Journal of Prime Research in Mathematics Vol. 15 *(2019),* 129-146

SOME NEW ESTIMATES OF GENERALIZED(h_1, h_2)-CONVEX FUNCTIONS

FARHAT SAFDAR[1], MUHAMMAD ASLAM NOOR[2], KHALIDA INAYAT NOOR[3], SAIMA RASHID[4]

Abstract. In this article, we introduce a new class of generalized convex functions involving two arbitrary auxiliary functions $h_1, h_2 : I = [a, b] \subseteq \mathbb{R} \to \mathbb{R}$, which is called generalized (h_1, h_2) convex functions. We obtain several new classes of convex functions as special cases. We derive some new integral inequalities for generalized convex functions. Several special cases are also discussed. Results proved in this paper can be viewed as new significant contributions in this area.

Key words: Generalized convex functions, generalized (h_1, h_2)-convex functions, Hermite-Hadamard inequalities.
AMS Subject: Primary:14H50, 14H20, Secondary: 32S15.

1. Introduction

Hermite Hadamard inequalities for convex functions are used to find the estimates of the mean value of continuous convex function. It is well known that a function f is convex, if and only if, it satisfies Hermite hadamard inequality.

Let I be an interval and $f : I = [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function. Then

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2} \quad \forall a, b \in I, t \in [0, 1]. \tag{1}$$

This double inequality is called the Hermite-Hadamard integral inequality for convex functions, see [11, 12, 15]. Related to integral inequalities, we have variational inequalities, which can be viewed as novel extension of the variational principles. It is also well known that the optimality conditions can be

[1] SBK Women's University, Quetta. Pakistan. Email: farhat_900@yahoo.com
[2]COMSATS University Islamabad, Islamabad, Pakistan. Email: noormaslam@gmail.com
[3]COMSATS University Islamabad, Islamabad. Email: khalidan@gmail.com
[4] Government College University, Faisalabad, Pakistan. Email: saimarashid@gcuf.edu.pk.

characterized by the variational inequalities. In fact, the minimum $u \in K$ of a differentiable convex functions is equivalent to finding $u \in K$, such that

$$\langle f'(u), v - u \rangle \geq 0, \quad \forall v \in K, \tag{2}$$

is called the variational inequality, where K is a closed convex set in a normed space. For the applications, formulation, generalizations, extensions and other aspects of variational inequalities, see [3, 16, 17, 18, 19]. In recent years, these inequalities have triggered huge amount of attention and interest, and these are the most investigated inequalities, see [1, 4, 7, 8, 15, 27, 28].

In recent years, several new classes of convex functions and convex sets have been introduced and investigated. A significant generalization of convex functions is the introduction of h-convex functions by Varosanec [32]. She studied the basic properties and showed that h-convex functions include s-convex [2], p convex [7] and godunova-levine [10] functions are its special cases. For different properties and other aspects of h-convex functions, readers are referred to see [13, 14, 28, 29, 30, 31]. Gordji et al. [8] also introduced an other important class of convex functions, which is called the generalized convex (φ-convex) function. These generalized convex functions are nonconvex functions. For recent developments, see [5, 9, 20, 21, 22, 23, 26] and the references therein.

Inspired by the ongoing research, we introduce the notion of generalized (h_1, h_2)-convex functions involving two auxiliary functions namely $h_1, h_2 : J \to \mathbb{R}$. Our results include a wide class of known and new inequalities. Results obtained in this paper continue to hold for the various classes of convex functions, which can be obtained as special cases. The ideas and techniques may stimulate further research.

2. Preliminaries

Let $I = [a, b]$ and J be the intervals in real line $\mathbb{R}$, $[0, 1] \subseteq J$ and let $h_1, h_2 : J \to \mathbb{R}$ be two real functions. Let $f : I = [a, b] \to \mathbb{R}$ be a nonnegative and continuous functions and $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous bifunction.

Definition 1. [8] *A function $f : I = [a, b] \to \mathbb{R}$ is said to be generalized convex function with respect to a bifunction $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if*

$$f((1 - t)a + tb) \leq (1 - t)f(a) + t[f(a) + \eta(f(b), f(a))], \forall a, b \in I, t \in [0, 1].$$

Definition 2. [23] *Let $h : J \to \mathbb{R}$ be a non-negative function. A function $f : I = [a, b] \to \mathbb{R}$ is said to be generalized h-convex function with respect to a bifunction $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if*

$$f((1 - t)a + tb) \leq h(1 - t)f(a) + h(t)[f(a) + \eta(f(b), f(a))], \quad \forall a, b \in I, t \in [0, 1]. \tag{3}$$

Now we introduce a new class of generalized convex functions with respect to two arbitrary functions, which is the main motivation of this paper.

Definition 3. *Let $h_1, h_2 : J \to \mathbb{R}$ be a two real functions, $h_1, h_2 \neq 0$. A function $f : I = [a, b] \to \mathbb{R}$ is said to be generalized (h_1, h_2)-convex function with respect to a bifunction $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if*

$$f((1-t)a + tb) \leq h_1(1-t)h_2(t)f(a) + h_1(t)h_2(1-t)[f(a) + \eta(f(b), f(a))], \quad \forall a, b \in I, t \in [0, 1]. \tag{4}$$

If $t = \frac{1}{2}$, then (4) reduces to

$$\begin{aligned} f(\frac{a+b}{2}) &\leq h_1(\frac{1}{2})h_2(\frac{1}{2})f(a) + h_2(\frac{1}{2})h_1(\frac{1}{2})[f(a) + \eta(f(b), f(a))] \\ &= h_1(\frac{1}{2})h_2(\frac{1}{2})[2f(a) + \eta(f(b), f(a))]. \end{aligned} \tag{5}$$

The function f is known as generalized Jensen (h_1, h_2)-convex function.

Now we will discuss some special cases of generalized (h_1, h_2)-convex function.

(I). If $\eta(f(b), f(a)) = f(b) - f(a)$, then

Definition 4. [24] *Let $h_1, h_2 : J \to \mathbb{R}$ be a two real functions, $h_1, h_2 \neq 0$. A function $f : I = [a, b] \to \mathbb{R}$ is said to be (h_1, h_2)-convex function, if*

$$f((1-t)a + tb) \leq h_1(1-t)h_2 f(a) + h_1(t)h_2(1-t)f(b), \quad \forall a, b \in I, t \in [0, 1].$$

(II). If $h_2(t) = 1$ and $\eta(f(b), f(a)) = f(b) - f(a)$, then Definition 3 reduces to the definition for h-convex functions [32].

(III). If $h_1(t) = 1 = h_2(t)$ and $\eta(f(b), f(a)) = f(b) - f(a)$, then Definition 3 reduces to the definition for p-functions [7].

(IV). If $h_1(t) = t^s = h_2(t)$, then we obtain the following new definition of s-convex functions of third kind.

Definition 5. *A function $f : I = [a, b] \to \mathbb{R}$ is said to be generalized (s, tgs)-convex for $s \in [0, 1]$ with respect to a bifunction $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if*

$$f((1-t)a + tb) \leq t^s(1-t)^s[2f(a) + \eta(f(b), f(a))], \forall a, b \in I, t \in [0, 1].$$

Note that if we take $s = 1$ and $\eta(f(b), f(a)) = f(b) - f(a)$, then we have the definition of tgs-convex functions [31].

(V). If $h_1(t) = t^{-s} = h_2(t)$, then we have the following new class of generalized s-Godunova-Levin-Dragomir tgs-convex functions.

Definition 6. *A function $f : I = [a, b] \to \mathbb{R}$ is said to be generalized s-Godunova-Levin-Dragomir tgs-convex functions for $s \in (0, 1]$ with respect to a bifunction $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if*

$$f((1-t)a + tb) \leq \frac{1}{t^s} \frac{1}{(1-t)^s}[2f(a) + \eta(f(b), f(a))], \quad \forall a, b \in I, t \in [0, 1].$$

Note that if we take $s = 1$ and $\eta(f(b), f(a)) = f(b) - f(a)$, then we have the definition of Godunova type of tgs-convex functions.

(V). If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$, then we have the following new class of generalized Breckner type of (s_1, s_2)-convex functions.

Definition 7. *A function $f : I = [a, b] \to \mathbb{R}$ is said to be generalized Breckner type of (s_1, s_2)-convex functions for $s_1, s_2 \in (0, 1]$, with respect to a bifunction $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if*

$$f((1-t)a + tb) \leq t^{s_1}(1-t)^{s_2}[f(a)] + t^{s_2}(1-t)^{s_1}[f(a) + \eta(f(b), f(a))], \quad \forall a, b \in I, t \in [0, 1].$$

(VI). If $h_1(t) = t^{-s_1}$ and $h_2(t) = t^{-s_2}$, then we have the following new class of generalized Godunova-Levin-Dragomir type of (s_1, s_2)-convex functions.

Definition 8. *A function $f : I = [a, b] \to \mathbb{R}$ is said to be generalized Godunova-Levin-Dragomir type of (s_1, s_2)-convex functions for $s_1, s_2 \in (0, 1]$, with respect to a bifunction $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if*

$$f((1-t)a + tb) \leq \frac{1}{t^{s_1}} \frac{1}{(1-t)^{s_2}}[f(a)] + \frac{1}{t^{s_2}} \frac{1}{(1-t)^{s_1}}[f(a) + \eta(f(b), f(a))], \quad \forall a, b \in I, t \in [0, 1].$$

Hence for different suitable choice of function (h_1, h_2), we can obtain several new and known class of convex functions. This shows that the concept of generalized (h_1, h_2)-convex function is quite general and unifying one.

Lemma 1. [27] *Let $f : I = [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in L(a, b)$, then*

$$\int_a^b (u-a)^\alpha (b-u)^\beta f(u)\mathrm{d}u = (b-a)^{\alpha+\beta+1} \int_0^1 t^\alpha (1-t)^\beta f((1-t)a + tb)\mathrm{d}t.$$

Lemma 2. [14] *If $f^{(n)}$ for $n \in \mathbb{N}$ exists and is integrable on $[a, b]$, then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) = \frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)f^{(n)}((1-t)a + tb)\mathrm{d}t.$$

3. Main results

In this section, we establish our main results.

Theorem 3. *Let $h : J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function. Let $f : I = [a, b] \to \mathbb{R}$ be generalized (h_1, h_2)-convex function on I. If $f \in L(a, b)$ and $h_1(\frac{1}{2})h_2(\frac{1}{2}) \neq 0$. Then*

$$\begin{aligned}\frac{f(\frac{a+b}{2})}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} &- \frac{1}{2(b-a)}\int_a^b [\eta(f(a+b-x), f(x))]\mathrm{d}x \\ &\leq \frac{1}{(b-a)}\int_a^b f(x)\mathrm{d}x \\ &\leq [2f(a) + \eta(f(b), f(a))]\int_0^1 h_1(t)h_2(1-t)\mathrm{d}t.\end{aligned}$$

Proof. Let f be a generalized (h_1, h_2)-convex function. Then

$$\begin{aligned}f((1-t)a + tb)) &\leq h_1(t)h_2(1-t)[f(a)] \\ &+ h_2(t)h_1(1-t)[f(a) + \eta\big(f(b), f(a)\big)]. \qquad (6)\end{aligned}$$

Integrating (6) with respect to t on $[0, 1]$, we have

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq [2f(a) + \eta(f(b), f(a))]\int_0^1 h_1(t)h_2(1-t)\mathrm{d}t. \qquad (7)$$

Using (5) and substituting $x = (1-t)a + tb$ and $y = ta + (1-t)b$, we have

$$f(\frac{a+b}{2}) \leq h_1(\frac{1}{2})h_2(\frac{1}{2})[2f((1-t)a + tb) + \eta(f(ta + (1-t)b), f((1-t)a + t))].$$

Integrating the above inequality with respect to t on [0,1], we have

$$f(\frac{a+b}{2}) \leq \frac{h_1(\frac{1}{2})h_2(\frac{1}{2})}{(b-a)}\int_a^b [2f(x) + \eta(f(a+b-x), f(x))]\mathrm{d}x. \qquad (8)$$

This implies

$$\frac{f(\frac{a+b}{2})}{h_1(\frac{1}{2})h_2(\frac{1}{2})} - \frac{1}{(b-a)}\int_a^b [\eta(f(a+b-x), f(x))]\mathrm{d}x \leq \frac{2}{(b-a)}\int_a^b f(x)\mathrm{d}x. \qquad (9)$$

From (7) and (9), we have

$$\begin{aligned}\frac{f(\frac{a+b}{2})}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} &- \frac{1}{2(b-a)}\int_a^b [\eta(f(a+b-x), f(x))]\mathrm{d}x \\ &\le \frac{1}{(b-a)}\int_a^b f(x)\mathrm{d}x \\ &\le [2f(a)+\eta(f(b), f(a))]\int_0^1 h_1(t)h_2(1-t)\mathrm{d}t.\end{aligned}$$

which is the required result. □

Corollary 4. *If $\eta(f(b), f(a)) = f(b) - f(a)$, then, under the assumptions of Theorem 3, we have a known result* [24].

$$\begin{aligned}\frac{f(\frac{a+b}{2})}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} &\le \frac{1}{(b-a)}\int_a^b f(x)\mathrm{d}x \\ &\le [f(a)+f(b)]\int_0^1 h_1(t)h_2(1-t)\mathrm{d}t.\end{aligned}$$

Corollary 5. *If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$, then, under the assumptions of Theorem 3, we have a result for generalized Breckner type of (s_1, s_2)-convex functions.*

$$\begin{aligned}\frac{f(\frac{a+b}{2})}{2^{1-s_1-s_2}} &- \frac{1}{2(b-a)}\int_a^b [\eta(f(a+b-x), f(x))]\mathrm{d}x \\ &\le \frac{1}{(b-a)}\int_a^b f(x)\mathrm{d}x \\ &\le [2f(a)+\eta(f(b), f(a))]\mathbb{B}(s_1+1, s_2+1).\end{aligned}$$

Corollary 6. *If $h_1(t) = t^{s_1}$, $h_2(t) = t^{s_2}$ and $\eta(f(b), f(a)) = f(b) - f(a)$ then, under the assumptions of Theorem 3, we have a known result* [23].

$$\begin{aligned}\frac{f(\frac{a+b}{2})}{2^{1-s_1-s_2}} &\le \frac{1}{(b-a)}\int_a^b f(x)\mathrm{d}x \\ &\le [f(a)+f(b)]\mathbb{B}(s_1+1, s_2+1).\end{aligned}$$

Corollary 7. *If $h_1(t) = t^{-s_1}$ and $h_2(t) = t^{-s_2}$, then, under the assumptions of Theorem 3, we have a result for generalized Godunova- Levin-Dragomir type* *of (s_1, s_2)-convex functions.*

$$\begin{aligned} \frac{f(\frac{a+b}{2})}{2^{1+s_1+s_2}} &- \frac{1}{2(b-a)}\int_a^b [\eta(f(a+b-x), f(x))]\mathrm{d}x \\ &\leq \frac{1}{(b-a)}\int_a^b f(x)\mathrm{d}x \\ &\leq [2f(a)+\eta(f(b), f(a))]\mathbb{B}(1-s_1, 1-s_2). \end{aligned}$$

Corollary 8. *If $h_1(t) = t^{-s_1}$, $h_2(t) = t^{-s_2}$, and $\eta(f(b), f(a)) = f(b) - f(a)$ then, under the assumptions of Theorem 3, we have a known result* [23].

$$\begin{aligned} \frac{f(\frac{a+b}{2})}{2^{1+s_1+s_2}} &\leq \frac{1}{(b-a)}\int_a^b f(x)\mathrm{d}x \\ &\leq [f(a)+f(b)]\mathbb{B}(1-s_1, 1-s_2). \end{aligned}$$

Theorem 9. *Let $f, g : I = [a, b] \to \mathbb{R}$ be generalized (h_1, h_2)-convex functions such that $h_1^2(\frac{1}{2})h_2^2(\frac{1}{2}) \neq 0$. If $fg \in L(a, b)$, then we have*

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x &\leq M(a,b)\int_0^1 h_1^2(t)h_2^2(1-t)\mathrm{d}t \\ &+N(a,b)\int_0^1 h_1(t)h_2(1-t)h_2(t)h_1(1-t)\mathrm{d}t, \end{aligned}$$

where

$$M(a,b) = \Big[[f(a)g(a)] + [f(a)+\eta\big(f(b), f(a)\big)][g(a)+\eta\big(g(b), g(a)\big)]\Big]. \quad (10)$$

$$N(a,b) = \Big[f(a)[g(a)+\eta\big(g(b), g(a)\big)] + g(a)[f(a)+\eta\big(f(b), f(a)\big)]\Big], \quad (11)$$

respectively.

Proof. Let f, g be a generalized (h_1, h_2)-convex functions. Then

$$\begin{aligned} f((1-t)a+tb)) &\leq h_1(t)h_2(1-t)[f(a)] + h_2(t)h_1(1-t)[f(a)+\eta\big(f(b), f(a)\big)]. \\ g((1-t)a+tb)) &\leq h_1(t)h_2(1-t)[g(a)] + h_2(t)h_1(1-t)[g(a)+\eta\big(g(b), g(a)\big)]. \end{aligned}$$

Consider

$$\begin{aligned}
& f((1-t)a+tb))g((1-t)a+tb)) \\
\leq\ & h_1(t)h_2(1-t)[f(a)]+h_2(t)h_1(1-t)[f(a)+\eta\big(f(b),f(a)\big)][h_1(t)h_2(1-t)[g(a)] \\
& +h_2(t)h_1(1-t)[g(a)+\eta\big(g(b),g(a)\big)]] \\
=\ & h_1^2(t)h_2^2(1-t)[f(a)g(a)]+h_1(t)h_2(1-t)h_2(t)h_1(1-t)[f(a)\{[g(a)+\eta\big(g(b),g(a)\big)] \\
& +g(a)[f(a)+\eta\big(f(b),f(a)\big)]\}] \\
& +h_2^2(t)h_1^2(1-t)[[f(a)+\eta\big(f(b),f(a)\big)][g(a)+\eta\big(g(b),g(a)\big)]] \\
=\ & h_1^2(t)h_2^2(1-t)\Big[[f(a)g(a)]+[f(a)+\eta\big(f(b),f(a)\big)][g(a)+\eta\big(g(b),g(a)\big)]\Big] \\
& +h_1(t)h_2(1-t)h_2(t)h_1(1-t)\Big[f(a)[g(a)+\eta\big(g(b),g(a)\big)] \\
& +g(a)[f(a)+\eta\big(f(b),f(a)\big)]\Big].
\end{aligned} \tag{12}$$

Integrating (12) with respect to t on $[0,1]$, we have

$$\begin{aligned}
& \int_0^1 f((1-t)a+tb))g((1-t)a+tb))\mathrm{d}t \\
\leq\ & \Big[[f(a)g(a)]+[f(a)+\eta\big(f(b),f(a)\big)][g(a)+\eta\big(g(b),g(a)\big)]\Big]\int_0^1 h_1^2(t)h_2^2(1-t)\mathrm{d}t \\
& +\Big[f(a)[g(a)+\eta\big(g(b),g(a)\big)]+g(a)[f(a)+\eta\big(f(b),f(a)\big)]\Big] \\
& \int_0^1 h_1(t)h_2(1-t)h_2(t)h_1(1-t)\mathrm{d}t.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \\
\leq\ & M(a,b)\int_0^1 h_1^2(t)h_2^2(1-t)\mathrm{d}t \\
& +N(a,b)\int_0^1 h_1(t)h_2(1-t)h_2(t)h_1(1-t)\mathrm{d}t.
\end{aligned}$$

which is the required result. □

Corollary 10. *If $h_1(t)=t^{s_1}$ and $h_2(t)=t^{s_2}$, then, under the assumptions of Theorem 9, we have a result for generalized Breckner type of (s_1,s_2)-convex* *functions, which is given as*

$$\frac{1}{b-a}\int_a^b [f(x)g(x)]\mathrm{d}x \leq M(a,b)\mathbb{B}(2s_1+1,2s_2+1)+N(a,b)\mathbb{B}(s_1+s_2+1,s_1+s_2+1).$$

where M(a,b) and N(a,b) and given by (10) *and* (11).

Corollary 11. *If* $h_1(t)=t^{-s_1}$ *and* $h_2(t)=t^{-s_2}$*, then, under the assumptions of Theorem 9, we have a result for generalized Godunova- Levin-Dragomir type of* (s_1,s_2)*-convex functions, which is given as*

$$\frac{1}{b-a}\int_a^b [f(x)g(x)]\mathrm{d}x \leq M(a,b)\mathbb{B}(1-2s_1,1-2s_2)+N(a,b)\mathbb{B}(1-s_1-s_2,1-s_1-s_2).$$

where M(a,b) and N(a,b) and given by (10) *and* (11).

Theorem 12. *Let* $f: I=[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ *be a continuous function such that* $f\in L(a,b)$*. If* f *is a generalized* (h_1,h_2) *convex function, then*

$$\int_a^b (u-a)^{\alpha}(b-u)^{\beta}f(u)\mathrm{d}u \leq (b-a)^{\alpha+\beta+1}[\gamma_1(t)f(a)+\gamma_2 t[f(a)+\eta\big(f(b),f(a)\big)],$$

where

$$\gamma_1(t)=\int_0^1 t^{\alpha}(1-t)^{\beta}h_1(t)h_2(1-t)\mathrm{d}t. \quad (13)$$

$$\gamma_2(t)=\int_0^1 t^{\alpha}(1-t)^{\beta}h_2(t)h_1(1-t)\mathrm{d}t, \quad (14)$$

respectively.

Proof. Using Lemma 1 and the fact that f is a generalized (h_1,h_2)-convex function, we have

$$\begin{aligned}\int_a^b (u-a)^{\alpha}(b-u)^{\beta}f(u)\mathrm{d}u &= (b-a)^{\alpha+\beta+1}\int_0^1 t^{\alpha}(1-t)^{\beta}f((1-t)a+tb)\mathrm{d}t\\ &\leq (b-a)^{\alpha+\beta+1}\int_0^1 t^{\alpha}(1-t)^{\beta}\Big(h_1(t)h_2(1-t)[f(a)]\\ &\quad +h_2(t)h_1(1-t)[f(a)+\eta\big(f(b),f(a)\big)]\Big)\mathrm{d}t\\ &= (b-a)^{\alpha+\beta+1}[\gamma_1(t)f(a)+\gamma_2 t[f(a)+\eta\big(f(b),f(a)\big)].\end{aligned}$$

which is the required result. □

Corollary 13. *If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$, then, under the assumptions of Theorem 12, we have a result for generalized Breckner type of (s_1, s_2)-convex functions, which is given as*

$$\int_a^b (u-a)^\alpha (b-u)^\beta f(u)\mathrm{d}u \leq (b-a)^{\alpha+\beta+1}[\acute{\gamma_1}(t) f(a) + \acute{\gamma_2} t[f(a) + \eta\big(f(b), f(a)\big)],$$

where

$$\acute{\gamma_1}(t) = \mathbb{B}(\alpha + s_1 + 1, \beta + s_2 + 1). \tag{15}$$
$$\acute{\gamma_2}(t) = \mathbb{B}(\alpha + s_2 + 1, \beta + s_1 + 1), \tag{16}$$

respectively.

Corollary 14. *If $h_1(t) = t^{-s_1}$ and $h_2(t) = t^{-s_2}$, then, under the assumptions of Theorem 12, we have a result for generalized Godunova- Levin-Dragomir type of (s_1, s_2)-convex functions, which is given as*

$$\int_a^b (u-a)^\alpha (b-u)^\beta f(u)\mathrm{d}u \leq (b-a)^{\alpha+\beta+1}[\acute{\gamma_1}(t) f(a) + \acute{\gamma_2} t[f(a) + \eta\big(f(b), f(a)\big)],$$

where

$$\acute{\gamma_1}(t) = \mathbb{B}(\alpha - s_1 + 1, \beta - s_2 + 1). \tag{17}$$
$$\acute{\gamma_2}(t) = \mathbb{B}(\alpha - s_2 + 1, \beta - s_1 + 1), \tag{18}$$

respectively.

Theorem 15. *Let $f : I = [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in L(a, b)$. If $|f|^{\frac{r}{r-1}}$ is generalized (h_1, h_2) convex function, then*

$$\begin{aligned} &\int_a^b (u-a)^\alpha (b-u)^\beta f(u)\mathrm{d}u \\ \leq\ & (b-a)^{\alpha+\beta+1}\Big[\mathbb{B}(r\alpha+1, r\beta+1)\Big]^{\frac{1}{r}} \bigg(\Big[|[f(a)]|^{\frac{r}{r-1}} + |[f(a) + \eta\big(f(b), f(a)\big)]|^{\frac{r}{r-1}}\Big] \\ & \int_0^1 h_1(t) h_2(1-t)\mathrm{d}t\bigg)^{\frac{r-1}{r}}. \end{aligned}$$

Proof. Using Lemma 1, Holder's inequality and the fact that $|f|^{\frac{r}{r-1}}$ is a generalized (h_1, h_2)-convex function. Then we have

$$\begin{aligned}
&\int_a^b (u-a)^\alpha (b-u)^\beta f(u)\mathrm{d}u \\
\leq\ & (b-a)^{\alpha+\beta+1}\left(\int_0^1 t^{r\alpha}(1-t)^{r\beta}\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_0^1 |f((1-t)a+tb)|^{\frac{r}{r-1}}\mathrm{d}t\right)^{\frac{r-1}{r}} \\
\leq\ & (b-a)^{\alpha+\beta+1}\left[\beta(r\alpha+1, r\beta+1)\right]^{\frac{1}{r}}\left[\int_0^1 \{(h_1(t)h_2(1-t)|[f(a)]|^{\frac{r}{r-1}}\right. \\
&\left. +h_2(t)h_1(1-t)|[f(a)+\eta(f(b), f(a))]|^{\frac{r}{r-1}}\}\mathrm{d}t\right]^{\frac{r-1}{r}} \\
=\ & (b-a)^{\alpha+\beta+1}\left[\mathbb{B}(r\alpha+1, r\beta+1)\right]^{\frac{1}{r}}\left(\left[|[f(a)]|^{\frac{r}{r-1}}+|[f(a)+\eta(f(b), f(a))]|^{\frac{r}{r-1}}\right]\right. \\
&\left.\int_0^1 h_1(t)h_2(1-t)\mathrm{d}t\right)^{\frac{r-1}{r}}.
\end{aligned}$$

which is the required result. ☐

Corollary 16. *If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$, then, under the assumptions of Theorem 15, we have a result for generalized Breckner type of (s_1, s_2)-convex function, where $s_1, s_2 \in [0, 1]$, then we have*

$$\begin{aligned}
&\int_a^b (u-a)^\alpha (b-u)^\beta f(u)\mathrm{d}u \\
\leq\ & (b-a)^{\alpha+\beta+1}\left[\mathbb{B}(r\alpha+1, r\beta+1)\right]^{\frac{1}{r}}\left(\left[|[f(a)]|^{\frac{r}{r-1}}+|[f(a)+\eta(f(b), f(a))]|^{\frac{r}{r-1}}\right]\right. \\
&\left.\mathbb{B}(\alpha+s_1+1, \beta+s_2+1)\right)^{\frac{r-1}{r}}.
\end{aligned}$$

Corollary 17. *If $h_1(t) = t^{-s_1}$ and $h_2(t) = t^{-s_2}$, then, under the assumptions of Theorem 15, we have a result for generalized Godunova- Levin-Dragomir type of (s_1, s_2)-convex function, where $s_1, s_2 \in [0, 1]$, then we have*

$$\begin{aligned}
&\int_a^b (u-a)^\alpha (b-u)^\beta f(u)\mathrm{d}u \\
\leq\ & (b-a)^{\alpha+\beta+1}\left[\mathbb{B}(1-r\alpha, 1-r\beta)\right]^{\frac{1}{r}}\left(\left[|[f(a)]|^{\frac{r}{r-1}}+|[f(a)+\eta(f(b), f(a))]|^{\frac{r}{r-1}}\right]\right. \\
&\left.\mathbb{B}(1-\alpha-s_1, 1-\beta-s_2+1)\right)^{\frac{r-1}{r}}.
\end{aligned}$$

Theorem 18. *Let $f : I = [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in L(a, b)$. If $|f|^r$ is generalized (h_1, h_2) convex function, then*

$$\begin{aligned}
&\int_a^b (u-a)^\alpha (b-u)^\beta f(u)\mathrm{d}u \\
\leq\ & (b-a)^{\alpha+\beta+1}\Big[\mathbb{B}(\alpha+1, \beta+1)\Big]^{\frac{r-1}{r}} \Big[\gamma_1(t)|[f(a)]|^r + \gamma_2(t)|[f(a)+\eta\big(f(b), f(a)\big)]|^r\Big]^{\frac{1}{r}}.
\end{aligned}$$

where $\gamma_1(t)$ and $\gamma_2(t)$ are given by (13) *and* (14).

Proof. Using Lemma 1, Holder's inequality and the fact that $|f|^r$ is a generalized (h_1, h_2)-convex function. Then we have

$$\begin{aligned}
&\int_a^b (u-a)^\alpha (b-u)^\beta f(u)\mathrm{d}u \\
\leq\ & (b-a)^{\alpha+\beta+1}\left(\int_0^1 t^\alpha (1-t)^\beta \mathrm{d}t\right)^{\frac{r-1}{r}} \left(\int_0^1 t^\alpha (1-t)^\beta |f((1-t)a+tb)|^r \mathrm{d}t\right)^{\frac{1}{r}} \\
\leq\ & (b-a)^{\alpha+\beta+1}\Big[\mathbb{B}(\alpha+1, \beta+1)\Big]^{\frac{r-1}{r}} \Big[t^\alpha (1-t)^\beta \int_0^1 \big\{(h_1(t)h_2(1-t)|[f(a)]|^r \\
& +h_2(t)h_1(1-t)|[f(a)+\eta\big(f(b), f(a)\big)]|^r\big\}\mathrm{d}t\Big]^{\frac{1}{r}} \\
=\ & (b-a)^{\alpha+\beta+1}\Big[\mathbb{B}(\alpha+1, \beta+1)\Big]^{\frac{r-1}{r}} \Big[\gamma_1(t)|[f(a)]|^r + \gamma_2(t)|[f(a)+\eta\big(f(b), f(a)\big)]|^r\Big]^{\frac{1}{r}}.
\end{aligned}$$

which is the required result. □

Corollary 19. *If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$, then, under the assumptions of Theorem 18, we have a result for generalized Breckner type of (s_1, s_2)-convex function, where $s_1, s_2 \in [0, 1]$, then we have*

$$\begin{aligned}
&\int_a^b (u-a)^\alpha (b-u)^\beta f(u)\mathrm{d}u \\
\leq\ & (b-a)^{\alpha+\beta+1}\Big[\mathbb{B}(\alpha+1, \beta+1)\Big]^{\frac{r-1}{r}} \Big[\acute{\gamma_1}(t)|[f(a)]|^r + \acute{\gamma_2}(t)|[f(a)+\eta\big(f(b), f(a)\big)]|^r\Big]^{\frac{1}{r}}.
\end{aligned}$$

where $\acute{\gamma_1}(t)$ and $\acute{\gamma_2}(t)$ are given by (15) *and* (16).

Corollary 20. *If $h_1(t) = t^{-s_1}$ and $h_2(t) = t^{-s_2}$, then, under the assumptions of Theorem 18, we have a result for generalized Godunova- Levin-Dragomir* *type of (s_1, s_2)-convex function, where $s_1, s_2 \in [0,1]$, then we have*

$$\begin{aligned} &\int_a^b (u-a)^{\alpha}(b-u)^{\beta} f(u)\mathrm{d}u \\ \leq\ & (b-a)^{\alpha+\beta+1}\Big[\mathbb{B}(\alpha+1,\beta+1)\Big]^{\frac{r-1}{r}}\Big[\acute{\gamma_1}(t)|[f(a)]|^r+\acute{\gamma_2}(t)|[f(a)+\eta\big(f(b),f(a)\big)]|^r\Big]^{\frac{1}{r}}. \end{aligned}$$

where $\acute{\gamma_1}(t)$ and $\acute{\gamma_2}(t)$ are given by (17) and (18).

Theorem 21. *Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ and $|f|^n \in L[a,b]$. If $|f^n|^q$ exists and is generalized (h_1, h_2)-convex function, then for n, $q \geq 1$, we have*

$$\begin{aligned} &\left|\tfrac{f(a)+f(b)}{2} - \tfrac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \sum_{k=2}^{n-1}\tfrac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| \\ \leq\ & \tfrac{(b-a)^n}{2n!}\left(\tfrac{n-1}{n+1}\right)^{1-\frac{1}{q}}\left[\psi_1(t)|f^{(n)}(a)|^q + \psi_2(t)[|f^{(n)}(a)|^q + \eta(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q)\right]^{\frac{1}{q}}, \end{aligned}$$

where

$$\psi_1(t) = \int_0^1 t^{n-1}(n-2t)h_1(1-t)h_2(t)\mathrm{d}t,$$

and

$$\psi_2(t) = \int_0^1 t^{n-1}(n-2t)h_1(t)h_2(1-t)\mathrm{d}t,$$

respectively.

Proof. Using Lemma 2, power mean inequality and the fact that $|f^n|$ is generalized (h_1, h_2)-convex function, we have

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)|$$

$$\leq \frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)|f^{(n)}((1-t)a+tb)|\mathrm{d}t$$

$$\leq \frac{(b-a)^n}{2n!}\Big(\int_0^1 t^{n-1}(n-2t)\mathrm{d}t\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t^{n-1}(n-2t)|f^{(n)}((1-t)a+tb)|^q\mathrm{d}t\Big)^{\frac{1}{q}}$$

$$\leq \frac{(b-a)^n}{2n!}(\frac{n-1}{n+1})^{1-\frac{1}{q}} \times \Big(|f^{(n)}(a)|^q\int_0^1 t^{n-1}(n-2t)h_1(1-t)h_2(t)\mathrm{d}t$$

$$+[|f^{(n)}(a)|^q + \eta(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q)]\int_0^1 t^{n-1}(n-2t)h_1(t)h_2(1-t)\mathrm{d}t\Big)^{\frac{1}{q}}$$

$$= \frac{(b-a)^n}{2n!}(\frac{n-1}{n+1})^{1-\frac{1}{q}} \times \Big(|f^{(n)}(a)|^q\psi_1(t) + [|f^{(n)}(a)|^q + \eta(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q)]\psi_2(t)\Big)^{\frac{1}{q}}.$$

□

Corollary 22. *If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$, then, under the assumptions of Theorem 21, we have a result for generalized Breckner type of (s_1, s_2)-convex function, where $s_1, s_2 \in [0,1]$, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$

$$\leq \frac{(b-a)^n}{2n!}(\frac{n-1}{n+1})^{1-\frac{1}{q}}\left[\acute{\psi}_1(t)|f^{(n)}(a)|^q + \acute{\psi}_2(t)[|f^{(n)}(a)|^q + \eta(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q)\right]^{\frac{1}{q}},$$

where

$$\acute{\psi}_1(t) = n\mathbb{B}(n+s_1, s_2+1) - 2\mathbb{B}(n+s_1+1, s_2+1),$$

and

$$\acute{\psi}_2(t) = n\mathbb{B}(n+s_2, s_1+1) - 2\mathbb{B}(n+s_2+1, s_2+1),$$

Corollary 23. *If $h_1(t) = t^{-s_1}$ and $h_2(t) = t^{-s_2}$, then, under the assumptions of Theorem 21, we have a result for generalized Godunova- Levin-Dragomir type of (s_1, s_2)-convex function, where $s_1, s_2 \in [0,1]$, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$

$$\leq \frac{(b-a)^n}{2n!}(\frac{n-1}{n+1})^{1-\frac{1}{q}}\left[\acute{\psi_1}(t)|f^{(n)}(a)|^q + \acute{\psi_2}(t)[|f^{(n)}(a)|^q + \eta(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q)\right]^{\frac{1}{q}},$$

where

$$\acute{\psi_1}(t) = n\mathbb{B}(n - s_1, 1 - s_2) - 2\mathbb{B}(n - s_1 + 1, 1 - s_2),$$

and

$$\acute{\psi_2}(t) = n\mathbb{B}(n - s_2, 1 - s_1) - 2\mathbb{B}(n - s_2 + 1, 1 - s_2),$$

respectively.

Theorem 24. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$and $|f^n|^q \in L[a, b]$. If $|f^n|^q$ exists and is generalized (h_1, h_2)-convex function, then for n, $q \geq 1$, we have*

$$\left|\tfrac{f(a)+f(b)}{2} - \tfrac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \sum_{k=2}^{n-1} \tfrac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a)\right|$$
$$\leq \tfrac{(b-a)^n}{2n(n-1)^{\frac{1}{q}}} \left[\Omega_1(t)|f^{(n)}(a)|^q + \Omega_2(t)[|f^{(n)}(a)|^q + \eta(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q)\right]^{\frac{1}{q}},$$

where

$$\Omega_1(t) = \int_0^1 t^{q(n-1)}(n - 2t)h_1(1 - t)h_2(t)\mathrm{d}t,$$

and

$$\Omega_2(t) = \int_0^1 t^{q(n-1)}(n - 2t)h_1(t)h_2(1 - t)\mathrm{d}t,$$

respectively.

Proof. Using Lemma 2, Holder's inequality and the fact that $|f^n|^q$ is generalized (h_1, h_2)-convex function, we have

$$|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a)|$$
$$\leq \frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n - 2t)|f^{(n)}((1 - t)a + tb)|\mathrm{d}t$$
$$\leq \frac{(b-a)^n}{2n!}\left(\int_0^1 (n - 2t)\mathrm{d}t\right)^{1-\frac{1}{q}} \left(\int_0^1 t^{q(n-1)}(n - 2t)|f^{(n)}((1 - t)a + tb)|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leq \frac{(b-a)^n}{2n!}(n-1)^{1-\frac{1}{q}} \times \left(|f^{(n)}(a)|^q \int_0^1 t^{q(n-1)}(n-2t)h_1(1-t)h_2(t)\mathrm{d}t\right.$$

$$\left.+[|f^{(n)}(a)|^q + \eta(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q)] \int_0^1 t^{q(n-1)}(n-2t)h_1(t)h_2(1-t)\mathrm{d}t\right)^{\frac{1}{q}}.$$

$$= \frac{(b-a)^n}{2n(n-1)^{\frac{1}{q}}} \times \left(|f^{(n)}(a)|^q\Omega_1(t) + [|f^{(n)}(a)|^q + \eta(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q)]\Omega_2(t)\right)^{\frac{1}{q}}.$$

□

Corollary 25. *If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$, then, under the assumptions of Theorem 24, we have a result for generalized Breckner type of (s_1, s_2)-convex function, where $s_1, s_2 \in [0,1]$, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a)\right|$$

$$\leq \frac{(b-a)^n}{2n(n-1)^{\frac{1}{q}}} \left[\Omega_1(t)|f^{(n)}(a)|^q + \Omega_2(t)[|f^{(n)}(a)|^q + \eta(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q)\right]^{\frac{1}{q}},$$

where

$$\Omega_1(t) = n\mathbb{B}(nq + s_2 - q + 1, s - 1 + 1) - 2\mathbb{B}(nq + s_1 - q + 2, s_1 + 1),$$

and

$$\Omega_2(t) = n\mathbb{B}(nq + s_1 - q + 1, s_2 + 1) - 2\mathbb{B}(nq + s_1 - q + 2, s_1 + 1),$$

respectively.

Corollary 26. *If $h_1(t) = t^{-s_1}$ and $h_2(t) = t^{-s_2}$, then, under the assumptions of Theorem 24, we have a result for generalized Godunova- Levin-Dragomir type of (s_1, s_2)-convex function, where $s_1, s_2 \in [0,1]$, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a)\right|$$

$$\leq \frac{(b-a)^n}{2n(n-1)^{\frac{1}{q}}} \left[\Omega'_1(t)|f^{(n)}(a)|^q + \Omega'_2(t)[|f^{(n)}(a)|^q + \eta(|f^{(n)}(b)|^q, |f^{(n)}(a)|^q)\right]^{\frac{1}{q}},$$

where

$$\Omega'_1(t) = n\mathbb{B}(nq - s_2 - q + 1, 1 - s_1) - 2\mathbb{B}(nq - s_2 - q + 2, 1 - s1),$$

and

$$\Omega'_2(t) = n\mathbb{B}(nq - s_1 - q + 1, 1 - s_2) - 2\mathbb{B}(nq - s_1 - q + 2, 1 - s_2),$$

respectively.

Conclusion

In this paper, we have introduced and studied a new class of generalized (h_1, h_2)-convex functions. Several new integral inequalities for these generalized functions have been derived, which have important applications in physics and material sciences. These estimates also useful in numerical analysis for finding the error bounds for the approximate solution. We have also discussed important several special cases, which can be obtained from our results.

Acknowledgements

The authors would like to thank the Vice Chancellor and the Rector, SBK Women's University Quetta. Pakistan and COMSATS University Islamabad, Pakistan for providing excellent research and academic environments. Authors are also grateful to the referees and editor for their valuable and constructive suggestions.

References

[1] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Generalized convexity and inequalities, J. Math. Anal. Appl, 335(2), 1294-1308, (2007).

[2] W.W. Breckner, Stetigkeitsaussagen fiir eine Klasse verallgemeinerter convexer funktionen in topologischen linearen Raumen, Pupl. Inst. Math, 23, 13-20, (1978).

[3] B. B. Mohsen, M. A. Noor, K. I. Noor and M. Postolache, Strongly convex functions of higher order involving mifunction, Mathematics, 5, xx-xx, (2019),

[4] G. Cristescu, L. Lupsa, Non-connected Convexities and Applications, Kluwer Academic Publishers, Dordrechet, Holland, (2002).

[5] M. R. Delavar and S. S. Dragomir, On η-convexity, Math. Inequal. Appl, 20(1), 203-216, (2017).

[6] S. S. Dragomir and B. Mond, Integral inequalities of Hadamard type for log-convex functions, Demonstratio, 31, 354-364, (1998).

[7] S. S. Dragomir and C. E. M. Pearce, Selected topics on Hermite-Hadamard inequalities and applications, Victoria University, Australia, (2000).

[8] M. E. Gordji, M. R. Delavar and M. D. Sen, On φ convex functions, J. Math. Inequal, 10(1), 173-183, (2016).

[9] M. E. Gordji, M. R. Delavar and S. S. Dragomir, An inequality related to η-convex functions (II), Int. J. Nonlinear. Anal. Appl, 6(2), 27-33, (2015).

[10] E. K. Godunova and V. I. Levin, Neravenstva dlja funkcii sirokogo klassa soderzascego vypuklye monotonnye i nekotorye drugie vidy funkii, Vycislitel. Mat. i.Fiz. Mezvuzov. Sb. Nauc. MGPI Moskva.138-142. in Russian.(1985).

[11] J. Hadamard, Etude sur les proprietes des fonctions entieres et en particulier dune fonction consideree par Riemann, J. Math. Pure. Appl, 58, 171-215, (1893).

[12] C. Hermite, Sur deux limites d'une integrale definie, Mathesis, 3, 82, (1883).

[13] D. H. Hyers and S. M. Ulam, Approximately convex functions, Proc. Amer. Math. Soc, 3, 821-828, (1952).

[14] D.-Y. Hwang, Some inequalities for n-time differentiable mappings and applications, Kyungpook Math. J, 43, 335-343, (2003).
[15] C. P. Niculescu and L. E. Persson, Convex Functions and Their Applications. Springer-Verlag, New York, (2018).
[16] M. A. Noor, New approximation schemes for general variatiomal inequalities, J. Math. Anal. Appl, 251, 271-229. (2000).
[17] M. A. Noor, Some developments in general variational inequalities, Appl. Math. Comput., 152, 199-277, (2004).
[18] M. A. Noor, Extended general variational inequalities, Appl. Math. Letters, 22, 182-186, (2009).
[19] M. A. Noor and K. I. Noor, New classes of strongly exponentially preinvex functions, AIMS Mathematics, 4(6): 1554-1568, (2019).
[20] M. A. Noor, K. I. Noor, M. U. Awan and F. Safdar, On strongly generalized convex functions, Filomat, 31(18), 5783-5790, (2017).
[21] M. A. Noor, K. I. Noor and F. Safdar, Generalized geometrically convex functions and inequalities, J. Inequal. Appl, (2017), (2017):202.
[22] M. A. Noor, K. I. Noor and F. Safdar, Integral inequaities via generalized convex functions, J. Math. Computer Sci, 17, 465-476, (2017).
[23] M. A. Noor, K. I. Noor and F. Safdar, Inequalities via generalized h-convex functions. Probl. Anal. Issues Anal, 7(25), No:2, 112-130, (2018).
[24] M. A. Noor, K. I. Noor, M. U. Awan, Some new classes of convex functions and inequalities, Miskolc Math. Notes, 19(1), 77-94, (2018).
[25] M. A. Noor, K. I. Noor, S. Iftikhar, F. Safdar, Integral inequaities for relative harmonic (s,η)-convex functions, Appl. Math. Computer. Sci, 1(1), 27-34, (2015).
[26] M. A. Noor, K. I. Noor and F. Safdar, Integral inequaities via generalized (α,m)-convex functions, J. Nonl. Funct. Anal, 2017, (2017), Article ID: 32.
[27] M. E. Ozdemir, E. Set, and M. Alomari, Integral inequalities via several kinds of convexity, Creat. Math, Inform, 20, 6273, (2011).
[28] J. E. Pecaric , F. Proschan and Y.L. Tong, Convex Functions, Partial Orderings and Statistical Applications, Academic Press, Boston, (1992).
[29] M. Z. Sarikaya, A. Saglam and H. Yildirim, On some Hadamard-type inequalities for h-convex functions. Jour. Math. Ineq. 2(3), 335-341, (2008).
[30] G. Toader, Some generalizations of the convexity, Proceedings of the Colloquium on Approximation and Optimization, Univ. Cluj-Napoca, 329-338, (1985).
[31] M. Tunc, E. Gov, and U. Sanal, On tgs-convex function and their inequalities, Facta Univ(NIS) Ser. Math. Inform, 30, 679691, (2015).
[32] S. Varosanec, On h-convexity, J. Math. Anal. Appl, 326, 303-311, (2007).