V(H_2) \mid F(H_1) + \mid V(H_1) \mid F(H_2) + 24 \mid E(H_2) \mid M_1(H_1) + 12 \mid E(H_1) \mid \\
&M_1(H_2) + 8 \mid E(H_1) \parallel V(H_2) \mid + 8(k-1) \mid V(H_2) \parallel E(H_1) \mid.
\end{aligned}$$

Proof.

Consider,

$$\begin{aligned}
 F(H_1 +_{R_k} H_2) &= \sum_{(s_1, t_1)(s_2, t_2) \in E(H_1 +_{R_k} H_2)} [d(s_1, t_1)^2 + d(s_2, t_2)^2] \\
 &= \sum_{s \in V(H_1)} \sum_{t_1 t_2 \in E(H_2)} [d(s, t_1)^2 + d(s, t_2)^2] + \sum_{t \in V(H_2)} \sum_{s_1 s_2 \in E(R_k(H_1))} [d(s_1, t)^2 + d(s_2, t)^2] \\
 &= \sum 1 + \sum 2.
 \end{aligned}$$

Consider,

$$\begin{aligned}
 \sum 1 &= \sum_{s \in V(H_1)} \sum_{t_1 t_2 \in E(H_2)} [(d_{R_k(H_1)}(s) + d_{H_2}(t_1))^2 + (d_{R_k(H_1)}(s) + d_{H_2}(t_2))^2] \\
 &= \sum_{s \in V(H_1)} \sum_{t_1 t_2 \in E(H_2)} [2d_{R_k(H_1)}(s)^2 + (d_{H_2}(t_1)^2 + d_{H_2}(t_2)^2) + 2d_{R_k(H_1)}(s)(d_{H_2}(t_1) \\
 &\quad + d_{H_2}(t_2))] \\
 &= \sum_{s \in V(H_1)} \sum_{t_1 t_2 \in E(H_2)} [8d_{H_1}(s)^2 + (d_{H_2}(t_1)^2 + d_{H_2}(t_2)^2) + 4d_{H_1}(s)(d_{H_2}(t_1) \\
 &\quad + d_{H_2}(t_2))] \\
 &= \sum_{s \in V(H_1)} [8 | E(H_2) | d_{H_1}(s)^2 + F(H_2) + 4M_1(H_2)d_{H_1}(s_1)] \\
 &= 8 | E(H_2) | M_1(H_1) + | V(H_1) | F(H_2) + 8 | E(H_1) | M_1(H_2),
 \end{aligned}$$

and

$$\begin{aligned}
 \sum 2 &= \sum_{t \in V(H_2)} \sum_{s_1 s_2 \in E(R_k(H_1))} [d(s_1, t)^2 + d(s_2, t)^2] \\
 &= \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(R_k(H_1)) \\ s_1, s_2 \in V(H_1)}} [d(s_1, t)^2 + d(s_2, t)^2] \\
 &+ \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(R_k(H_1)) \\ s_1 \in V(H_1) \\ s_2 \in V(R_k(H_1)) - V(H_1)}} [d(s_1, t)^2 + d(s_2, t)^2] \\
 &+ \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(R_k(H_1)) \\ s_1, s_2 \in V(R_k(H_1)) - V(H_1)}} [d(s_1, t)^2 + d(s_2, t)^2]
 \end{aligned}$$

$$= \sum' 2 + \sum'' 2 + \sum''' 2.$$

Consider for $s_1 s_2 \in V(H_1)$, we have $s_1 s_2 \in E(R_k(H_1))$ if and only if $s_1 s_2 \in E(H_1)$; for $s_1 \in V(H_1)$, we obtain $d_{R_k(H_1)}(s_1) = 2d_{H_1}(s_1)$ and for $s_2 \in V(R_k(H_1)) - V(H_1)$, we have $d_{R_k(H_1)}(s_2) = 2$. Now, consider

$$\begin{aligned} \sum' 2 &= \sum_{t \in V(H_2)} \sum_{s_1 s_2 \in E(H_1)} [d(s_1, t)^2 + d(s_2, t)^2] \\ &= \sum_{t \in V(H_2)} \sum_{s_1 s_2 \in E(H_1)} [(d_{R_k(H_1)}(s_1) + d_{H_2}(t))^2 + (d_{R_k(H_1)}(s_2) + d_{H_2}(t))^2] \\ &= \sum_{t \in V(H_2)} \sum_{s_1 s_2 \in E(H_1)} [d_{R_k(H_1)}(s_1)^2 + 2d_{H_2}(t)^2 + d_{R_k(H_1)}(s_2)^2 + 2d_{H_2}(t)(d_{R_k(H_1)}(s_1) \\ &\quad + d_{R_k(H_1)}(s_2))] \\ &= \sum_{t \in V(H_2)} \sum_{s_1 s_2 \in E(H_1)} [4d_{H_1}(s_1)^2 + 2d_{H_2}(t)^2 + 4d_{H_1}(s_2)^2 + 2d_{H_2}(t)(2d_{H_1}(s_1) + 2d_{H_1}(s_2))] \\ &= \sum_{t \in V(H_2)} [4F(H_1) + 2 | E(H_1) | d_{H_2}(t)^2 + 4M_1(H_1)d_{H_2}(t)] \\ &= 4 | V(H_2) | F(H_1) + 2 | E(H_1) | M_1(H_2) + 8 | E(H_2) | M_1(H_1), \end{aligned}$$

$$\begin{aligned} \sum'' 2 &= \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(R_k(H_1)) \\ s_1 \in V(H_1) \\ s_2 \in V(R_k(H_1)) - V(H_1)}} [(d_{R_k(H_1)}(s_1) + d_{H_2}(t))^2 + d_{R_k(H_1)}(s_2)^2] \\ &= \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(R_k(H_1)) \\ s_1 \in V(H_1) \\ s_2 \in V(R_k(H_1)) - V(H_1)}} [d_{R_k(H_1)}(s_1)^2 + d_{H_2}(t)^2 + 2(d_{R_k(H_1)}(s_1)d_{H_2}(t)) \\ &\quad + d_{R_k(H_1)}(s_2)^2] \\ &= \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(R_k(H_1)) \\ s_1 \in V(H_1) \\ s_2 \in V(R_k(H_1)) - V(H_1)}} [4d_{H_1}(s_1)^2 + d_{H_2}(t)^2 + 4(d_{H_1}(s_1)d_{H_2}(t)) + 4] \\ &= \sum_{t \in V(H_2)} [4F(H_1) + 2 | E(H_1) | d_{H_2}(t)^2 + 4M_1(H_1)d_{H_2}(t) + 4 \times 2 | E(H_1) |] \end{aligned}$$

$$= 4 \mid V(H_2) \mid F(H_1)+2 \mid E(H_1) \mid M_1(H_2)+8 \mid E(H_2) \mid M_1(H_1)+8 \\ \mid V(H_2) \parallel E(H_1) \mid ,$$

and

$$\begin{aligned} & \sum''' 2 = \\ & (k-1) \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(R_k(H_1)) \\ s_1, s_2 \in V(R_k(H_1)) - V(H_1)}} [d_{R_k(H_1)}(s_1)^2 + d_{R_k(H_1)}(s_2)^2] \\ & = 8(k-1) \mid V(H_2) \parallel E(H_1) \mid . \end{aligned}$$

Hence

$$\begin{aligned} F(H_1 +_{R_k} H_2) &= 8 \mid V(H_2) \mid F(H_1) + \mid V(H_1) \mid F(H_2) + 24 \mid E(H_2) \mid M_1(H_1) \\ &+ 12 \mid E(H_1) \mid M_1(H_2) + 8 \mid E(H_1) \parallel V(H_2) \mid + 8(k-1) \mid V(H_2) \parallel E(H_1) \mid . \end{aligned}$$

Theorem 3.3.

For $\mid V(H_1) \mid, \mid V(H_2) \mid \geq 4$ and $k \geq 1$, the F-index of $F(H_1 +_{Q_k} H_2)$ is

$$\begin{aligned} & 3 \mid V(H_2) \mid F(H_1) + \mid V(H_1) \mid F(H_2) + 6 \mid E(H_2) \mid M_1(H_1) + 6 \mid E(H_1) \mid \\ & M_1(H_2) + 4 \mid V(H_2) \mid M_2(H_1) + (k) \mid V(H_2) \mid [M_4(H_1) - F(H_1) - 4M_2(H_1) \\ & + \sum_{u \in V(H_1)} \left(\sum_{v \in N_{H_1}(u)} d_{H_1}(u)(d_{H_1}(v) - 1) \right) + \sum_{uv \in E(H_1)} d_{H_1}(u)d_{H_1}(v)(d_{H_1}(u) \\ & + d_{H_1}(v))] + 2(k-1) \mid V(H_2) \mid F(H_1). \end{aligned}$$

Proof.

Consider,

$$\begin{aligned} F(H_1 +_{Q_k} H_2) &= \sum_{(s_1, t_1)(s_2, t_2) \in E(H_1 +_{Q_k} H_2)} [d(s_1, t_1)^2 + d(s_2, t_2)^2] \\ &= \sum_{s \in V(H_1)} \sum_{t_1 t_2 \in E(H_2)} [d(s, t_1)^2 + d(s, t_2)^2] + \sum_{t \in V(H_2)} \sum_{s_1 s_2 \in E(Q_k(H_2))} [d(s_1, t)^2 + d(s_2, t)^2] \\ &= \sum 1 + \sum 2. \end{aligned}$$

Now

$$\begin{aligned} & \sum 1 = \\ & \sum_{s \in V(H_1)} \sum_{t_1 t_2 \in E(H_2)} [(d_{H_1}(s) + d_{H_2}(t_1))^2 + (d_{H_1}(s) + d_{H_2}(t_2))^2] \\ &= \sum_{s \in V(H_1)} \sum_{t_1 t_2 \in E(H_2)} [2d_{H_1}(s)^2 + (d_{H_2}(t_1)^2 + d_{H_2}(t_2)^2) + 2d_{H_1}(s)(d_{H_2}(t_1) + d_{H_2}(t_2))] \end{aligned}$$

$$\begin{aligned}
&= \sum_{s \in V(H_1)} [|E(H_2)| \times 2d_{H_1}(s)^2 + F(H_2) + 2M_1(H_2)d_{H_1}(s)] \\
&= 2 [|E(H_2)| M_1(H_1) + |V(H_1)| F(H_2) + 4 |E(H_1)| M_1(H_2)],
\end{aligned}$$

and

$$\begin{aligned}
&\sum 2 = \\
&\sum_{t \in V(H_2)} \sum_{s_1 s_2 \in E(Q_k(H_1))} [d(s_1, t)^2 + d(s_2, t)^2] \\
&= \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(Q_k(H_1)) \\ s_1 \in V(H_1) \\ s_2 \in V(Q_k(H_1)) - V(H_1)}} [d(s_1, t)^2 + d(s_2, t)^2] \\
&+ \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(Q_k(H_1)) \\ s_1 s_2 \in V(Q_k(H_1)) - V(H_1)}} [d(s_1, t)^2 + d(s_2, t)^2] \\
&= \sum' 2 + \sum'' 2.
\end{aligned}$$

Now we talk,

$$\begin{aligned}
&\sum' 2 = \\
&\sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(Q_k(H_2)) \\ s_1 \in V(H_1) \\ s_2 \in V(Q_k(H_1)) - V(H_1)}} [(d_{Q_k(H_1)}(s_1) + d_{H_2}(t))^2 + d_{Q_k(H_1)}(s_2)^2] \\
&= \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(Q_k(H_2)) \\ s_1 \in V(H_1) \\ s_2 \in V(Q_k(H_1)) - V(H_1)}} [d_{Q_k(H_1)}(s_1)^2 + d_{H_2}(t)^2 + 2d_{Q_k(H_1)}(s_1)d_{H_2}(t) \\
&+ d_{Q_k(H_1)}(s_2)^2] \\
&= \sum_{t \in V(H_2)} \sum_{s_1 \in V(H_1)} [d_{Q_k(H_1)}(s_1)^2 + d_{H_2}(t)^2 + 2d_{Q_k(H_1)}(s_1)d_{H_2}(t)] \\
&+ \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(Q_k(H_2)) \\ s_1 \in V(H_1) \\ s_2 \in V(Q_k(H_1)) - V(H_1)}} [d_{Q_k(H_1)}(s_2)^2] \\
&= |V(H_2)| F(H_1) + 2 |E(H_1)| M_1(H_2) + 4 |E(H_2)| M_1(H_1) + \sum_{t \in V(H_2)}
\end{aligned}$$

$$\sum_{\substack{s_1 s_2 \in E(Q_k(H_1)) \\ s_1 \in V(H_1) \\ s_2 \in V(Q_k(H_1)) - V(H_1)}} [d_{Q_k(H_1)}(s_2)^2]$$

For $s_2 \in V(Q_k(H_1)) - V(H_1)$ that is introduced into the edges uv of (H_1) , we have $d_{Q_k(H_1)}(s_2) = d_{H_1}(u) + d_{H_1}(v)$. This gives

$$\begin{aligned} \sum_{\substack{s_1 s_2 \in E(Q_k(H_1)) \\ s_1 \in V(H_1) \\ s_2 \in V(Q_k(H_1)) - V(H_1)}} [d_{Q_k(H_1)}(s_2)^2] &= 2 \sum_{uv \in E(H_1)} [d_{H_1}(u) + d_{H_1}(v)]^2 \\ &= 2 \sum_{uv \in E(H_1)} [d_{H_1}(u)^2 + d_{H_1}(v)^2 + 2d_{H_1}(u)d_{H_1}(v)] = 2F(H_1) + 4M_2(H_1) \end{aligned}$$

and

$$\begin{aligned} &= \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(Q_k(H_1)) \\ s_1 \in V(H_1) \\ s_2 \in V(Q_k(H_1)) - V(H_1)}} [d_{Q_k(H_1)}(s_2)]^2 \\ &= 2 \mid V(H_2) \mid F(H_1) + 4 \mid V(H_2) \mid M_2(H_1) \end{aligned}$$

so,

$$\begin{aligned} \sum' 2 &= \mid V(H_2) \mid F(H_1) + 2 \mid E(H_1) \mid M_1(H_2) + 4 \mid E(H_2) \mid M_1(H_1) + 2 \mid V(H_2) \mid F(H_1) \\ &\quad + 4 \mid V(H_2) \mid M_2(H_1), \\ &= 3 \mid V(H_2) \mid F(H_1) + 2 \mid E(H_1) \mid M_1(H_2) + 4 \mid E(H_2) \mid M_1(H_1) + 4 \mid V(H_2) \mid \\ &\quad M_2(H_1), \end{aligned}$$

and

$$\begin{aligned} \sum'' 2 &= \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(Q_k(H_1)) \\ s_1, s_2 \in V(Q_k(H_1)) - V(H_1)}} [d(s_1, t)^2 + d(s_2, t)^2] \end{aligned}$$

Now we divide $\sum'' 2$ into $\sum 3$ and $\sum 4$ for the nodes, s_1 and s_2 , where $s_1 s_2 \in V(Q_k(H_1)) - V(H_1)$. So $\sum'' 2 = \sum 3 + \sum 4$, where $\sum 3$ include the edges of $Q_k(H_1)$ having the similar edges of H_1 and $\sum 4$ of $Q_k(H_1)$ in two dissimilar adjacent edges of H_1 .

$$\sum 3 =$$

$$\begin{aligned}
& \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(Q_k(H_1)) \\ s_1 s_2 \in V(Q_k(H_1)) - V(H_1)}} [d_{Q_k(H_1)}(s_1)^2 + d_{Q_k(H_1)}(s_2)^2] \\
&= (k-1)2 \sum_{t \in V(H_2)} \sum_{uv \in E(H_1)} [d_{H_1}(u)^2 + d_{H_1}(v)^2] = 2(k-1) |V(H_2)| |F(H_1)|,
\end{aligned}$$

and

$$\begin{aligned}
& \sum 4 = \\
& \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(Q_k(H_1)) \\ s_1, s_2 \in V(Q_k(H_1)) - V(H_1)}} [d_{Q_k(H_1)}(s_1)^2 + d_{Q_k(H_1)}(s_2)^2] \\
&= (k) \sum_{t \in V(H_2)} \sum_{\substack{uv \in E(H_1) \\ vw \in E(H_2)}} [(d_{H_1}(u) + d_{H_1}(v))^2 + (d_{H_1}(v) + d_{H_1}(w))^2],
\end{aligned}$$

where s_1 and s_2 are the nodes that are introduced into the edge set of uv and vw of H_1 .

$$\begin{aligned}
&= (k) |V(H_2)| \left[\sum_{u \in V(H_1)} d_{H_1}(v)^4 - d_{H_1}(v)^3 + \sum_{v \in V(H_1)} (d_{H_1}(v) - 1) \sum_{\substack{u \in V(H_1) \\ uv \in E(H_1)}} d_{H_1}(u)^2 \right. \\
&+ 2 \sum_{v \in V(H_1)} d_{H_1}(v)(d_{H_1}(v) - 1) \sum_{\substack{u \in V(H_1) \\ uv \in E(H_1)}} d_{H_1}(u) \left. \right] \\
&= (k) |V(H_2)| [M_4(H_1) - F(H_1) - 4M_2(H_1) + \sum_{u \in V(H_1)} \left(\sum_{v \in N_{H_1}(u)} d_{H_1}(u) \right. \\
&\left. (d_{H_1}(v) - 1) + \sum_{uv \in E(H_1)} d_{H_1}(u)d_{H_1}(v)(d_{H_1}(u) + d_{H_1}(v)) \right)],
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
& 3 |V(H_2)| |F(H_1)| + |V(H_1)| |F(H_2)| + 6 |E(H_2)| |M_1(H_1)| + 6 |E(H_1)| |M_1(H_2)| \\
& + 4 |V(H_2)| |M_2(H_1)| + (k) |V(H_2)| [M_4(H_1) - F(H_1) - 4M_2(H_1) \\
& + \sum_{u \in V(H_1)} \left(\sum_{v \in N_{H_1}(u)} d_{H_1}(u)(d_{H_1}(v) - 1) \right) + \sum_{uv \in E(H_1)} d_{H_1}(u)d_{H_1}(v)(d_{H_1}(u) \\
& + d_{H_1}(v))] + 2(k-1) |V(H_2)| |F(H_1)|.
\end{aligned}$$

Theorem 3.4.

For $|V(H_1)|, |V(H_2)| \geq 4$ and $k \geq 1$, the F-index of $F(H_1 +_{T_k} H_2)$ is $9 |V(H_2)| |F(H_1)| + |V(H_1)| |F(H_2)| + 24 |E(H_2)| |M_1(H_1)| + 12 |E(H_1)|$

$$\begin{aligned}
 & M_1(H_2)+4 \mid V(H_2) \mid M_2(H_1)+(k) \mid V(H_2) \mid [M_4(H_1)-F(H_1)-4M_2(H_1) \\
 & + \sum_{u \in V(H_1)} \left(\sum_{v \in N_{H_1}(u)} d_{H_1}(u)(d_{H_1}(v)-1) \right) + \sum_{uv \in E(H_1)} d_{H_1}(u)d_{H_1}(v)(d_{H_1}(u) \\
 & + d_{H_1}(v))] + 2(k-1) \mid V(H_2) \mid F(H_1).
 \end{aligned}$$

Proof. Proof is same as of Theorem 4.2 and Theorem 4.3.

4. CONCLUSION

In this paper, we computed the F-index of the generalized F-sum graphs. For $m_1, m_2 \geq 4$ and $k = 4$, assume that $H_1 = P_{m_1}$ and $H_2 = P_{m_2}$ are particular alkane called as paths of orders m_1 and m_2 respectively. Then the following outcomes are the direct consequences of the achieved results.

- $F(P_{m_1+S_4}P_{m_2}) = 88m_1m_2 - 90m_1 - 66m_2 + 56$,
- $F(P_{m_1+R_4}P_{m_2}) = 248m_1m_2 - 336m_1 - 182m_2 + 216$,
- $F(P_{m_1+Q_4}P_{m_2}) = 264m_1m_2 - 522m_1 - 74m_2 + 72$,
- $F(P_{m_1+T_4}P_{m_2}) = 416m_1m_2 - 655m_1 - 182m_2 + 216$.

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