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Mathematical Properties of Inverse Sum Index Eccentric Coindices of Graphs

M. R. Farahani[a], K. Pattabiraman[b], S. Sudharsan[c], S. V. Patil[d], M. Alaeiyan[a,*], M. Cancan[e]

[a]*Department of Mathematics and Computer Science, University of Science and Technology (IUST), Narmak, Tehran, 16844, Iran*
[b]*Department of Mathematics, Government Arts College Kumbakonam 612 002, India*
[c]*Department of Mathematics, Annamalai University Annamalainagar 608 002, India*
[d]*Department of Mathematics, KLE Dr. M. S. Sheshgiri College of Engineering and Technology, Belagavi, Karnataka 590001,India*
[e]*Department of Mathematics, Faculty of Education,Yuzuncu Yil University, Van Turkey 65090, Turkey*

Abstract

Essential and widely studied topological indices, including the well-known Zagreb indices (M_1 and M_2), and the newly proposed Inverse Sum Indeg Eccentricity Index (ξ_{ISI}), To ensure the contributions of all edges within a graph are effectively considered. By emphasizing on the total eccentricity of non-adjacent vertices, Hua et al. introduced the eccentric connectivity coindex ($\overline{\xi^c}$). Inspired by their contributions, we introduce the inverse sum indeg eccentric coindex ($\overline{\xi_{ISI}}$), which is defined as the ratio of the product of the eccentricities to the sum of the eccentricities for all isolated pair of vertex in a connected graph. This study primarily aims to establish various bounds for $\overline{\xi_{ISI}}$ in finite simple graphs and derives the values of the proposed indices for two specific graph constructions. Additionally, we present a comprehensive set of relationships for $\overline{\xi_{ISI}}$ using several graph products.

Keywords: Topological index, Eccentricity of a vertex, Graph products

1. Introduction

A graph invariant, often referred to as a topological index, is a numerical parameter obtained directly from the structural properties of a molecule graph. These indices are widely used in theoretical chemistry to model various molecular properties, including physico-chemical, biological, and pharmaceutical characteristics. Numerous graph invariants linked to the graph-theoretic notion of eccentricity has been previously proposed also utilized in QSAR/QSPR studies. Many of these indices have proven to be effective tools for predicting pharmaceutical properties.

*Corresponding author
Email addresses: mrfarahani88@gmail.com (M. R. Farahani), pramank@gmail.com (K. Pattabiraman), sudharsanmaths1999@gmail.com (S. Sudharsan), shobhap49@gmail.com (S. V. Patil), alaeiyan@iust.ac.ir (M. Alaeiyan), m_cancan@yyu.edu.tr, m_cencen@yahoo.com (M. Cancan)

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In this study, we concentrate on the properties of simple connected graphs. Consider such a graph be $\widetilde{\Lambda}$, the set of vertices and edges, are denoted by $\mathfrak{Ve}(\widetilde{\Lambda})$ and $\mathfrak{Ed}(\widetilde{\Lambda})$ correspondingly. The graph $\widetilde{\Lambda}$ has n vertices and m edges. An edge connecting two vertices x and y is denoted as xy. The complement of $\widetilde{\Lambda}$, denoted by $\overline{\widetilde{\Lambda}}$, is a graph with the same vertex set $\mathfrak{Ve}(\widetilde{\Lambda})$, where two vertices are adjacent if and only if they are not adjacent in $\widetilde{\Lambda}$. The degree of a vertex x, denoted by $d(x)$, is the number of vertices directly connected to x in $\widetilde{\Lambda}$. The first and second Zagreb indices are defined as

$$M_1(\widetilde{\Lambda}) = \sum_{x\in\mathfrak{Ve}(\widetilde{\Lambda})} d(x)^2 = \sum_{xy\in\mathfrak{Ed}(\widetilde{\Lambda})} (d(x)+d(y))$$
$$M_2(\widetilde{\Lambda}) = \sum_{xy\in\mathfrak{Ed}(\widetilde{\Lambda})} d(x)d(y)$$

such a topological indices were first introduced in [11, 12]. For detailed information on their properties and additional references, see [7, 9, 13, 6, 14, 23, 31, 33, 34]. For historical context, refer to [10].

In 2008, Doslic introduced a pair of Zagreb coindices [8], building on the earlier definitions.

$$\overline{M_1}(\widetilde{\Lambda}) = \sum_{xy\notin\mathfrak{Ed}(\widetilde{\Lambda})} (d(x)+d(y)) \text{ and } \overline{M_2}(\widetilde{\Lambda}) = \sum_{xy\notin\mathfrak{Ed}(\widetilde{\Lambda})} d(x)d(y)$$

Recently studied some detail by the Zagreb coindices in [1, 2, 17, 18, 19].

The distance between the two vertices such as x and y in graph $\widetilde{\Lambda}$ is denoted as

$$d_{\widetilde{\Lambda}}(x,y)$$

The eccentricity ϵ_x of $x \in \mathfrak{Ve}(\widetilde{\Lambda})$ is defined as

$$\epsilon_x = \max\{d_{\widetilde{\Lambda}}(x,y) \mid x,y \in \mathfrak{Ve}(\widetilde{\Lambda})\}.$$

In all the vertices of graph $\widetilde{\Lambda}$ the diameter and radius is notated as $\rho(\widetilde{\Lambda})$ and $r(\widetilde{\Lambda})$ correspondingly. If $\rho(\widetilde{\Lambda}) = r(\widetilde{\Lambda})$, then $\widetilde{\Lambda}$ is self-centred graph denoted as $s-c$ graph. A vertex $x \in \mathfrak{Ve}(\widetilde{\Lambda})$ with $d(x) = n-1$ is known as a universal vertex.

In 1997, Sharma et al. [28] recommended the eccentric connectivity index of $\widetilde{\Lambda}$ as

$$\xi^c(\widetilde{\Lambda}) = \sum_{x\in\mathfrak{Ve}(\widetilde{\Lambda})} d(x)\epsilon_x = \sum_{xy\in\mathfrak{Ed}(\widetilde{\Lambda})} \epsilon_x + \epsilon_y$$

The invariants $\tau(\widetilde{\Lambda})$ and $\xi^2(\widetilde{\Lambda})$ defined as

$$\tau(\widetilde{\Lambda}) = \sum_{x\in\mathfrak{Ve}(\widetilde{\Lambda})} \epsilon_x \text{ and}$$
$$\xi^2(\widetilde{\Lambda}) = \sum_{x\in\mathfrak{Ve}(\widetilde{\Lambda})} d(x)\ \epsilon_x^2 = \sum_{xy\in\mathfrak{Ed}(\widetilde{\Lambda})} (\epsilon_x^2 + \epsilon_y^2)$$

are respectively called the total eccentricity and second eccentric connectivity index [3] of $\widetilde{\Lambda}$.

The first and second Zagreb eccentricity indices [29, 30] of $\widetilde{\Lambda}$ are respectively, defined as

$$\xi_1(\widetilde{\Lambda}) = \sum_{x\in\mathfrak{Ve}(\widetilde{\Lambda})} \epsilon_x^2 \text{ and } \xi_{M_2}(\widetilde{\Lambda}) = \sum_{xy\in\mathfrak{Ed}(\widetilde{\Lambda})} \epsilon_x\epsilon_y.$$

In 2019, Hua and Miao [20] proposed the eccentric connectivity coindex of $\widetilde{\Lambda}$ as

$$\overline{\xi^c}(\widetilde{\Lambda}) = \overline{\xi_{M_1}}(\widetilde{\Lambda}) = \sum_{xy\notin\mathfrak{Ed}(\widetilde{\Lambda})} (\epsilon_x + \epsilon_y).$$

In 2021, Mahdieh Azari [22] proposed the second Zagreb eccentricity coindex of $\widetilde{\Lambda}$ as

$$\overline{\xi_{M_2}}(\widetilde{\Lambda}) = \overline{E_2}(\widetilde{\Lambda}) = \sum_{xy \notin \mathfrak{Ed}(\widetilde{\Lambda})} \epsilon_x \epsilon_y$$

The Harmonic eccentric index and its coindex [12] of $\widetilde{\Lambda}$ are respectively, defined as

$$\xi_H(\widetilde{\Lambda}) = \sum_{xy \in \mathfrak{Ed}(\widetilde{\Lambda})} \frac{2}{\epsilon_x + \epsilon_y} \quad \text{and} \quad \overline{\xi_H}(\widetilde{\Lambda}) = \sum_{xy \notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{2}{\epsilon_x + \epsilon_y}.$$

The status or transmission [16] of a vertex x in the graph $\widetilde{\Lambda}$ is defined as

$$D_x = \sum_{\{x,y\} \in \mathfrak{Ve}(\widetilde{\Lambda})} d_{\widetilde{\Lambda}}(x, y)$$

The status connectivity indices first and second of a graph $\widetilde{\Lambda}$, introduced by Ramane et al.[26] are respectively, defined as

$$S_1(\widetilde{\Lambda}) = \sum_{xy \in \mathfrak{Ed}(\widetilde{\Lambda})} (D_x + D_y) \text{ and } S_2(\widetilde{\Lambda}) = \sum_{xy \in \mathfrak{Ed}(\widetilde{\Lambda})} D_x D_y$$

In 2018, Ramane et al. introduced the concepts of the first and second status connectivity coindices [27], which are defined respectively as follows:

$$\overline{S_1}(\widetilde{\Lambda}) = \sum_{xy \notin \mathfrak{Ed}(\widetilde{\Lambda})} (D_x + D_y) \text{ and } \overline{S_2}(\widetilde{\Lambda}) = \sum_{xy \notin \mathfrak{Ed}(\widetilde{\Lambda})} D_x D_y$$

Ease way of recognize ISI and $\overline{ISI}$ are just the first representation of a class of topological indices and coindex of the form

$$ISI_{general}(\widetilde{\Lambda}) = \sum_{xy \in \mathfrak{Ed}(\widetilde{\Lambda})} \frac{Q_x Q_y}{Q_x + Q_y}$$
$$\text{and} \quad \overline{ISI}_{general}(\widetilde{\Lambda}) = \sum_{xy \notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{Q_x Q_y}{Q_x + Q_y} \tag{1.1}$$

respectively, Here, Q_x represents a specific quantity that can be uniquely and effectively associated with the vertex x of the graph $\widetilde{\Lambda}$.

In this study, we turn our focus to another member of this class, denoted by $\overline{ISI_3}$, which can provisionally be referred to as the third inverse sum indeg coindex. This index is formulated in such a way that it is associated with $\overline{\xi}^c$ and $\overline{\xi_{M_2}}$.

Based on the general formula (1.1), we define the third ISI and $\overline{ISI}$ indices as:

$$\xi_{ISI}(\widetilde{\Lambda}) = ISI_3(\widetilde{\Lambda}) = \sum_{xy \in \mathfrak{Ed}(\widetilde{\Lambda})} \frac{\epsilon_x \epsilon_y}{\epsilon_x + \epsilon_y} \quad \text{and} \quad \overline{\xi_{ISI}}(\widetilde{\Lambda}) = \overline{ISI_3}(\widetilde{\Lambda}) = \sum_{xy \notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{\epsilon_x \epsilon_y}{\epsilon_x + \epsilon_y}, \quad \text{respectively}$$

Hua and Miao [20] explored several extremal problems concerning the eccentric connectivity coindex and established various lower bounds for this invariant based on distinct graph parameters. Hayat [15] derived precise lower bounds for the second Zagreb eccentricity index in the context of n-vertex cacti graphs. Azari [4] examined the eccentric connectivity coindex's properties for different graph products. Additionally, numerous bounds on the eccentric connectivity coindex, expressed in terms of existing invariants, as well as its values for specific graph constructions, were discussed in [5].

The following concepts are included in the paper:

In Section 2, derive significant bounds for $\overline{\xi_{ISI}}(\widetilde{\Lambda})$ across various classes of graphs, offering a deeper understanding of the range and behavior of this coindex within different graph structures. Section 4 investigates how $\overline{\xi_{ISI}}(\widetilde{\Lambda})$ interacts with graph

2. Bounds on $\overline{\xi_{ISI}}$

The $\overline{\widetilde{\Lambda}}$ with $\mathfrak{Ve}(\widetilde{\Lambda})$ is set of all vertices is the complement of $\widetilde{\Lambda}$ and any two vertices of $\overline{\widetilde{\Lambda}}$ are adjust if and only if they are not adjacent in $\widetilde{\Lambda}$. The number of edges of $\overline{\widetilde{\Lambda}}$ is denoted by $\overline{\widetilde{m}}$, that is

$$\overline{\widetilde{m}} = \frac{|\mathfrak{Ve}(\widetilde{\Lambda})| \quad (|\mathfrak{Ve}(\widetilde{\Lambda})|-1)}{2} - |\mathfrak{Ed}(\widetilde{\Lambda})|.$$

In this section, we establish some new relation a $\overline{\xi_{ISI}}$ index and other graph parameters.

Theorem 2.1. *Let $\widetilde{\Lambda}$ be a graph on n vertices and m edges. Then*

$$\frac{\overline{\widetilde{m}} r(\widetilde{\Lambda})^2}{2\ \rho(\widetilde{\Lambda})} \leq \overline{\xi_{ISI}}(\widetilde{\Lambda}) \leq \frac{\overline{\widetilde{m}} \rho(\widetilde{\Lambda})^2}{2\ r(\widetilde{\Lambda})}. \tag{2.1}$$

The equality on the left-hand side of (2.1) is satisfied if and only if $\widetilde{\Lambda}$ is self-centered, while the equality on the right-hand side holds if and only if $\widetilde{\Lambda}$ is self-centered or $r(\widetilde{\Lambda}) = 1$ and $\rho(\widetilde{\Lambda}) = 2$.

Proof. Note that for each $x \in \mathfrak{Ve}(\widetilde{\Lambda})$, $r(\widetilde{\Lambda}) \leq \epsilon_x \leq \rho(\widetilde{\Lambda})$. Therefore,

$$\overline{\widetilde{m}} \left(\frac{r(\widetilde{\Lambda})^2}{2\ \rho(\widetilde{\Lambda})} \right) = \sum_{xy \notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{r(\widetilde{\Lambda}) r(\widetilde{\Lambda})}{\rho(\widetilde{\Lambda}) + \rho(\widetilde{\Lambda})} \leq \sum_{xy \notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{\epsilon_x \epsilon_y}{\epsilon_x + \epsilon_y} \leq \sum_{xy \notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{\rho(\widetilde{\Lambda}) \rho(\widetilde{\Lambda})}{r(\widetilde{\Lambda}) + r(\widetilde{\Lambda})} = \overline{\widetilde{m}} \left(\frac{\rho(\widetilde{\Lambda})^2}{2\ r(\widetilde{\Lambda})} \right).$$

The equality on the left-hand side of (2.1) holds if and only if, for every pair of vertices $xy \notin \mathfrak{Ed}(\widetilde{\Lambda})$, $\epsilon_x = \epsilon_y = r(\widetilde{\Lambda}) = \rho(\widetilde{\Lambda})$. If $\widetilde{\Lambda}$ is self-centered, then the left-hand side equality in (2.1) is trivially satisfied. Now, assume that the left-hand side equality holds in (2.1). If $\widetilde{\Lambda} \cong K_n$, then $\widetilde{\Lambda}$ is a self-centered graph. Let $\widetilde{\Lambda} \cong K_n$, and consider $x \in \mathfrak{Ve}(\widetilde{\Lambda})$ such that $\epsilon_x = \rho(\widetilde{\Lambda}) \geq 2$. Then there exists a vertex $y \in \mathfrak{Ve}(\widetilde{\Lambda})$ with $xy \notin \mathfrak{Ed}(\widetilde{\Lambda})$. Consequently, $\rho(\widetilde{\Lambda}) = \epsilon_x = \epsilon_y = r(\widetilde{\Lambda})$, which implies that $\widetilde{\Lambda}$ is a self-centered graph.

Similarly, the equality on the right-hand side of (2.1) holds if and only if, for every pair of vertices $xy \notin \mathfrak{Ed}(\widetilde{\Lambda})$, $\epsilon_x = \epsilon_y = r(\widetilde{\Lambda}) = \rho(\widetilde{\Lambda})$. If $\widetilde{\Lambda}$ is self-centered or satisfies $r(\widetilde{\Lambda}) = 1$ and $\rho(\widetilde{\Lambda}) = 2$, then the right-hand side equality in (2.1) holds trivially. Let us assume the right-hand side equality holds in (2.1). If $\rho(\widetilde{\Lambda}) = r(\widetilde{\Lambda}) = 1$, then $\widetilde{\Lambda} \cong K_n$, which is self-centered. If $r(\widetilde{\Lambda}) = 1$ and $\rho(\widetilde{\Lambda}) = 2$, there is nothing further to prove. Let $r(\widetilde{\Lambda}) \geq 2$. Consider $x \in \mathfrak{Ve}(\widetilde{\Lambda})$ such that $\epsilon_x = r(\widetilde{\Lambda}) \geq 2$. Then there exists a vertex $y \in \mathfrak{Ve}(\widetilde{\Lambda})$ such that $xy \notin \mathfrak{Ed}(\widetilde{\Lambda})$. Hence, $r(\widetilde{\Lambda}) = \epsilon_x = \epsilon_y = \rho(\widetilde{\Lambda})$, which confirms that $\widetilde{\Lambda}$ is a self-centered graph. □

Lemma 2.2. *[21] For a vertex x in a connected graph $\widetilde{\Lambda}$ with n vertices, $\epsilon_x \leq n - d(x)$ with equality if and only if $\widetilde{\Lambda} \cong P_4$ or $\widetilde{\Lambda} \cong K_n - iK_2$, $\quad 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$. The graph $K_n - iK_2$ is obtained from K_n by removing i independent edges.*

Theorem 2.3. *For a graph $\widetilde{\Lambda}$ with n vertices,*

$$\overline{\xi_{ISI}}(\widetilde{\Lambda}) \quad \leq \quad \frac{1}{2\ r(\widetilde{\Lambda})} \left[\frac{n^2(n(n-1) - 2m)}{2} - n\ \overline{M_1}(\widetilde{\Lambda}) + \overline{M_2}(\widetilde{\Lambda}) \right]. \tag{2.2}$$

with equality if and only if $\widetilde{\Lambda} \cong P_4$ (or) $\widetilde{\Lambda} \cong K_n - iK_2$, $\quad 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

Proof. By the Definition of $\overline{\xi_{ISI}}$ and Lemma 2.2, we obtain

$$\begin{aligned} \overline{\xi_{ISI}}(\widetilde{\Lambda}) &\leq \sum_{xy \notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{(n - d(x))(n - d(y))}{2\ r(\widetilde{\Lambda})} \\ &= \sum_{xy \notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{n^2 - n(d(x) + d(y)) + d(x)d(y)}{2\ r(\widetilde{\Lambda})} \end{aligned} \tag{2.3}$$

$$
\begin{aligned}
&= \frac{n^2}{2\,r(\widetilde{\Lambda})}\left(\frac{n(n-1)}{2}-m\right)-\frac{n}{2\,r(\widetilde{\Lambda})}\overline{M_1}(\widetilde{\Lambda})+\frac{\overline{M_2}(\widetilde{\Lambda})}{2\,r(\widetilde{\Lambda})}\\
&\le \frac{1}{2\,r(\widetilde{\Lambda})}\left[\frac{n^2(n(n-1)-2m)}{2}-n\,\overline{M_1}(\widetilde{\Lambda})+\overline{M_2}(\widetilde{\Lambda})\right].
\end{aligned}
$$

The equality in (2.2) holds if and only if, for any edge $xy \notin \mathfrak{Ed}(\widetilde{\Lambda})$, $\epsilon_x = n - d(x)$ and $\epsilon_y = n - d(y)$. This implies that for each non-trivial vertex $x \in \mathfrak{Ve}(\widetilde{\Lambda})$, $\epsilon_x = n - d(x)$. The equality $\epsilon_x = n - d(x)$ also holds for any universal vertex $x \in \mathfrak{Ve}(\widetilde{\Lambda})$. Therefore, $\epsilon_x = n - d(x)$ holds for all $x \in \mathfrak{Ve}(\widetilde{\Lambda})$. According to Lemma 2.2, the equality in (2.2) holds if and only if $\widetilde{\Lambda} \cong P_4$ or $\widetilde{\Lambda} \cong K_n - iK_2$ for $0 \le i \le \lfloor \frac{n}{2} \rfloor$. □

The proof of this lemma is straightforward.

Lemma 2.4. *For a $(n,m)-$graph $\widetilde{\Lambda}$*

$$\overline{M_1}(\widetilde{\Lambda}) = 2m(n-1) - M_1(\widetilde{\Lambda}) \text{ and } \overline{M_2}(\widetilde{\Lambda}) = 2m^2 - \tfrac{M_1(\widetilde{\Lambda})}{2} - M_2(\widetilde{\Lambda}).$$

By using Lemma 2.4 and Theorem 2.3, we obtain the following.

Corollary 2.5. *For a $(n,m)-$graph $\widetilde{\Lambda}$,*

$$\overline{\xi_{ISI}}(\widetilde{\Lambda}) \le \tfrac{1}{4\,r(\widetilde{\Lambda})}\left[n(n-1)(n^2-4m)+2m(2m-n^2)+M_1(\widetilde{\Lambda})-2M_2(\widetilde{\Lambda})\right].$$

Theorem 2.6. *For a graph $\widetilde{\Lambda}$ with n vertices,*

$$\overline{\xi_{ISI}}(\widetilde{\Lambda}) \ge \tfrac{\overline{S_2}(\widetilde{\Lambda})}{2\,\rho(\widetilde{\Lambda})(n-1)^2}$$

with equality if and only if $\widetilde{\Lambda} \cong K_n$.

Proof. For every vertex $x \in \mathfrak{Ve}(\widetilde{\Lambda})$, $D_x = \sum\limits_{x\in\mathfrak{Ve}(\widetilde{\Lambda})|\{y\}} d(x,y) \le \sum\limits_{x\in\mathfrak{Ve}(\widetilde{\Lambda})|\{y\}} \epsilon_{xy} = (n-1)\epsilon_y$, with equality if and only if for every $y \in \mathfrak{Ve}(\widetilde{\Lambda})|\{x\}, d(x,y) = \epsilon_y$, that is $\epsilon_y = 1$. Therefore,

$$
\begin{aligned}
\overline{\xi_{ISI}}(\widetilde{\Lambda}) &\ge \sum_{xy\in\mathfrak{Ed}(\widetilde{\Lambda})} \frac{\left(\frac{D_x}{n-1}\right)\left(\frac{D_y}{n-1}\right)}{2\,\rho(\widetilde{\Lambda})}\\
&= \frac{1}{2\,\rho(\widetilde{\Lambda})(n-1)^2}\sum_{xy\notin\mathfrak{Ed}(\widetilde{\Lambda})} D_xD_y\\
&= \frac{\overline{S_2}(\widetilde{\Lambda})}{2\,\rho(\widetilde{\Lambda})(n-1)^2}
\end{aligned}
$$

with equality if and only if $\widetilde{\Lambda} \cong K_n$. □

Lemma 2.7. *For a graph $\widetilde{\Lambda}$ with n vertices,*

$$\overline{S_2}(\widetilde{\Lambda}) = 2W(\widetilde{\Lambda})^2 - \tfrac{1}{2}\sum_{x\in\mathfrak{Ve}(\widetilde{\Lambda})} D_x^2 - S_2(\widetilde{\Lambda}).$$

By using Theorem 2.6 and Lemma 2.7, we obtain the following result in terms of Weiner index and the second status connectivity index.

Corollary 2.8. *For a $(n,m)-$graph $\widetilde{\Lambda}$,*

$$\overline{\xi_{ISI}}(\widetilde{\Lambda}) \ge \frac{2W(\widetilde{\Lambda})^2 - \frac{1}{2}\sum\limits_{x\in\mathfrak{Ve}(\widetilde{\Lambda})} D_x^2 - S_2(\widetilde{\Lambda})}{2\,\rho(\widetilde{\Lambda})(n-1)^2} = \frac{1}{4\,\rho(\widetilde{\Lambda})(n-1)^2}\left[4W(\widetilde{\Lambda})^2 - \sum_{x\in\mathfrak{Ve}(\widetilde{\Lambda})} D_x^2 - 2S_2(\widetilde{\Lambda})\right]$$

Theorem 2.9. *For a $(n,m)-$ graph $\widetilde{\Lambda}$,*

$$\overline{\xi_{ISI}}(\widetilde{\Lambda}) \leq \frac{1}{2\ r(\widetilde{\Lambda})}\left((n-1)\xi_1(\widetilde{\Lambda}) - \xi^2(\widetilde{\Lambda})\right). \tag{2.4}$$

with equality if and only if $\widetilde{\Lambda}$ is $s-c$ graph.

Proof. One can easily verify that for any edge $xy \notin \mathfrak{Ed}(\widetilde{\Lambda})$, $\epsilon_x^2 + \epsilon_y^2 \geq 2\epsilon_x\epsilon_y$ with equality if and only if $\epsilon_x = \epsilon_y$. Using this fact, we obtain

$$\begin{aligned}
2\overline{\xi_{ISI}}(\widetilde{\Lambda}) &\leq \sum_{xy\notin\mathfrak{Ed}(\widetilde{\Lambda})} \frac{2\epsilon_x\epsilon_y}{2\ r(\widetilde{\Lambda})} \leq \sum_{xy\notin\mathfrak{Ed}(\widetilde{\Lambda})} \frac{(\epsilon_x^2+\epsilon_y^2)}{2\ r(\widetilde{\Lambda})} \\
&= \sum_{x\in\mathfrak{Ve}(\widetilde{\Lambda})} \frac{(n-1-d(x))}{2\ r(\widetilde{\Lambda})}\epsilon_x^2 \\
&= \left(\frac{n-1}{2\ r(\widetilde{\Lambda})}\right)\sum_{x\in\mathfrak{Ve}(\widetilde{\Lambda})}\epsilon_x^2 - \frac{1}{2\ r(\widetilde{\Lambda})}\sum_{x\in\mathfrak{Ve}(\widetilde{\Lambda})} d(x)\epsilon_x^2 \\
&= \left(\frac{n-1}{2\ r(\widetilde{\Lambda})}\right)\xi_1(\widetilde{\Lambda}) - \frac{1}{2\ r(\widetilde{\Lambda})}\xi^2(\widetilde{\Lambda}). \\
&= \frac{1}{2\ r(\widetilde{\Lambda})}\left[(n-1)\xi_1(\widetilde{\Lambda}) - \xi^2(\widetilde{\Lambda})\right].
\end{aligned}$$

The equality in Equation (2.4) holds if and only if, for every pair of vertices $xy \notin \mathfrak{Ed}(\widetilde{\Lambda})$, $\epsilon_x = \epsilon_y$. It is straightforward to verify that the equality in Equation (2.4) holds if and only if $\widetilde{\Lambda}$ is self-centered or $r(\widetilde{\Lambda}) = 1$ and $\rho(\widetilde{\Lambda}) = 2$. □

The product version of eccentricity based topological indices are defined as

$$\xi_1^*(\widetilde{\Lambda}) = \prod_{x\in\mathfrak{Ve}(\widetilde{\Lambda})}\epsilon_x^2 \text{ and } \xi_2^*(\widetilde{\Lambda}) = \prod_{x\in\mathfrak{Ve}(\widetilde{\Lambda})}\epsilon_x^{d(x)}.$$

Theorem 2.10. *For a (n,m)-graph $\widetilde{\Lambda}$,*

$$\overline{\xi_{ISI}}(\widetilde{\Lambda}) \geq \frac{\overline{\overline{m}}}{2\ \rho(\widetilde{\Lambda})}\left(\frac{\xi_1^*(\widetilde{\Lambda})^{\frac{n-1}{2}}}{\xi_2^*(\widetilde{\Lambda})}\right)^{\frac{1}{\overline{\overline{m}}}}, \tag{2.5}$$

where $\overline{\overline{m}} = \frac{n(n-1)}{2} - m$ holds with equality if and only if $\widetilde{\Lambda}$ is self-centered or $r(\widetilde{\Lambda}) = 1$ and $\rho(\widetilde{\Lambda}) = 2$.

Proof. By arithmetic-geometric mean inequality

$$\begin{aligned}
\frac{\overline{\xi_{ISI}}(\widetilde{\Lambda})}{\overline{\overline{m}}} &\geq \frac{\frac{1}{2\ \rho(\widetilde{\Lambda})}\sum\limits_{xy\notin\mathfrak{Ed}(\widetilde{\Lambda})}\epsilon_x\epsilon_y}{\overline{\overline{m}}}. \\
\frac{\overline{\xi_{ISI}}(\widetilde{\Lambda})}{\overline{\overline{m}}} &\geq \frac{1}{2\ \rho(\widetilde{\Lambda})}\left(\prod_{xy\notin\mathfrak{Ed}(\widetilde{\Lambda})}\epsilon_x\epsilon_y\right)^{\frac{1}{\overline{\overline{m}}}}.
\end{aligned} \tag{2.6}$$

For every vertex $x \in \mathfrak{Ve}(\widetilde{\Lambda})$, the factor ϵ_x appears $(n-1-d(x))$ times in $\prod_{xy \notin \mathfrak{Ed}(\widetilde{\Lambda})} \epsilon_x\epsilon_y$. Thus

$$\begin{aligned} \frac{\overline{\xi_{ISI}}(\widetilde{\Lambda})}{\ddot{\overline{m}}} &\geq \frac{1}{2\ \rho(\widetilde{\Lambda})}\left(\prod_{x\in \mathfrak{Ve}(\widetilde{\Lambda})} \epsilon_x^{(n-1-d(x))}\right)^{\frac{1}{\ddot{\overline{m}}}} \\ &= \frac{1}{2\ \rho(\widetilde{\Lambda})}\left(\frac{\prod_{x\in \mathfrak{Ve}(\widetilde{\Lambda})} (\epsilon_x^2)^{\frac{n-1}{2}}}{\prod_{x\in \mathfrak{Ve}(\widetilde{\Lambda})} \epsilon_x^{d(x)}}\right)^{\frac{1}{\ddot{\overline{m}}}} \\ &= \frac{1}{2\ \rho(\widetilde{\Lambda})}\left(\frac{(\xi_1^*(\widetilde{\Lambda}))^{\frac{n-1}{2}}}{\xi_2^*(\widetilde{\Lambda})}\right)^{\frac{1}{\ddot{\overline{m}}}}. \end{aligned}$$

Therefore,

$$\overline{\xi_{ISI}}(\widetilde{\Lambda}) \geq \frac{\ddot{\overline{m}}}{2\ \rho(\widetilde{\Lambda})}\left(\frac{(\xi_1^*(\widetilde{\Lambda}))^{\frac{n-1}{2}}}{\xi_2^*(\widetilde{\Lambda})}\right)^{\frac{1}{\ddot{\overline{m}}}}.$$

The equality holds if and only if, for each edge $xy \notin \mathfrak{Ed}(\widetilde{\Lambda})$, $\epsilon_x\epsilon_y$ is constant. If $\widetilde{\Lambda}$ is a self-centered graph or $r(\widetilde{\Lambda}) = 1$ and $\rho(\widetilde{\Lambda}) = 2$, then the equality in (2.5) holds trivially. Suppose that the equality holds in (2.5), and $r(\widetilde{\Lambda}) \geq 2$ with $\widetilde{\Lambda}$ being non-self-centered. Since $r(\widetilde{\Lambda}) \geq 2$, there exist vertices $s, t \in \mathfrak{Ve}(\widetilde{\Lambda})$ such that $st \notin \mathfrak{Ed}(\widetilde{\Lambda})$ and $\epsilon_s = r(\widetilde{\Lambda})$. Given that $\widetilde{\Lambda}$ is non-complete, there also exist vertices $a, b \in \mathfrak{Ve}(\widetilde{\Lambda})$ such that $ab \notin \mathfrak{Ed}(\widetilde{\Lambda})$ and $\epsilon_a = \epsilon_b = \rho(\widetilde{\Lambda})$. Because $\widetilde{\Lambda}$ is non-self-centered, we have $r(\widetilde{\Lambda}) < \rho(\widetilde{\Lambda})$. This leads to $\epsilon_s\epsilon_t = r(\widetilde{\Lambda})\epsilon_t < \rho(\widetilde{\Lambda})^2 = \epsilon_a\epsilon_b$, which results in a contradiction. Therefore, equality in (2.5) holds if and only if $\widetilde{\Lambda}$ is self-centered or $r(\widetilde{\Lambda}) = 1$ and $\rho(\widetilde{\Lambda}) = 2$. □

Theorem 2.11. *For a connected graph $\widetilde{\Lambda}$, $\overline{\xi_{ISI}}(\widetilde{\Lambda}) \leq \frac{\rho(\widetilde{\Lambda})}{4\ r(\widetilde{\Lambda})}\ \overline{\xi^c}(\widetilde{\Lambda})$ equality holds if and only if $\widetilde{\Lambda}$ is $s-c$ graph.*

Proof. For a vertex $x \in \mathfrak{Ve}(\widetilde{\Lambda})$, $\epsilon_x \leq \rho(\widetilde{\Lambda})$. Thus

$$\begin{aligned} \overline{\xi_{ISI}}(\widetilde{\Lambda}) &\leq \sum_{xy\notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{\epsilon_x\epsilon_y}{2\ r(\widetilde{\Lambda})} = \sum_{xy\notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{\sqrt{\epsilon_x\epsilon_y}\sqrt{\epsilon_x\epsilon_y}}{2\ r(\widetilde{\Lambda})} \\ &\leq \sum_{xy\notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{\sqrt{\rho(\widetilde{\Lambda})^2}\sqrt{\epsilon_x\epsilon_y}}{2\ r(\widetilde{\Lambda})} \\ &\leq \frac{\rho(\widetilde{\Lambda})}{2\ r(\widetilde{\Lambda})} \sum_{xy\notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{\epsilon_x+\epsilon_y}{2} \\ &= \frac{\rho(\widetilde{\Lambda})}{4\ r(\widetilde{\Lambda})}\ \overline{\xi^c}(\widetilde{\Lambda}) \end{aligned}$$

with equality if and only if, for every pair of vertices $xy \notin \mathfrak{Ed}(\widetilde{\Lambda})$, $\epsilon_x = \epsilon_y = \rho(\widetilde{\Lambda})$. Thus, $\widetilde{\Lambda}$ is self-centered. □

Theorem 2.12. *For any graph $\widetilde{\Lambda}$ with n vertices and m edges,*

$$\overline{\xi_{ISI}}(\widetilde{\Lambda}) \geq \frac{n(n-1)-2m}{2} \tag{2.7}$$

Proof. One can see that for any $xy \notin \mathfrak{E}\mathfrak{d}(\widetilde{\Lambda})$, $\epsilon_x^{\widetilde{\Lambda}}, \epsilon_y^{\widetilde{\Lambda}} \geq 2$. Hence, by the definition of $\overline{\xi_{ISI}}$,

$$\overline{\xi_{ISI}}(\widetilde{\Lambda}) = \sum_{xy\notin\mathfrak{E}\mathfrak{d}(\widetilde{\Lambda})} \frac{\epsilon_x\epsilon_y}{\epsilon_x+\epsilon_y} \geq \sum_{xy\notin\mathfrak{E}\mathfrak{d}(\widetilde{\Lambda})} \frac{2(2)}{2+2} = \frac{n(n-1)}{2} - m.$$

The equality occurs in (2.7) if and only if for any $xy \notin \mathfrak{E}\mathfrak{d}(\widetilde{\Lambda})$, $\epsilon_x = \epsilon_y = 2$, which is equivalent to $\rho(\widetilde{\Lambda}) \leq 2$. The following results is immediately from Theorem 2.12. □

Corollary 2.13. *For any connected unicyclic graph $\widetilde{\Lambda}$ with $n \geq 6$ vertices, $\overline{\xi_{ISI}}(\widetilde{\Lambda}) \geq \frac{n(n-3)}{2}$ with equality if and only if $\widetilde{\Lambda}$ is obtained from S_n by joining two of its pendent vertices with an edge.*

Corollary 2.14. *Let $\widetilde{\Lambda}$ be $(n,m)-$graph. If $\overline{\widetilde{\Lambda}}$ is a connected then complement of $\widetilde{\Lambda}$,*

$$\overline{\xi_{ISI}}(\widetilde{\Lambda}) + \overline{\xi_{ISI}}(\overline{\widetilde{\Lambda}}) \geq \frac{n(n-1)}{2} \tag{2.8}$$

with equality if and only if $\widetilde{\Lambda}$ and $\overline{\widetilde{\Lambda}}$ are $s-c$ with $r(\widetilde{\Lambda}) = \rho(\widetilde{\Lambda}) = 2$.

Proof. Since $|\mathfrak{E}\mathfrak{d}(\widetilde{\Lambda})| = m$. By Theorem 2.12, we get

$$\overline{\xi_{ISI}}(\widetilde{\Lambda}) + \overline{\xi_{ISI}}(\overline{\widetilde{\Lambda}}) \geq \frac{n(n-1)}{2} - m + m = \frac{n(n-1)}{2}.$$

The equality in (2.8) holds if and only if $\rho(\widetilde{\Lambda}), \rho(\overline{\widetilde{\Lambda}}) \leq 2$. Since both $\widetilde{\Lambda}$ and $\overline{\widetilde{\Lambda}}$ are connected and contain no universal vertices, the equality in (2.8) occurs if and only if both $\widetilde{\Lambda}$ and $\overline{\widetilde{\Lambda}}$ are self-centered graphs. □

3. Inequalities between $\overline{\xi_{ISI}}$ and $\overline{\xi_H}$.

In this section, we establish attractive relation between $\overline{\xi_{ISI}}$ and $\overline{\xi_H}$ of graph.

Theorem 3.1. *For a $(n,m)-$connected graph $\widetilde{\Lambda}$, then*

$$\overline{\xi_{ISI}}(\widetilde{\Lambda}) + \frac{(r(\widetilde{\Lambda}))^2}{2}\overline{\xi_H}(\widetilde{\Lambda}) \geq \frac{r(\widetilde{\Lambda})}{2}(n(n-1) - 2m) \tag{3.1}$$

and

$$\overline{\xi_{ISI}}(\widetilde{\Lambda}) + \frac{(\rho(\widetilde{\Lambda}))^2}{2}\overline{\xi_H}(\widetilde{\Lambda}) \geq \frac{\rho(\widetilde{\Lambda})}{2}(n(n-1) - 2m) \tag{3.2}$$

Equality holds in (3.1) if and only if each case of $\widetilde{\Lambda}$ is incident to atleast one vertex of eccentricity $r(\widetilde{\Lambda})$ and holds in (3.2) if and only if each edge of $\widetilde{\Lambda}$ incident to atleast one vertex of eccentricity $\rho(\widetilde{\Lambda})$.

Proof. Let $x_1, x_2, ...x_n$ be the vertices of $\widetilde{\Lambda}$ and let $(\epsilon_{x_1}, \epsilon_{x_2}, ..., \epsilon_{x_n})$ be the sequence of vertex eccentricities of $\widetilde{\Lambda}$ satisfying

$$\rho(\widetilde{\Lambda}) = \epsilon_1 \geq \epsilon_2 ... \geq \epsilon_n = r(\widetilde{\Lambda}).$$

One can see that the following relations holds for any i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$;

$$(\epsilon_{x_i} - r(\widetilde{\Lambda}))(\epsilon_{x_j} - r(\widetilde{\Lambda})) \geq 0 \text{ and } (\rho(\widetilde{\Lambda}) - \epsilon_{x_i})(\rho(\widetilde{\Lambda}) - \epsilon_{x_j}) \geq 0$$

That is,

$$\epsilon_{x_i}\epsilon_{x_j} + (r(\widetilde{\Lambda}))^2 \geq r(\widetilde{\Lambda})(\epsilon_{x_i} + \epsilon_{x_j}) \tag{3.3}$$

and

$$\epsilon_{x_i}\epsilon_{x_j} + (\rho(\widetilde{\Lambda}))^2 \geq \rho(\widetilde{\Lambda})(\epsilon_{x_i} + \epsilon_{x_j}) \tag{3.4}$$

Multiplied by $\frac{1}{\epsilon_{x_i}+\epsilon_{x_j}}$ an both the equation 3.3 and 3.4, we have

$$\frac{\epsilon_{x_i}\epsilon_{x_j}}{\epsilon_{x_i} + \epsilon_{x_j}} + \frac{r(\widetilde{\Lambda})^2}{\epsilon_{x_i} + \epsilon_{x_j}} \geq r(\widetilde{\Lambda}) \tag{3.5}$$

and

$$\frac{\epsilon_{x_i}\epsilon_{x_j}}{\epsilon_{x_i} + \epsilon_{x_j}} + \frac{\rho(\widetilde{\Lambda})^2}{\epsilon_{x_i} + \epsilon_{x_j}} \geq \rho(\widetilde{\Lambda}) \tag{3.6}$$

The summation of (3.5) and (3.6) over all pairs of $nm-$adjacent vertices x_i and x_j in $\widetilde{\Lambda}$, we obtain

$$\sum_{x_ix_j \notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{\epsilon_{x_i}\epsilon_{x_j}}{\epsilon_{x_i} + \epsilon_{x_j}} + \frac{(r(\widetilde{\Lambda}))^2}{2} \sum_{x_ix_j \notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{2}{\epsilon_{x_i} + \epsilon_{x_j}} \geq r(\widetilde{\Lambda})\overline{\ddot{m}}$$

and

$$\sum_{x_ix_j \notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{\epsilon_{x_i}\epsilon_{x_j}}{\epsilon_{x_i} + \epsilon_{x_j}} + \frac{(\rho(\widetilde{\Lambda}))^2}{2} \sum_{x_ix_j \notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{2}{\epsilon_{x_i} + \epsilon_{x_j}} \geq \rho(\widetilde{\Lambda})\overline{\ddot{m}}$$

Hence,

$$\overline{\xi_{ISI}}(\widetilde{\Lambda}) + \frac{r(\widetilde{\Lambda})^2}{2}\ \overline{\xi_H}(\widetilde{\Lambda}) \geq \frac{r(\widetilde{\Lambda})}{2}\ (n(n-1) - 2m)$$

and

$$\overline{\xi_{ISI}}(\widetilde{\Lambda}) + \frac{\rho(\widetilde{\Lambda})^2}{2}\ \overline{\xi_H}(\widetilde{\Lambda}) \geq \frac{\rho(\widetilde{\Lambda})}{2}\ (n(n-1) - 2m)$$

□

Theorem 3.2. *For a $(n, m)-$connected graph $\widetilde{\Lambda}$,*

$$2\overline{\xi_{ISI}}(\widetilde{\Lambda}) + r(\widetilde{\Lambda})\ \rho(\widetilde{\Lambda})\ \overline{\xi_H}(\widetilde{\Lambda}) \leq \frac{r(\widetilde{\Lambda}) + \rho(\widetilde{\Lambda})}{2}\ (n(n-1) - m). \tag{3.7}$$

with inequality if and only if $\widetilde{\Lambda}$ is $s-c$.

Proof. Let $x_1, x_2, ...x_n$ be the vertices of $\widetilde{\Lambda}$ and let $(\epsilon_{x_1}, \epsilon_{x_2}, ..., \epsilon_{x_n})$ be the sequence of vertex eccentricities of $\widetilde{\Lambda}$ satisfying

$$\rho(\widetilde{\Lambda}) = \epsilon_1 \geq \epsilon_2 ... \geq \epsilon_n = r(\widetilde{\Lambda}).$$

One can see that the following relations holds for any i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$;

$$(\rho(\widetilde{\Lambda}) - \epsilon_{x_i})\ (\epsilon_{x_j} - r(\widetilde{\Lambda})) \geq 0$$

and $$(\epsilon_{x_i} - r(\widetilde{\Lambda}))\ (\rho(\widetilde{\Lambda}) - \epsilon_{x_j}) \geq 0$$

This implies that

$$\epsilon_{x_i}\epsilon_{x_j} + r(\widetilde{\Lambda})\ \rho(\widetilde{\Lambda}) \leq r(\widetilde{\Lambda})\epsilon_{x_i} + \rho(\widetilde{\Lambda})\epsilon_{x_j} \tag{3.8}$$

and $$\epsilon_{x_i}\epsilon_{x_j} + r(\widetilde{\Lambda})\ \rho(\widetilde{\Lambda}) \leq r(\widetilde{\Lambda})\epsilon_{x_j} + \rho(\widetilde{\Lambda})\epsilon_{x_i} \tag{3.9}$$

By summing up the Equations in (3.8) and (3.9), we get

$$2\epsilon_{x_i}\epsilon_{x_j} + 2\ r(\widetilde{\Lambda})\ \rho(\widetilde{\Lambda}) \leq (r(\widetilde{\Lambda}) + \rho(\widetilde{\Lambda}))(\epsilon_{x_i} + \epsilon_{x_j}) \tag{3.10}$$

Divided by $(\epsilon_{x_i} + \epsilon_{x_j})$ on both sides on (3.10), we have

$$2\frac{\epsilon_{x_i}\epsilon_{x_j}}{\epsilon_{x_i} + \epsilon_{x_j}} + \frac{2\ r(\widetilde{\Lambda})\ \rho(\widetilde{\Lambda})}{\epsilon_{x_i} + \epsilon_{x_j}} \leq r(\widetilde{\Lambda}) + \rho(\widetilde{\Lambda}).$$

Taking summation over all pairs of non-adjacent pair of vertices x_i and x_j in $\widetilde{\Lambda}$, we have

$$2 \sum_{x_ix_j \notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{\epsilon_{x_i}\epsilon_{x_j}}{\epsilon_{x_i} + \epsilon_{x_j}} + \sum_{x_ix_j \notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{2\ r(\widetilde{\Lambda})\ \rho(\widetilde{\Lambda})}{\epsilon_{x_i} + \epsilon_{x_j}} \leq \sum_{x_ix_j \notin \mathfrak{Ed}(\widetilde{\Lambda})} (r(\widetilde{\Lambda}) + \rho(\widetilde{\Lambda})).$$

$$\text{Hence,} \quad 2\overline{\xi_{ISI}}(\widetilde{\Lambda}) + r(\widetilde{\Lambda})\ \rho(\widetilde{\Lambda})\ \overline{\xi_H}(\widetilde{\Lambda}) \leq \frac{(r(\widetilde{\Lambda}) + \rho(\widetilde{\Lambda}))}{2}\ (n(n-1) - m).$$

□

4. Some Graph Constructions

In this section, we calculate the inverse sum indeg eccentric coindices for various graph constructions. These graph models are crucial for understanding how the $\overline{\xi_{ISI}}(\widetilde{\Lambda})$ behaves in graphs with increased complexity and symmetry.

4.1. Double graph

Let $\widetilde{\Lambda}$ be a graph with the vertex set $\mathfrak{Ve}(\widetilde{\Lambda}) = \{v_1, v_2, \ldots, v_n\}$. Consider two sets of vertices, $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ of $\widetilde{\Lambda}$, preserving the original edge set of each version and adding the edges x_iy_j and x_jy_i for every edge $v_iv_j \in \mathfrak{Ed}(\widetilde{\Lambda})$. The result is a new graph, $\widetilde{\Lambda}^*$, known as the double graph of $\widetilde{\Lambda}$.

Theorem 4.1. *For a double graph $\widetilde{\Lambda}^*$ of $\widetilde{\Lambda}$,*

$$\overline{\xi_{ISI}}(\widetilde{\Lambda^*}) = 4\overline{\xi_{ISI}}(\widetilde{\Lambda}) + \tfrac{\tau(\widetilde{\Lambda})}{2}.$$

Proof. From the definition of double graph $\widetilde{\Lambda}^*$, we have

$$\epsilon_{\widetilde{\Lambda}^*}(x_i) = \epsilon_{\widetilde{\Lambda}^*}(y_i) = \begin{cases} 2 & if\ \epsilon_{\widetilde{\Lambda}}(v_i) = 1, \\ \epsilon_{\widetilde{\Lambda}}(v_i) & if\ \epsilon_{\widetilde{\Lambda}}(v_i) \geq 2. \end{cases}$$

Hence,

$$\begin{aligned}
\overline{\xi_{ISI}}(\widetilde{\Lambda}) &= \sum_{xy \notin \mathfrak{Ed}(\widetilde{\widetilde{\Lambda^*}})} \frac{\epsilon_x^{\widetilde{\Lambda^*}}\epsilon_y^{\widetilde{\Lambda^*}}}{\epsilon_x^{\widetilde{\Lambda^*}} + \epsilon_y^{\widetilde{\Lambda^*}}} \\
&= \sum_{v_iv_j \notin \mathfrak{Ed}(\widetilde{\Lambda})} \left[\frac{\epsilon_{x_i}^{\widetilde{\Lambda^*}}\epsilon_{x_j}^{\widetilde{\Lambda^*}}}{\epsilon_{x_i}^{\widetilde{\Lambda^*}} + \epsilon_{x_j}^{\widetilde{\Lambda^*}}} + \frac{\epsilon_{y_i}^{\widetilde{\Lambda^*}}\epsilon_{y_j}^{\widetilde{\Lambda^*}}}{\epsilon_{y_i}^{\widetilde{\Lambda^*}} + \epsilon_{y_j}^{\widetilde{\Lambda^*}}} + \frac{\epsilon_{x_i}^{\widetilde{\Lambda^*}}\epsilon_{y_j}^{\widetilde{\Lambda^*}}}{\epsilon_{x_i}^{\widetilde{\Lambda^*}} + \epsilon_{y_j}^{\widetilde{\Lambda^*}}} + \frac{\epsilon_{x_j}^{\widetilde{\Lambda^*}}\epsilon_{y_i}^{\widetilde{\Lambda^*}}}{\epsilon_{x_j}^{\widetilde{\Lambda^*}} + \epsilon_{y_i}^{\widetilde{\Lambda^*}}} \right] + \sum_{i=1}^{n} \frac{\epsilon_{x_i}^{\widetilde{\Lambda^*}}\epsilon_{y_i}^{\widetilde{\Lambda^*}}}{\epsilon_{x_i}^{\widetilde{\Lambda^*}} + \epsilon_{y_i}^{\widetilde{\Lambda^*}}} \\
&= 4 \sum_{v_iv_j \notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{\epsilon_{v_i}^{\Lambda}\epsilon_{v_j}^{\Lambda}}{\epsilon_{v_i}^{\Lambda} + \epsilon_{y_j}^{\Lambda}} + \sum_{i=1} \frac{(\epsilon_{v_i}^{\widetilde{\Lambda}})^2}{2\epsilon_{v_i}} \\
&= 4\overline{\xi_{ISI}}(\widetilde{\Lambda}) + \frac{1}{2}\sum_{i=1}^{n} \epsilon_{v_i}^{\widetilde{\Lambda}} \\
&= 4\ \overline{\xi_{ISI}}(\widetilde{\Lambda}) + \frac{1}{2}\tau(\widetilde{\Lambda}).
\end{aligned}$$

□

4.2. Extended double graph

Let $\widetilde{\Lambda}$ be a graph with $\mathfrak{Ve}(\widetilde{\Lambda}) = \{v_1, v_2, \ldots, v_n\}$. Consider a bipartite graph $\widetilde{\Lambda}^{**}$ with bipartition (X, Y) where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. In this graph, an edge x_i is adjacent to y_j if and only if $i = j$ or $v_iv_j \in \mathfrak{Ed}(\widetilde{\Lambda})$. This graph is known as the extended double graph of $\widetilde{\Lambda}$. It is clear from the above definition that for each $1 \leq i \leq n$,

$$\epsilon_{\widetilde{\Lambda}^{**}}(y_i) = \epsilon_{\widetilde{\Lambda}}(v_i) + 1.$$

Theorem 4.2. *For a extended double graph $\widetilde{\Lambda}^{**}$ of $\widetilde{\Lambda}$,*

$$\begin{aligned} \overline{\xi_{ISI}}(\widetilde{\Lambda}^{**}) &\leq \frac{1}{2}\left[\overline{\xi_{ISI}}(\widetilde{\Lambda}) + \bar{\xi}_H(\widetilde{\Lambda}) + \frac{\overline{\xi_{M_1}}(\widetilde{\Lambda})}{2} + \frac{\overline{\xi_{M_2}}(\widetilde{\Lambda})}{2}\right] + \frac{n(n-1)}{2} - 4m \\ &\quad + \frac{1}{r(\widetilde{\Lambda})+1}\left[\frac{1}{2}(\tau(\widetilde{\Lambda})^2 - E_1(\widetilde{\Lambda})) + (n-1)\tau(\widetilde{\Lambda}) + \frac{n(n-1)}{2}\right]. \end{aligned}$$

Proof. By the definition of $\overline{\xi_{ISI}}$, we obtain

$$\begin{aligned} \overline{\xi_{ISI}}(\widetilde{\Lambda}^{**}) &= \sum_{xy \notin \mathfrak{Ed}(\widetilde{\Lambda}^{**})} \frac{\epsilon_x^{\widetilde{\Lambda}^{**}} \epsilon_y^{\widetilde{\Lambda}^{**}}}{\epsilon_x^{\widetilde{\Lambda}^{**}} + \epsilon_y^{\widetilde{\Lambda}^{**}}} \\ &= \sum_{v_iv_j \notin \mathfrak{Ed}(\widetilde{\Lambda})} \left[\frac{\epsilon_{x_i}^{\widetilde{\Lambda}^{**}} \epsilon_{y_j}^{\widetilde{\Lambda}^{**}}}{\epsilon_{x_i}^{\widetilde{\Lambda}^{**}} + \epsilon_{y_j}^{\widetilde{\Lambda}^{**}}} + \frac{\epsilon_{x_j}^{\widetilde{\Lambda}^{**}} \epsilon_{y_i}^{\widetilde{\Lambda}^{**}}}{\epsilon_{x_j}^{\widetilde{\Lambda}^{**}} + \epsilon_{y_j}^{\widetilde{\Lambda}^{**}}}\right] + \sum_{1 \leq i < j \leq n} \left[\frac{\epsilon_{x_i}^{\widetilde{\Lambda}^{**}} \epsilon_{x_j}^{\widetilde{\Lambda}^{**}}}{\epsilon_{x_i}^{\widetilde{\Lambda}^{**}} + \epsilon_{x_j}^{\widetilde{\Lambda}^{**}}} + \frac{\epsilon_{y_i}^{\widetilde{\Lambda}^{**}} \epsilon_{y_j}^{\widetilde{\Lambda}^{**}}}{\epsilon_{y_i}^{\widetilde{\Lambda}^{**}} + \epsilon_{y_j}^{\widetilde{\Lambda}^{**}}}\right] \\ &= 2 \sum_{v_iv_j \notin \mathfrak{Ed}(\widetilde{\Lambda})} \frac{(\epsilon_{v_j}^{\widetilde{\Lambda}} + 1)(\epsilon_{v_j}^{\widetilde{\Lambda}} + 1)}{\epsilon_{v_i}^{\widetilde{\Lambda}} + 1 + \epsilon_{v_j}^{\widetilde{\Lambda}} + 1} + 2 \sum_{1 \leq i < j \leq n} \frac{(\epsilon_{v_j}^{\widetilde{\Lambda}} + 1)(\epsilon_{v_j}^{\widetilde{\Lambda}} + 1)}{\epsilon_{v_i}^{\widetilde{\Lambda}} + 1 + \epsilon_{v_j}^{\widetilde{\Lambda}} + 1} \\ &= A_1 + A_2, \end{aligned}$$

where,

$$A_1 = 2 \sum_{v_iv_j \notin \mathfrak{Ed}(\widetilde{\Lambda})} \left[\frac{\epsilon_{v_i}^{\widetilde{\Lambda}} \epsilon_{v_j}^{\widetilde{\Lambda}} + (\epsilon_{v_i}^{\widetilde{\Lambda}} + \epsilon_{v_j}^{\widetilde{\Lambda}}) + 2}{(\epsilon_{v_i}^{\widetilde{\Lambda}} + \epsilon_{v_j}^{\widetilde{\Lambda}}) + 2}\right].$$

By Jenson's inequality, we obtain $\frac{1}{\epsilon_{v_i} + \epsilon_{v_j} + 2} \leq \frac{1}{4(\epsilon_{v_i} + \epsilon_{v_j})} + \frac{1}{8}$ with equality if and only if $\epsilon_{v_i}^{\widetilde{\Lambda}} + \epsilon_{v_j}^{\widetilde{\Lambda}} = 2$, for every $n - a$ (non-adjacent) pair of vertices $v_iv_j \notin \mathfrak{Ed}(\widetilde{\Lambda})$. Hence,

$$\begin{aligned} A_1 &\leq \frac{1}{2} \sum_{v_iv_j \notin \mathfrak{Ed}(\widetilde{\Lambda})} \left[\frac{\epsilon_{v_i}^{\widetilde{\Lambda}} \epsilon_{v_j}^{\widetilde{\Lambda}} + (\epsilon_{v_i}^{\widetilde{\Lambda}} + \epsilon_{v_j}^{\widetilde{\Lambda}}) + 2}{(\epsilon_{v_i}^{\widetilde{\Lambda}} + \epsilon_{v_j}^{\widetilde{\Lambda}})} + \frac{\epsilon_{v_i}^{\widetilde{\Lambda}} \epsilon_{v_j}^{\widetilde{\Lambda}} + (\epsilon_{v_i}^{\widetilde{\Lambda}} + \epsilon_{v_j}^{\widetilde{\Lambda}}) + 2}{2}\right] \\ &= \frac{1}{2}\left[\overline{\xi_{ISI}}(\widetilde{\Lambda}) + \overline{\overline{m}} + \bar{\xi}_H(\widetilde{\Lambda}) + \frac{\overline{\xi_{M_2}}(\widetilde{\Lambda})}{2} + \frac{\overline{\xi_{M_1}}(\widetilde{\Lambda})}{2} + \overline{\overline{m}}\right] \\ &= \frac{1}{2}\left[\overline{\xi_{ISI}}(\widetilde{\Lambda}) + \bar{\xi}_H(\widetilde{\Lambda}) + \frac{\overline{\xi_{M_2}}(\widetilde{\Lambda})}{2} + \frac{\overline{\xi_{M_1}}(\widetilde{\Lambda})}{2} + n(n-1) - 2m\right]. \end{aligned}$$

Similarly, we can prove

$$\begin{aligned} A_2 &= 2\sum_{1\le i<j\le n}\frac{(\epsilon_{v_i}^{\widetilde{\Lambda}}+1)(\epsilon_{v_j}^{\widetilde{\Lambda}}+1)}{\epsilon_{v_i}^{\widetilde{\Lambda}}+\epsilon_{v_j}^{\widetilde{\Lambda}}+2} \\ &\le \frac{1}{r(\widetilde{\Lambda})+1}\left[\frac{(\tau(\widetilde{\Lambda})^2-E_1(\widetilde{\Lambda}))}{2}+(n-1)\tau(\widetilde{\Lambda})+\binom{n}{2}\right]. \end{aligned}$$

Hence,

$$\begin{aligned} \overline{\xi_{ISI}}(\widetilde{\Lambda}^{**}) &\le \frac{1}{2}\left[\overline{\xi_{ISI}}(\widetilde{\Lambda})+\bar{\xi}_H(\widetilde{\Lambda})+\frac{\overline{\xi_{M_1}}(\widetilde{\Lambda})}{2}+\frac{\overline{\xi_{M_2}}(\widetilde{\Lambda})}{2}\right]+\frac{n(n-1)}{2}-4m \\ &\quad +\frac{1}{r(\widetilde{\Lambda})+1}\left[\frac{1}{2}(\tau(\widetilde{\Lambda})^2-E_1(\widetilde{\Lambda}))+(n-1)\tau(\widetilde{\Lambda})+\frac{n(n-1)}{2}\right]. \end{aligned}$$

□

4.3. Generalized hierarchical product

Let $\phi \neq \mathbb{Y} \subseteq \mathfrak{Ve}(\widetilde{\Lambda_1})$. The generalized hierarchical product $\widetilde{\Lambda_1}(\mathbb{Y}) \sqcap \widetilde{\Lambda_2}$ of graphs $\widetilde{\Lambda_1}$ and $\widetilde{\Lambda_2}$ is a graph with vertex set $\mathfrak{Ve}(\widetilde{\Lambda_1}) \times \mathfrak{Ve}(\widetilde{\Lambda_2})$. Two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either $[u_1 = v_1 \in \mathbb{Y}$ and $u_2v_2 \in \mathfrak{Ed}(\widetilde{\Lambda_2})]$ or $[u_2 = v_2 \in \mathfrak{Ve}(\widetilde{\Lambda_2})$ and $u_1v_1 \in \mathfrak{Ed}(\widetilde{\Lambda_1})]$.

For $\phi \neq \mathbb{Y} \subseteq \mathfrak{Ve}(\widetilde{\Lambda})$, a path between vertices $u, v \in \mathfrak{Ve}(\widetilde{\Lambda})$ through $\mathbb{Y}$ is a uv-path in $\widetilde{\Lambda}$ containing some vertex $z \in \mathbb{Y}$ (where z could be either u or v). The distance between u and v through $\mathbb{Y}$, denoted by $d_{\widetilde{\Lambda}(\mathbb{Y})}(u, v)$, is the length of any shortest path between u and v that belongs to $\mathbb{Y}$. Thus, $d_{\widetilde{\Lambda}(\mathbb{Y})}(u, v) = d_{\widetilde{\Lambda}}(u, v)$. For $u \in \mathfrak{Ve}(\widetilde{\Lambda})$, we define $\epsilon_{\widetilde{\Lambda}(\mathbb{Y})}(u) = \max_{v\in\mathfrak{Ve}(\widetilde{\Lambda})} d_{\widetilde{\Lambda}(\mathbb{Y})}(u, v)$. The following invariants related to $\mathbb{Y}$ are defined for the convenience.

$$\begin{aligned} \tau(\mathbb{Y}) &= \sum_{u\in\mathbb{Y}}\epsilon_{\widetilde{\Lambda}(\mathbb{Y})}(u)=\sum_{u\in\mathbb{Y}}\epsilon_{\widetilde{\Lambda}}(u); \\ E_1(\mathbb{Y}) &= \sum_{u\in\mathbb{Y}}\epsilon_{\widetilde{\Lambda}(\mathbb{Y})}(u)^2=\sum_{u\in\mathbb{Y}}\epsilon_{\widetilde{\Lambda}}(u)^2; \\ \tau(\widetilde{\Lambda}(\mathbb{Y})) &= \sum_{u\in\mathfrak{Ve}(\widetilde{\Lambda})}\epsilon_{\widetilde{\Lambda}(\mathbb{Y})}(u); \\ E_1(\widetilde{\Lambda}(\mathbb{Y})) &= \sum_{u\in\widetilde{\Lambda}}\epsilon_{\widetilde{\Lambda}(\mathbb{Y})}(u)^2; \\ \overline{\xi^c}(\widetilde{\Lambda}(\mathbb{Y})) &= \sum_{uv\notin\mathfrak{Ed}(\widetilde{\Lambda})}(\epsilon_{\widetilde{\Lambda}(\mathbb{Y})}(u)+\epsilon_{\widetilde{\Lambda}(\mathbb{Y})}(v)); \\ \overline{E}_2(\widetilde{\Lambda}(\mathbb{Y})) &= \sum_{uv\notin\mathfrak{Ed}(\widetilde{\Lambda})}\epsilon_{\widetilde{\Lambda}(\mathbb{Y})}(u)\epsilon_{\widetilde{\Lambda}(\mathbb{Y})}(v). \end{aligned}$$

The eccentricity of a vertex (u_1, u_2) in $\widetilde{\Lambda}_1(\mathbb{Y}) \sqcap \widetilde{\Lambda_2}$as follows.

$$\epsilon_{\widetilde{\Lambda_1}(\mathbb{Y})\cup\widetilde{\Lambda_2}}(u_1,u_2)=\epsilon_{\widetilde{\Lambda_1}(\mathbb{Y})}(u_1)+\epsilon_{\widetilde{\Lambda_2}}(u_2).$$

$\widetilde{\Lambda_1}\Box\widetilde{\Lambda_2}$ is the cartesian product of graphs $\widetilde{\Lambda_1}$ and $\widetilde{\Lambda_2}$ is a graph with set of all vertices $\mathfrak{Ve}(\widetilde{\Lambda_1}) \times \mathfrak{Ve}(\widetilde{\Lambda_2})$. Two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either $[u_1 = v_1 \in \mathfrak{Ve}(\widetilde{\Lambda_1})$ and $u_2v_2 \in \mathfrak{Ed}(\widetilde{\Lambda_2})]$ or $[u_2 = v_2 \in \mathfrak{Ve}(\widetilde{\Lambda_2})$ and $u_1v_1 \in \mathfrak{Ed}(\widetilde{\Lambda_1})]$.

Theorem 4.3. *For a graph* $\widetilde{\Lambda_1}(u) \sqcap \widetilde{\Lambda_2}$,

$$\begin{aligned}\overline{\xi_{ISI}}(\widetilde{\Lambda_1}(\mathbb{Y}) \sqcap \widetilde{\Lambda_2}) \ \leq\ & \frac{1}{2(r_{\widetilde{\Lambda_1}} + r_{\widetilde{\Lambda_2}})}\Bigg(\overline{\ddot{m}_2}\xi(\mathbb{Y}) + \overline{\xi_{M_1}}(\widetilde{\Lambda_2})\tau(\mathbb{Y}) + |\mathbb{Y}|\overline{\xi_{M_2}}(\widetilde{\Lambda_2}) \\ & + \left[\overline{\ddot{m}_2}\xi_1(\widetilde{\Lambda_2}) + \overline{\xi_{M_1}}(\widetilde{\Lambda_1}(\mathbb{Y}))\tau(\widetilde{\Lambda_2}) + n_2\overline{\xi_{M_2}}(\widetilde{\Lambda_1}(\mathbb{Y}))\right] + \\ & \left[\binom{n_2}{2}\ (\xi_1(\widetilde{\Lambda_1}(\mathbb{Y})) - \xi_1(\mathbb{Y})) + (\tau(\widetilde{\Lambda_1}(\mathbb{Y})) - \tau(\mathbb{Y}))(n_2-1)\tau(\widetilde{\Lambda_2}) + \frac{1}{2}(n_1 - |\mathbb{Y}|)(\tau(\widetilde{\Lambda_2})^2 - \xi_1(\widetilde{\Lambda_2}))\right] \\ & + \left[\binom{n_2}{2}\ \left(\tau(\widetilde{\Lambda_1}(\mathbb{Y}))^2 - \xi_1(\widetilde{\Lambda_1}(\mathbb{Y}))\right) + \binom{n_1}{2}\left(\tau(\widetilde{\Lambda_2})^2 - \xi_1(\widetilde{\Lambda_2})\right) + (n_1-1)(n_2-1)\tau(\widetilde{\Lambda_1}(\mathbb{Y}))\tau(\widetilde{\Lambda_2})\right]\Bigg).\end{aligned}$$

Proof. By the definition of $\overline{\xi_{ISI}}$, we obtain

$$\overline{\xi_{ISI}}(\widetilde{\Lambda_1}(\mathbb{Y}) \sqcap \widetilde{\Lambda_2}) \ = \ \sum_{(x_1,y_1)(x_2,y_2)\notin\mathfrak{Ed}(\widetilde{\Lambda_1}(\mathbb{Y})\sqcap\widetilde{\Lambda_2})} \frac{\epsilon_{(x_1,y_1)}^{\widetilde{\Lambda_1}(\mathbb{Y})\sqcap\widetilde{\Lambda_2}}\ \epsilon_{(x_2,y_2)}^{\widetilde{\Lambda_1}(\mathbb{Y})\sqcap\widetilde{\Lambda_2}}}{\epsilon_{(x_1,y_1)}^{\widetilde{\Lambda_1}(\mathbb{Y})\sqcap\widetilde{\Lambda_2}} + \epsilon_{(x_2,y_2)}^{\widetilde{\Lambda_1}(\mathbb{Y})\sqcap\widetilde{\Lambda_2}}}$$

By analyzing the structure of $\widetilde{\Lambda_1}(\mathbb{Y}) \sqcap \widetilde{\Lambda_2}$, we obtain the following edge partition with respect to the eccentricity of the end vertices of the edges.

First, calculate the sum A_1 over all vertices $x_1 \in \mathbb{Y}$ and $n-a$ vertex pairs $y_1y_2 \notin \mathfrak{Ed}(\widetilde{\Lambda_2})$.

$$\begin{aligned}A_1 \ &= \ \sum_{x_1\in\mathbb{Y}}\sum_{y_1y_2\notin\mathfrak{Ed}(\widetilde{\Lambda_2})} \frac{(\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_1}^{\widetilde{\Lambda_2}})\ \ (\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_2}^{\widetilde{\Lambda_2}})}{(\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_1}^{\widetilde{\Lambda_2}}) + (\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_2}^{\widetilde{\Lambda_2}})} \\ &\leq \ \sum_{x_1\in\mathbb{Y}}\sum_{y_1y_2\notin\mathfrak{Ed}(\widetilde{\Lambda_2})} \frac{(\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})})^2 + \epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})}(\epsilon_{y_1}^{\widetilde{\Lambda_2}} + \epsilon_{y_2}^{\widetilde{\Lambda_2}}) + \epsilon_{y_1}^{\widetilde{\Lambda_2}}\epsilon_{y_2}^{\widetilde{\Lambda_2}}}{2r_{\widetilde{\Lambda_1}} + 2r_{\widetilde{\Lambda_2}}} \\ &= \ \frac{1}{2(r_{\widetilde{\Lambda_1}} + r_{\widetilde{\Lambda_2}})}\left[\overline{\ddot{m}_2}\sum_{x_1\in\mathbb{Y}}(\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})})^2 + \sum_{x_1\in\mathbb{Y}}\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})}\sum_{y_1y_2\notin\mathfrak{Ed}(\widetilde{\Lambda_2})}(\epsilon_{y_1}^{\widetilde{\Lambda_2}} + \epsilon_{y_2}^{\widetilde{\Lambda_1}}) + |\mathbb{Y}|\sum_{y_1y_2\notin\mathfrak{Ed}(\widetilde{\Lambda_2})}\epsilon_{y_1}^{\widetilde{\Lambda_2}}\epsilon_{y_2}^{\widetilde{\Lambda_2}}\right] \\ &= \ \frac{1}{2(r_{\widetilde{\Lambda_1}} + r_{\widetilde{\Lambda_2}})}\left[\overline{\ddot{m}_2}\xi(\mathbb{Y}) + \overline{\xi_{M_1}}(\widetilde{\Lambda_2})\tau(\mathbb{Y}) + |\mathbb{Y}|\overline{\xi_{M_2}}(\widetilde{\Lambda_2})\right].\end{aligned}$$

Next, we consider A_2 which is taken over all vertices $y_1 \in \mathfrak{Ve}(\widetilde{\Lambda_2})$ and $n-a$ vertex pairs $x_1x_2 \notin \mathfrak{Ed}(\widetilde{\Lambda_1})$.

$$\begin{aligned}A_2 \ &= \ \sum_{y_1\in\mathfrak{Ve}(\widetilde{\Lambda_2})}\sum_{x_1x_2\notin\mathfrak{Ed}(\widetilde{\Lambda_1})} \frac{(\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_1}^{\widetilde{\Lambda_2}})\ \ (\epsilon_{x_2}^{\widetilde{\Lambda_2}(\mathbb{Y})} + \epsilon_{y_1}^{\widetilde{\Lambda_2}})}{(\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_1}^{\widetilde{\Lambda_2}}) + (\epsilon_{x_2}^{\widetilde{\Lambda_2}(\mathbb{Y})} + \epsilon_{y_1}^{\widetilde{\Lambda_2}})} \\ &\leq \ \sum_{y_1\in\mathfrak{Ve}(\widetilde{\Lambda_2})}\sum_{x_1x_2\notin\mathfrak{Ed}(\widetilde{\Lambda_1})} \frac{(\epsilon_{y_1}^{\widetilde{\Lambda_2}})^2 + \epsilon_{y_1}^{\widetilde{\Lambda_2}}(\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{x_2}^{\widetilde{\Lambda_1}(\mathbb{Y})}) + \epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})}\epsilon_{x_2}^{\widetilde{\Lambda_1}(\mathbb{Y})}}{2(r_{\widetilde{\Lambda_1}} + r_{\widetilde{\Lambda_2}})} \\ &= \ \frac{1}{2(r_{\widetilde{\Lambda_1}} + r_{\widetilde{\Lambda_2}})}\left[\overline{\ddot{m}_2}\xi_1(\widetilde{\Lambda_2}) + \overline{\xi_{M_1}}(\widetilde{\Lambda_1}(\mathbb{Y}))\tau(\widetilde{\Lambda_2}) + n_2\overline{\xi_{M_2}}(\widetilde{\Lambda_1}(\mathbb{Y}))\right].\end{aligned}$$

The third sum A_3 is taken over all vertices $x_1 \in \mathfrak{Ve}(\widetilde{\Lambda_1})|\mathbb{Y}$ and unordered vertex pairs $\{y_1, y_2\} \subseteq \mathfrak{Ve}(\widetilde{\Lambda_2})$.

$$
\begin{aligned}
A_3 &= \sum_{x_1 \in \mathfrak{Ve}(\widetilde{\Lambda_1})|\mathbb{Y}} \sum_{y_1 y_2 \subseteq \mathfrak{Ve}(\widetilde{\Lambda_2})} \frac{(\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_1}^{\widetilde{\Lambda_2}}) \ (\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_2}^{\widetilde{\Lambda_2}})}{(\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_1}^{\widetilde{\Lambda_2}}) + (\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_2}^{\widetilde{\Lambda_2}})} \\
&\leq \sum_{x_1 \in \mathfrak{Ve}(\widetilde{\Lambda_1})|\mathbb{Y}} \sum_{y_1 y_2 \subseteq \mathfrak{Ve}(\widetilde{\Lambda_2})} \frac{(\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})})^2 + \epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})}(\epsilon_{y_1}^{\widetilde{\Lambda_2}} + \epsilon_{y_2}^{\widetilde{\Lambda_2}}) + \epsilon_{y_1}^{\widetilde{\Lambda_2}} \epsilon_{y_2}^{\widetilde{\Lambda_2}}}{2(r_{\widetilde{\Lambda_1}} + r_{\widetilde{\Lambda_2}})} = \frac{1}{2(r_{\widetilde{\Lambda_1}} + r_{\widetilde{\Lambda_2}})} \left[\binom{n_2}{2} \ (\xi_1(\widetilde{\Lambda_1}(\mathbb{Y})) \right. \\
&\left. -\xi_1(\mathbb{Y})) + (\tau(\widetilde{\Lambda_1}(\mathbb{Y})) - \tau(\mathbb{Y}))(n_2 - 1)\tau(\widetilde{\Lambda_2}) + \frac{1}{2}(n_1 - |\mathbb{Y}|)(\tau(\widetilde{\Lambda_2})^2 - \xi_1(\widetilde{\Lambda_2})) \right].
\end{aligned}
$$

The final sum A_4 is taken over all unordered vertex pairs $\{x_1, x_2\} \subseteq \mathfrak{Ve}(\widetilde{\Lambda_1})$ and $\{y_1, y_2\} \subseteq \mathfrak{Ve}(\widetilde{\Lambda_2})$.

$$
\begin{aligned}
A_4 &= \sum_{\{x_1,x_2\} \subseteq \mathfrak{Ve}(\widetilde{\Lambda_1})} \sum_{\{y_1 y_2\} \subseteq \mathfrak{Ve}(\widetilde{\Lambda_2})} \left[\frac{(\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_1}^{\widetilde{\Lambda_2}}) \ (\epsilon_{x_2}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_2}^{\widetilde{\Lambda_2}})}{(\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_1}^{\widetilde{\Lambda_2}}) + (\epsilon_{x_2}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_2}^{\widetilde{\Lambda_2}})} + \frac{(\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_2}^{\widetilde{\Lambda_2}}) \ (\epsilon_{x_2}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_1}^{\widetilde{\Lambda_2}})}{(\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_2}^{\widetilde{\Lambda_2}}) + (\epsilon_{x_2}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{y_1}^{\widetilde{\Lambda_2}})} \right] \\
&\leq \sum_{\{x_1,x_2\} \subseteq \mathfrak{Ve}(\widetilde{\Lambda_1})} \sum_{\{y_1 y_2\} \subseteq \mathfrak{Ve}(\widetilde{\Lambda_2})} \left[\frac{2\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})} \epsilon_{x_2}^{\widetilde{\Lambda_1}(\mathbb{Y})} + 2\epsilon_{y_1}^{\widetilde{\Lambda_2}} \epsilon_{y_2}^{\widetilde{\Lambda_2}} + (\epsilon_{x_1}^{\widetilde{\Lambda_1}(\mathbb{Y})} + \epsilon_{x_2}^{\widetilde{\Lambda_1}(\mathbb{Y})})(\epsilon_{y_1}^{\widetilde{\Lambda_2}} + \epsilon_{y_1}^{\widetilde{\Lambda_1}})}{2(r_{\widetilde{\Lambda_1}} + r_{\widetilde{\Lambda_2}})} \right] \\
&= \frac{1}{2(r_{\widetilde{\Lambda_1}} + r_{\widetilde{\Lambda_2}})} \left[\binom{n_2}{2} \ \left(\tau(\widetilde{\Lambda_1}(\mathbb{Y}))^2 - \xi_1(\widetilde{\Lambda_1}(\mathbb{Y}))\right) + \binom{n_1}{2} \left(\tau(\widetilde{\Lambda_2})^2 - \xi_1(\widetilde{\Lambda_2})\right) \right. \\
&\left. +(n_1 - 1)(n_2 - 1)\tau(\widetilde{\Lambda_1}(\mathbb{Y}))\tau(\widetilde{\Lambda_2}) \right].
\end{aligned}
$$

By adding A_1 to A_4 and simplifying we get the required result. □

4.4. Composition

$\widetilde{\Lambda_1}[\widetilde{\Lambda_2}]$ is the composition of $\widetilde{\Lambda_1}$ and $\widetilde{\Lambda_2}$ is a graph with vertex set $\mathfrak{Ve}(\widetilde{\Lambda_1}) \times \mathfrak{Ve}(\widetilde{\Lambda_2})$. Two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1 x_2 \in \mathfrak{Ed}(\widetilde{\Lambda_1})$ or $[x_1 = x_2 \in \mathfrak{Ve}(\widetilde{\Lambda})$ and $y_1 y_2 \in \mathfrak{Ed}(\widetilde{\Lambda_2})]$. The composition of two graphs is also known as their lexicographic product or wreath product.

The eccentricity of a vertex $(x_1, y_1) \in \widetilde{\Lambda_1}[\widetilde{\Lambda_2}]$ is given by

$$
\epsilon_{(x_1,y_1)}^{\epsilon_1[\widetilde{\Lambda_2}]} = \begin{cases} 1 & if \ \ \epsilon_{x_1}^{\widetilde{\Lambda_1}} = \epsilon_{y_1}^{\widetilde{\Lambda_2}} = 1, \\ 2 & if \ \ \epsilon_{x_1}^{\widetilde{\Lambda_1}} = \epsilon_{y_1}^{\widetilde{\Lambda_2}} \geq 2, \\ \epsilon_{x_1}^{\widetilde{\Lambda_1}} & \ \ if \ \ \epsilon_{x_1}^{\widetilde{\Lambda_1}} \geq 2. \end{cases}
$$

The number of universal vertices of $\widetilde{\Lambda}$ is denoted by $C(\widetilde{\Lambda})$.

Theorem 4.4. *The inverse sum indeg eccentric coindex of* $\widetilde{\Lambda_1}[\widetilde{\Lambda_2}]$ *is given by*

$$
\overline{\xi_{ISI}}(\widetilde{\Lambda_1}[\widetilde{\Lambda_2}]) = \frac{\overline{\overline{m}}}{2}(\tau(\widetilde{\Lambda_1}) + C(\widetilde{\Lambda_1})) + n_2^2 \ \ \overline{\xi_{ISI}}(\widetilde{\Lambda_1})
$$

Proof. By the definition of $\overline{\xi_{ISI}}$ of $\widetilde{\Lambda}_1[\widetilde{\Lambda_2}]$, we have

$$\begin{aligned}
\overline{\xi_{ISI}}(\widetilde{\Lambda_1}[\widetilde{\Lambda_2}]) &= \sum_{(x_1,y_1)(x_2,y_2)\notin \mathfrak{Ed}(\widetilde{\Lambda_1}[\widetilde{\Lambda_2}])} \frac{\epsilon_{(x_1,y_1)}^{\widetilde{\Lambda_1}[\widetilde{\Lambda_2}]}\ \epsilon_{(x_2,y_2)}^{\widetilde{\Lambda_1}[\widetilde{\Lambda_2}]}}{\epsilon_{(x_1,y_1)}^{\widetilde{\Lambda_1}[\widetilde{\Lambda_2}]} + \epsilon_{(x_2,y_2)}^{\widetilde{\Lambda_1}[\widetilde{\Lambda_2}]}} \\
&= \sum_{x_1\in\mathfrak{Ve}(\widetilde{\Lambda_1})} \epsilon_{x_1}^{\widetilde{\Lambda_1}} \sum_{y_1y_2\notin\mathfrak{Ed}(\widetilde{\Lambda_2})} \left(\frac{2(2)}{2+2}\right) + \sum_{\substack{x_1\in\mathfrak{Ve}(\widetilde{\Lambda_1})\\ \epsilon_{x_1}^{\widetilde{\Lambda_1}}\geq 2}} \sum_{y_1y_2\notin\mathfrak{Ed}(\widetilde{\Lambda_2})} \frac{(\epsilon_{x_1}^{\widetilde{\Lambda_1}})^2}{2\epsilon_{x_1}^{\widetilde{\Lambda_1}}} \\
&\quad + \sum_{y_1\in\mathfrak{Ve}(\widetilde{\Lambda_2})} \sum_{x_1x_2\notin\mathfrak{Ed}(\widetilde{\Lambda_1})} \frac{\epsilon_{x_1}^{\widetilde{\Lambda_1}}\epsilon_{x_2}^{\widetilde{\Lambda_1}}}{\epsilon_{x_1}^{\widetilde{\Lambda_1}}+\epsilon_{x_2}^{\widetilde{\Lambda_1}}} + 2\sum_{x_1x_2\notin\mathfrak{Ed}(\widetilde{\Lambda_1})} \sum_{\{y_1y_2\}\subseteq\mathfrak{Ve}(\widetilde{\Lambda_2})} \frac{\epsilon_{x_1}^{\widetilde{\Lambda_1}}\epsilon_{x_2}^{\widetilde{\Lambda_1}}}{\epsilon_{x_1}^{\widetilde{\Lambda_1}}+\epsilon_{x_2}^{\widetilde{\Lambda_2}}} \\
&= \sum_{y_1y_2\notin\mathfrak{Ed}(\widetilde{\Lambda_2})} C(\widetilde{\Lambda_1}) + \sum_{y_1y_2\notin\mathfrak{Ed}(\widetilde{\Lambda_2})} \sum_{\substack{x_1\in\mathfrak{Ve}(\widetilde{\Lambda_1})\\ \epsilon_{x_1}^{\widetilde{\Lambda_1}}\geq 2}} \left(\frac{\epsilon_{x_1}^{\widetilde{\Lambda_1}}}{2}\right) + \left(n_2 + 2\binom{n_2}{2}\right) \sum_{x_1x_2\notin\mathfrak{Ed}(\widetilde{\Lambda_1})} \frac{\epsilon_{x_1}^{\widetilde{\Lambda_1}}\ \epsilon_{x_1}^{\widetilde{\Lambda_1}}}{\epsilon_{x_1}^{\widetilde{\Lambda_1}}\ +\ \epsilon_{x_1}^{\widetilde{\Lambda_1}}} \\
&= \overline{\ddot{m}_2}\ C(\widetilde{\Lambda_1}) + \frac{\overline{\ddot{m}_2}}{2}\ (\tau(\widetilde{\Lambda_1}) - C(\widetilde{\Lambda_1})) + \left(n_2 + 2\ \binom{n_2}{2}\right)\ \overline{\xi_{ISI}}(\widetilde{\Lambda})
\end{aligned}$$

$$\begin{aligned}
&= \overline{\ddot{m}_2}\ C(\widetilde{\Lambda_1}) + n_2^2\ \ \overline{\xi_{ISI}}(\widetilde{\Lambda_1}) + \frac{\overline{\ddot{m}}}{2}\ C(\widetilde{\Lambda_1}) \\
&= \frac{\overline{\ddot{m}}}{2}\ \left(\tau(\widetilde{\Lambda_1}) + C(\widetilde{\Lambda_1})\right) + n_2^2\ \ \overline{\xi_{ISI}}(\widetilde{\Lambda_1}).
\end{aligned}$$

□

The following Corollary we obtain by Theorem 4.4.

Corollary 4.5. *If $\widetilde{\Lambda}_1$ has no universal vertices, then $\overline{\xi_{ISI}}(\widetilde{\Lambda_1}[\widetilde{\Lambda_2}]) = \frac{\overline{\ddot{m}}}{2}\ \tau(\widetilde{\Lambda_1}) + n_2^2\ \ \overline{\xi_{ISI}}(\widetilde{\Lambda_1})$.*

4.5. Strong Product

The strong product of two graphs $\widetilde{\Lambda_1}$ and $\widetilde{\Lambda_2}$ is a graph with vertex set $\mathfrak{Ve}(\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2}) = \mathfrak{Ve}(\widetilde{\Lambda_1})\boxtimes\mathfrak{Ve}(\widetilde{\Lambda_2})$. Two vertices (x_1, y_1) and (x_2, y_2) are adjacent in the strong product if $x_1 = x_2$ and $x_1x_2 \in E$.

Theorem 4.6. *If $r_{\widetilde{\Lambda_1}} \geq \rho_{\widetilde{\Lambda_2}}$, then $\overline{\xi_{ISI}}(\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2}) = \frac{\overline{\ddot{m}_2}}{2}\ \tau(\widetilde{\Lambda_1}) + 2\overline{\ddot{m}_2}\ \ \xi_{ISI}(\widetilde{\Lambda}) + n_2^2\ \ \overline{\xi_{ISI}}(\widetilde{\Lambda})$.*

Proof. Based on the condition $r_{\widetilde{\Lambda_1}} \geq \rho_{\widetilde{\Lambda_2}}$ for every vertex $x_1 \in \mathfrak{Ve}(\widetilde{\Lambda_2}), y_1 \in \mathfrak{Ve}(\widetilde{\Lambda_2}), \epsilon_{x_1}^{\widetilde{\Lambda_1}} \geq \epsilon_{y_1}^{\widetilde{\Lambda_2}}$, we have

$$\epsilon_{(x_1,y_1)}^{\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2}} = \epsilon_{x_1}^{\widetilde{\Lambda_1}}$$

Hence,

$$\begin{aligned}
\overline{\xi_{ISI}}(\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2}) &= \sum_{(x_1,y_1)(x_2,y_2)\notin\mathfrak{Ed}(\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2})} \frac{\epsilon_{(x_1,y_1)^{\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2}}}\,\epsilon_{(x_2,y_2)^{\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2}}}}{\epsilon_{(x_1,y_1)^{\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2}}} + \epsilon_{(x_2,y_2)^{\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2}}}} \\
&= \sum_{x_1\in\mathfrak{Ve}(\widetilde{\Lambda_1})} \sum_{y_1y_2\notin\mathfrak{Ed}(\widetilde{\Lambda_2})} \frac{(\epsilon_{x_1}^{\widetilde{\Lambda_1}})^2}{2\epsilon_{x_1}^{\widetilde{\Lambda_1}}} + \sum_{y_1\in\mathfrak{Ve}(\widetilde{\Lambda_1})} \sum_{x_1x_2\notin\mathfrak{Ed}(\widetilde{\Lambda_1})} \frac{\epsilon_{x_1}^{\widetilde{\Lambda_1}}\epsilon_{x_2}^{\widetilde{\Lambda_1}}}{\epsilon_{x_1}^{\widetilde{\Lambda_1}}+\epsilon_{x_2}^{\widetilde{\Lambda_1}}} \\
&\quad +2\sum_{x_1x_2\in\mathfrak{Ed}(\widetilde{\Lambda_1})} \sum_{y_1y_2\notin\mathfrak{Ed}(\widetilde{\Lambda_1})} \frac{\epsilon_{x_1}^{\widetilde{\Lambda_1}}\epsilon_{x_2}^{\widetilde{\Lambda_2}}}{\epsilon_{x_1}^{\widetilde{\Lambda_1}}+\epsilon_{x_2}^{\widetilde{\Lambda_2}}} + 2\sum_{x_1x_2\notin\mathfrak{Ed}(\widetilde{\Lambda_1})} \sum_{y_1y_2\notin\mathfrak{Ed}(\widetilde{\Lambda_2})} \frac{\epsilon_{x_1}^{\widetilde{\Lambda_1}}\epsilon_{x_2}^{\widetilde{\Lambda_1}}}{\epsilon_{x_1}^{\widetilde{\Lambda_1}}+\epsilon_{x_2}^{\widetilde{\Lambda_1}}}
\end{aligned} \tag{4.1}$$

$$
\begin{aligned}
&= \frac{\overline{\ddot{m}_2}}{2}\tau(\widetilde{\Lambda_1}) + n_2\overline{\xi_{ISI}}(\widetilde{\Lambda_1}) + 2\overline{\ddot{m}_2\xi_{ISI}}(\widetilde{\Lambda_1}) + 2\binom{n_2}{2}\overline{\xi_{ISI}}(\widetilde{\Lambda_1})\\
&= \frac{\overline{\ddot{m}_2}}{2}\tau(\widetilde{\Lambda_1}) + 2\overline{\ddot{m}_2\xi_{ISI}}(\widetilde{\Lambda_1}) + n_2^2\ \overline{\xi_{ISI}}(\widetilde{\Lambda_1}).
\end{aligned}
$$

□

Theorem 4.7. *The inverse sum indeg coindex of* $\widetilde{\Lambda}_1 \boxtimes \widetilde{\Lambda_2}$ *is given by*

$$
\begin{aligned}
\overline{\xi_{ISI}}(\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2}) \geq\ & \frac{1}{8(\rho_{\widetilde{\Lambda_1}}+\rho_{\widetilde{\Lambda_2}})}\Bigg(\Big[\overline{\ddot{m}_2}E_1(\widetilde{\Lambda_1}) + \tau(\widetilde{\Lambda_1})\overline{\xi_{M_1}}(\widetilde{\Lambda_2}) + n_1\overline{\xi_{M_2}}(\widetilde{\Lambda_2})\Big]\\
&+\Big[\overline{\ddot{m}_1}\ E_1(\widetilde{\Lambda_2}) + \tau(\widetilde{\Lambda_2})\ \overline{\xi_{M_1}}(\widetilde{\Lambda_1}) + n_1\ \ \overline{\xi_{M_2}}(\widetilde{\Lambda_1})\Big]\\
&+\Big[2\overline{\ddot{m}_2}\ \xi_{M_2}(\widetilde{\Lambda_1}) + 2m_1\ \overline{\xi_{M_2}}(\widetilde{\Lambda_2}) + \xi_{M_1}(\widetilde{\Lambda_1})\ \overline{\xi_{M_1}}(\widetilde{\Lambda_2})\Big]\\
&+\Big[2\binom{n_2}{2}\ \overline{\xi_{M_2}}(\widetilde{\Lambda_1}) + 2\ \overline{\ddot{m}_1}\Big(\xi_{M_2}(\widetilde{\Lambda_2}) + \overline{\xi_{M_2}}(\widetilde{\Lambda_2})\Big) + \overline{\xi_{M_1}}(\widetilde{\Lambda_2})\Big(\xi_{M_1}(\widetilde{\Lambda_2}) + \overline{\xi_{M_1}}(\widetilde{\Lambda_2})\Big)\Big]\Bigg).
\end{aligned}
$$

The equality in Equation (4.2) holds if and only if $\widetilde{\Lambda}_1$ *and* $\widetilde{\Lambda}_2$ *are* $s-c$ *and* $\rho_{\widetilde{\Lambda_1}} = \rho_{\widetilde{\Lambda_2}}$.

Proof. The eccentricity of a vertex $(x_1, y_1) \in \mathfrak{Ve}(\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2})$ is

$$\epsilon_{(x_1,y_1)}^{\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2}} = \max\{\epsilon_{x_1}^{\widetilde{\Lambda_1}}, \epsilon_{y_1}^{\widetilde{\Lambda_2}}\}.$$

We know that for any real numbers a, b, we have $\max\{a, b\} \geq \frac{a+b}{2}$ with equality if and only if $a = b$. Therefore,

$$
\begin{aligned}
\overline{\xi_{ISI}}(\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2}) &= \sum_{(x_1,y_1)(x_2,y_2)\notin\mathfrak{Ed}(\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2})} \frac{\epsilon_{(x_1,y_1)}^{\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2}}\ \epsilon_{(x_2,y_2)}^{\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2}}}{\epsilon_{(x_1,y_1)}^{\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2}} + \epsilon_{(x_2,y_2)}^{\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2}}}\\
&= \sum_{(x_1,y_1)(x_2,y_2)\notin\mathfrak{Ed}(\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2})} \frac{max\{\epsilon_{x_1}^{\widetilde{\Lambda_1}}, \epsilon_{y_1}^{\widetilde{\Lambda_2}}\}\ max\{\epsilon_{x_2}^{\widetilde{\Lambda_1}}, \epsilon_{y_2}^{\widetilde{\Lambda_2}}\}}{max\{\epsilon_{x_1}^{\widetilde{\Lambda_1}}, \epsilon_{y_1}^{\widetilde{\Lambda_2}}\} + max\{\epsilon_{x_2}^{\widetilde{\Lambda_1}}, \epsilon_{y_2}^{\widetilde{\Lambda_2}}\}}\\
&\geq \sum_{(x_1,y_1)(x_2,y_2)\notin\mathfrak{Ed}(\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2})} \frac{\left(\frac{\epsilon_{x_1}^{\widetilde{\Lambda_1}}+\epsilon_{y_1}^{\widetilde{\Lambda_2}}}{2}\right)\left(\frac{\epsilon_{x_2}^{\widetilde{\Lambda_1}}+\epsilon_{y_2}^{\widetilde{\Lambda_2}}}{2}\right)}{\left(\frac{\epsilon_{x_1}^{\widetilde{\Lambda_1}}+\epsilon_{y_1}^{\widetilde{\Lambda_2}}}{2}\right)+\left(\frac{\epsilon_{x_2}^{\widetilde{\Lambda_1}}+\epsilon_{y_2}^{\widetilde{\Lambda_2}}}{2}\right)}\\
&= \sum_{(x_1,y_1)(x_2,y_2)\notin\mathfrak{Ed}(\widetilde{\Lambda_1}\boxtimes\widetilde{\Lambda_2})} \frac{(\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_1}^{\widetilde{\Lambda_2}})(\epsilon_{x_2}^{\widetilde{\Lambda_1}} + \epsilon_{y_2}^{\widetilde{\Lambda_2}})}{4(\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_1}^{\widetilde{\Lambda_2}} + \epsilon_{x_2}^{\widetilde{\Lambda_1}} + \epsilon_{y_2}^{\widetilde{\Lambda_2}})} \qquad (4.2)\\
&= A_1 + A_2 + A_3 + A_4,
\end{aligned}
$$

where A_1= The sum of Equation (4.2) which in taken overall vertices $x_1 \in \mathfrak{Ve}(\widetilde{\Lambda_1})$ and $n-a$ vertex pairs $y_1y_2 \notin \mathfrak{Ed}(\widetilde{\Lambda_2})$.

A_2= The sum of Equation (4.2) which in taken overall vertices $y_1 \in \mathfrak{Ve}(\widetilde{\Lambda_2})$ and $n-a$ vertex pairs $x_1x_2 \notin \mathfrak{Ed}(\widetilde{\Lambda_2})$.

A_3= The sum of Equation (4.2) which in taken overall edges $x_1x_2 \in \mathfrak{Ed}(\widetilde{\Lambda_1})$ and $n-a$ vertex pairs $y_1y_2 \notin \mathfrak{Ed}(\widetilde{\Lambda_2})$.

and

A_4= The sum of Equation (4.2) which in taken overall $n-a$ vertices pairs $x_1x_2 \notin \mathfrak{Ed}(\widetilde{\Lambda_1})$ and $n-a$ vertex pairs $x_1x_2 \notin \mathfrak{Ed}(\widetilde{\Lambda_2})$ and unordered pairs of vertices $\{y_1, y_2\} \subseteq \mathfrak{Ve}(\widetilde{\Lambda_2})$.

Now we shall obtain the sums A_1 to A_4, separately.

$$\begin{aligned} A_1 &= \frac{1}{4} \sum_{x_1 \in \mathfrak{Ve}(\widetilde{\Lambda_1})} \sum_{y_1y_2 \notin \mathfrak{Ed}(\widetilde{\Lambda_2})} \frac{(\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_1}^{\widetilde{\Lambda_2}})(\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_2}^{\widetilde{\Lambda_2}})}{2\epsilon_{x_1}^{\widetilde{\Lambda_1}} + (\epsilon_{y_1}^{\widetilde{\Lambda_2}} + \epsilon_{y_2}^{\widetilde{\Lambda_2}})} \\ &\geq \frac{1}{8(\rho_{\widetilde{\Lambda_1}} + \rho_{\widetilde{\Lambda_2}})} \sum_{x_1 \in \mathfrak{Ve}(\widetilde{\Lambda_1})} \sum_{y_1y_2 \notin \mathfrak{Ed}(\widetilde{\Lambda_2})} \left[(\epsilon_{x_1}^{\widetilde{\Lambda_1}})^2 + (\epsilon_{y_1}^{\widetilde{\Lambda_2}} + \epsilon_{y_2}^{\widetilde{\Lambda_2}})\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_1}^{\widetilde{\Lambda_2}}\epsilon_{y_2}^{\widetilde{\Lambda_2}} \right] \\ &= \frac{1}{8(\rho_{\widetilde{\Lambda_1}} + \rho_{\widetilde{\Lambda_2}})} \left[\overline{\ddot{m}_2} E_1(\widetilde{\Lambda_1}) + \tau(\widetilde{\Lambda_1}) \overline{\xi_{M_1}}(\widetilde{\Lambda_2}) + n_1 \overline{\xi_{M_2}}(\widetilde{\Lambda_2}) \right]. \end{aligned}$$

$$A_2 = \frac{1}{4} \sum_{y_1 \notin \mathfrak{Ve}(\widetilde{\Lambda_2})} \sum_{x_1x_2 \notin \mathfrak{Ed}(\widetilde{\Lambda_1})} \frac{(\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_1}^{\widetilde{\Lambda_2}})(\epsilon_{x_2}^{\widetilde{\Lambda_1}} + \epsilon_{y_1}^{\widetilde{\Lambda_2}})}{\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{x_2}^{\widetilde{\Lambda_1}} + 2\epsilon_{y_1}^{\widetilde{\Lambda_2}}}$$

By symmetry of the sum A_1, we get

$$A_2 \geq \frac{1}{8(\rho_{\widetilde{\Lambda_1}} + \rho_{\widetilde{\Lambda_2}})} \left[\overline{\ddot{m}_1}\ E_1(\widetilde{\Lambda_2}) + \tau(\widetilde{\Lambda_2})\ \overline{\xi_{M_1}}(\widetilde{\Lambda_1}) + n_1\ \ \overline{\xi_{M_2}}(\widetilde{\Lambda_1}) \right].$$

$$\begin{aligned} A_3 &= \frac{1}{4} \sum_{x_1x_2 \in \mathfrak{Ed}(\widetilde{\Lambda_2})} \sum_{y_1y_2 \notin \mathfrak{Ed}(\widetilde{\Lambda_2})} \left[\frac{(\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_1}^{\widetilde{\Lambda_2}})(\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_2}^{\widetilde{\Lambda_2}})}{(\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_1}^{\widetilde{\Lambda_2}}) + (\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_2}^{\widetilde{\Lambda_2}})} + \frac{(\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_2}^{\widetilde{\Lambda_2}})(\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_1}^{\widetilde{\Lambda_2}})}{(\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_2}^{\widetilde{\Lambda_2}}) + (\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_1}^{\widetilde{\Lambda_2}})} \right] \\ &\geq \frac{1}{8(\rho_{\widetilde{\Lambda_1}} + \rho_{\widetilde{\Lambda_2}})} \sum_{x_1x_2 \in \mathfrak{Ed}(\widetilde{\Lambda_2})} \sum_{y_1y_2 \notin \mathfrak{Ed}(\widetilde{\Lambda_2})} \left[2\epsilon_{x_1}^{\widetilde{\Lambda_1}}\epsilon_{x_2}^{\widetilde{\Lambda_1}} + 2\epsilon_{y_1}^{\widetilde{\Lambda_2}}\epsilon_{y_2}^{\widetilde{\Lambda_2}} + (\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{x_2}^{\widetilde{\Lambda_1}})(\epsilon_{y_1}^{\widetilde{\Lambda_2}} + \epsilon_{y_2}^{\widetilde{\Lambda_2}}) \right] \\ &= \frac{1}{8(\rho_{\widetilde{\Lambda_1}} + \rho_{\widetilde{\Lambda_2}})} \left[2\overline{\ddot{m}_2}\ \xi_{M_2}(\widetilde{\Lambda_1}) + 2m_1\ \overline{\xi_{M_2}}(\widetilde{\Lambda_2}) + \xi_{M_1}(\widetilde{\Lambda_1})\ \overline{\xi_{M_1}}(\widetilde{\Lambda_2}) \right]. \end{aligned}$$

Similarly, we compute

$$\begin{aligned} A_4 &= \frac{1}{4} \sum_{x_1x_2 \notin \mathfrak{Ed}(\widetilde{\Lambda_1})} \sum_{\{y_1,y_2\} \subseteq \mathfrak{Ve}(\widetilde{\Lambda_2})} \frac{(\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_1}^{\widetilde{\Lambda_2}})(\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_2}^{\widetilde{\Lambda_2}})}{(\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_1}^{\widetilde{\Lambda_2}}) + (\epsilon_{x_1}^{\widetilde{\Lambda_1}} + \epsilon_{y_2}^{\widetilde{\Lambda_2}})} \\ &\geq \frac{1}{8(\rho_{\widetilde{\Lambda_1}} + \rho_{\widetilde{\Lambda_2}})} \left[2\binom{n_2}{2}\ \overline{\xi_{M_2}}(\widetilde{\Lambda_1}) + 2\ \overline{\ddot{m}_1} \left(\xi_{M_2}(\widetilde{\Lambda_2}) + \overline{\xi_{M_2}}(\widetilde{\Lambda_2}) \right) + \overline{\xi_{M_1}}(\widetilde{\Lambda_2}) \left(\xi_{M_1}(\widetilde{\Lambda_2}) + \overline{\xi_{M_1}}(\widetilde{\Lambda_2}) \right) \right] \end{aligned}$$

By adding the sums A_1 to A_4, we get the inequalities which is our required result. The equality holds in (4.2) if and only if for each edge $(x_1, y_1)(x_2, y_2) \notin \mathfrak{Ed}(\widetilde{\Lambda_1 \boxtimes \Lambda_2})$, $\epsilon_{x_1}^{\widetilde{\Lambda_1}} = \epsilon_{x_2}^{\widetilde{\Lambda_1}}$ and $\epsilon_{y_1}^{\widetilde{\Lambda_2}} = \epsilon_{y_2}^{\widetilde{\Lambda_2}}$. If $\widetilde{\Lambda}_1$ and $\widetilde{\Lambda}_2$ are $s-c$ graph and $\rho_{\widetilde{\Lambda_1}} = \rho_{\widetilde{\Lambda_2}}$, then the equality in (4.2) holds trivially.

Let the equality in (4.2) holds. Then for each $x_1 \in \mathfrak{Ve}(\widetilde{\Lambda_1}), y_1y_2 \notin \mathfrak{Ed}(\widetilde{\Lambda_2})$, $\epsilon_{x_1}^{\widetilde{\Lambda_1}} = \epsilon_{y_1}^{\widetilde{\Lambda_2}} = \epsilon_{y_2}^{\widetilde{\Lambda_2}}$ and for each vertex $y_1 \in \mathfrak{Ve}(\widetilde{\Lambda_2})$, $x_1x_2 \in \mathfrak{Ed}(\widetilde{\Lambda_1})$, $\epsilon_{y_1}^{\widetilde{\Lambda_2}} = \epsilon_{x_1}^{\widetilde{\Lambda_1}} = \epsilon_{x_2}^{\widetilde{\Lambda_2}}$. This gives that both $\widetilde{\Lambda}_1$ and $\widetilde{\Lambda}_2$ are $s-c$ graphs and $\rho_{\widetilde{\Lambda_1}} = \rho_{\widetilde{\Lambda_2}}$. □

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