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ZAGREB CONNECTION INDICES OF DISJUNCTION AND SYMMETRIC DIFFERENCE OPERATIONS ON GRAPHS

USMAN ALI[1], M. JAVAID[1,*]

ABSTRACT. In this paper, we compute the general results in the form of the upper bounds of the Zagreb indices based on connection number called as first Zagreb connection index, second Zagreb connection index and modified first Zagreb connection index for the resultant graphs which are obtained by applying the product-related operations such as disjunction (co-normal product) and symmetric difference. At the end, some applications of the obtained results with a comparison between exact and computed values of the aforesaid Zagreb connection indices for the particular classes of alkanes are also included.

Key words: Zegreb indices, connection number, disjunction, symmetric difference.
MSC: 05C05, 05C12, 05C35, 05C90.

1. Introduction

Topological indices (TI's) are the fixed real numbers directly linked with the molecular graphs which are used to predict the physicochemical properties of the chemical compounds such as critical temperature, connectivity, density, stability, chirality, ZE-isomerism, correlation coefficient, boiling point, see [1, 2]. Medicinal chemistry, drugs and bio-informatics current trends in drugs discovery & crystalline materials have studied with the help of various TI's [3–10]. In addition, the quantitative structures activity relationships (QSAR) and quantitative structures property relationships (QSPR) are used to compute relationship between molecular graphs to their biological reactions. For more details, we refer to [11–13] and Todeschini and Consonni [14] reported that these relationships play an important role in the subject of cheminformatics.

[1] Department of Mathematics, School of Science, University of Management and Technology, Lahore, Pakistan.
[*] Corresponding Author: javaidmath@gmail.com .

In history, the first TI was studied by famous chemist Harold Wiener (1947) [15] when he worked successfully on the boiling point of paraffin. First time in 1972, Gutman and Trinajstic [16] defined degree based TI named first Zagreb index. Gutman and Ruscic extended their work and developed a new TI named second Zagreb index. Now at present time, so many revolutionized work have been presented on these TI's, see [17–19]. After a long space, Furtula and Gutman [20] studied one more TI and called it as forgotten index. Currently, many worked related to TI's with various properties have been studied, see [2, 21].

In 1972 [16], Gutman defined another descriptor which depends on that degree whose distance from corresponding vertex must be two. No one researchers had put their eyes or attention to this descriptor till 2017. Recently, Ali and Trinajstic [22] reopened this idea and called it as modified first Zagreb connection index. They also reported that it has better absolute value of correlation coefficient for the following thirteen physicochemical properties of octane isomers. Now-a-days, this descriptor are being used in various disciplines of graph theory especially in chemical graph theory to predict the physicochemical properties of chemical compounds liked alkanes, extremal alkanes and cycloalkanes etc. For more detail, see [23–25].

Operations on graphs have paid very important position in the study of chemical graphs. A simple graph can be changed or designed into various chemical structures by applying any one product-ralated operations such as: a cycloalkanes in the molecular graph obtained by the corona product of N_2 with any graph Q, a lader graph is a chemical graph obtained by the cartesian product of P_{n+1} & P_2 and the C_4 nanotube $TUC_4(m, n)$ is designed by the cartesian product of P_n & P_m and so on. So, many resultant graphs which are designed by any one producted-related operations are shown in the literature [26–37].

In this paper, we compute the general results in the form of upper bounds of the first Zagreb connection index, second Zagreb connection index and modified first Zagreb connection index of the resultant graphs which are obtained by applying various product-related operations such as disjunction and symmetric difference. The rest of the paper is settle as: Section II represents the preliminary definitions and results. Section III covers the general results of product-related graphs and Section IV includes the applications & conclusion.

2. Preliminaries

Let $Q = (V(Q), E(Q))$ be a simple and connected molecular graph with vertex set $V(Q)$ and edge set $E(Q) \subseteq V(Q) \times V(Q)$ such that their cardinalities are called order and size of Q respectively. Todesehini and Consonni [38] defined the generalized form of the degrees of vertices of the graph Q as ${}^{k}f_Q(b) = |{}^{k}N_Q(b)|$, where ${}^{k}N_Q(b) = \{a \in V(Q) : d(a, b) = k\}$. For k=1 and

k=2, ${}^1f_Q(b) = d_G(b)$ and ${}^2f_Q(b) = \tau_G(b)$ are called the degree and connection number of the vertex b in the graph Q. In chemical modeling of molecular descriptors, chemical terms atom and bond are equal to graphical terms vertex and edge respectively. Now, throughout the paper, we assume that Q_1 and Q_2 are two connected graphs such that $|V(Q_1)| = n_1$, $|V(Q_2)| = n_2$, $|E(Q_1)| = e_1$ and $|E(Q_2)| = e_2$.

Definition 1. *For a graph Q, the first Zagreb index* $(M_1(Q))$*, second Zagreb index* $(M_2(Q))$ *and their coindices are defined as:*

$$M_1(Q) = \sum_{b\in V(Q)} [d_Q(b)]^2 = \sum_{ab\in E(Q)} [d_Q(a) + d_Q(b)],$$

$$M_2(Q) = \sum_{ab\in E(Q)} [d_Q(a) \times d_Q(b)],$$

$$\bar{M_1}(Q) = \sum_{ab\notin E(Q)} [d_Q(a) + d_Q(b)]$$

and

$$\bar{M_2}(Q) = \sum_{ab\notin E(Q)} [d_Q(a) \times d_Q(b)].$$

The two TI's defined in above Definition 1 are a few decades old and two are recently degree based version descriptors for the chemical structure defined by Gutman, Trinajstic, Ruscic and Ashrafi see [16, 39, 40]. But these are very popular descriptors to compute the activity & quantitative structural properties of the molecular graphs such as chirality, heterosystems, branching, total π-electron energy and complexity, see [11, 12]. Corresponding to these degree-based TI's, the connection-based TI's are defined in Definition 2. For further studies of connection-based TI's, see [22, 25, 26, 41].

Definition 2. *For a graph Q, the first Zagreb connection index* $(ZC_1(Q))$ *, second Zagreb connection index* $(ZC_2(Q))$ *and modified first Zagreb connection index* $(ZC_1^*(Q))$ *are defined as;*

$$ZC_1(Q) = \sum_{b\in V(Q)} [\tau_Q(b)]^2,$$

$$ZC_2(Q) = \sum_{ab\in E(Q)} [\tau_Q(a) \times \tau_Q(b)]$$

and

$$ZC_1^*(Q) = \sum_{b\in V(Q)} d_Q(b)\tau_Q(b) = \sum_{ab\in E(Q)} [\tau_Q(a) + \tau_Q(b)].$$

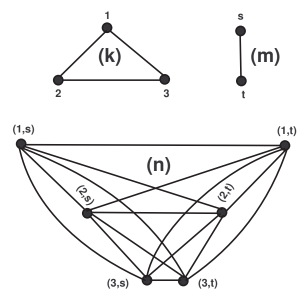

FIGURE 1. (k)$Q_1 \cong C_3$, (m)$Q_2 \cong P_2$ and (n)$Q_1 \vee Q_2$.

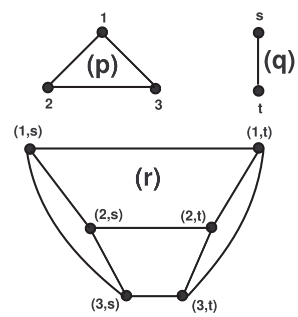

FIGURE 2. (p)$Q_1 \cong C_3$, (q)$Q_2 \cong P_2$ and (r)$Q_1 \oplus Q_2$.

Definition 3. *The disjunction* $(Q_1 \vee Q_2)$ *and symmetric difference* $(Q_1 \oplus Q_2)$ *of two graphs* Q_1 *and* Q_2 *are obtained by taking vertex set :* $V(Q_1 \times Q_2) = V(Q_1) \times V(Q_2)$ *and edge set:* $E(Q_1 \times Q_2) = \{(a_1, b_1)(a_2, b_2)$; *where* $(a_1, b_1), (a_2, b_2) \in V(Q_1) \times V(Q_2)\}$ *with conditions:* $[a_1a_2 \in E(Q_1)]$ *or* $[b_1b_2 \in E(Q_2)]$, *and* $[a_1a_2 \in E(Q_1)]$ *or* $[b_1b_2 \in E(Q_2)]$, *but not both hold at the same time, respectively.*

For more detail, see Figures 1 & 2.

Lemma 1. *[42] Let* Q *be a connected graph and* $\bar{Q}$ *be its complement. Then*
(i) $\sum_{b \in V(Q)} d_Q(b) = 2e$,
(ii) $d_{\bar{Q}}(b) = (n-1) - d_Q(b)$ *and*
(iii) $M_1(\bar{Q}) = M_1(Q) - 4$, *where* $|V(Q)| = n$ *and* $|E(Q)| = e$.

Lemma 2. *[18] Let* Q *be a connected graph with* n *vertices and* e *edges. Then* $\tau_Q(a) + d_Q(a) \leq \sum_{b \in N_Q(a)} (d_Q(b)$, *where equality holds if and only if* Q *is a* $\{C_3, C_4\}-$ *free graph.*

Lemma 3. *[26] Let Q be a connected and $\{C_3, C_4\}-$ free graph with n vertices and e edges. Then,* $\sum\limits_{a\in V(Q)} \tau_Q(a) = M_1(Q) - 2e.$

3. Main Results

The main results have been started here.

Theorem 4. *Let Q_1 and Q_2 be two connected graphs. Then, ZC_1 of the disjunction (co-normal product) of Q_1 and Q_2 are as follows:*
$ZC_1(Q_1 \vee Q_2) \leq n_2M_1(Q_1) + 4ZC_1(Q_1)ZC_1(Q_2) + n_1M_1(Q_2) + n_1n_2 + 4n_1e_2 +4ZC_1^*(Q_1)[M_1(Q_2)-2e_2]+4ZC_1^*(Q_2)[M_1(Q_1)-2e_1]+4[M_1(Q_1)-2e_1][M_1(Q_2) - 2e_2] + 8e_1e_2 + 4n_2e_1.$

Proof. If for any $a \in V(Q_1)$, $b \in V(Q_2)$ and $\tau_{Q_1\vee Q_2}(a, b) \leq d_{Q_1}(a) + 2\tau_{Q_1}(a) \times \tau_{Q_2}(b) + d_{Q_2}(b) + 1$.
$ZC_1(Q_1 \vee Q_2) = \sum_{(a,b)\in V(Q_1\vee Q_2)}[\tau_{Q_1\vee Q_2}(a, b)]^2$
$\leq \sum_{a\in V(Q_1)} \sum_{b\in V(Q_2)}[d_{Q_1}(a) + 2\tau_{Q_1}(a)\tau_{Q_2}(b) + \{d_{Q_2}(b) + 1\}]^2$
$= \sum_{a\in V(Q_1)}[n_2d^2_{Q_1}(a) + 4\tau^2_{Q_1}(a)ZC_1(Q_2) + M_1(Q_2) + n_2 + 4e_2 + 4d_{Q_1}(a)\tau_{Q_1}(a) \times \{M_1(Q_2) - 2e_2\} + 4\tau_{Q_1}(a)ZC_1^*(Q_2) + 4\tau_{Q_1}(a)\{M_1(Q_2) - 2e_2\} + 4e_2d_{Q_1}(a) + 2n_2d_{Q_1}(a)]$
$= n_2M_1(Q_1) + 4ZC_1(Q_1)ZC_1(Q_2) + n_1M_1(Q_2) + n_1n_2 + 4n_1e_2 + 4ZC_1^*(Q_1) \times [M_1(Q_2) - 2e_2] + 4ZC_1^*(Q_2)[M_1(Q_1) - 2e_1] + 4[M_1(Q_1) - 2e_1][M_1(Q_2) - 2e_2] + 8e_1e_2 + 4n_2e_1.$ □

Theorem 5. *Let Q_1 and Q_2 be two connected graphs. Then, ZC_2 of the disjunction (co-normal product) of Q_1 and Q_2 are as follows:*
$ZC_2(Q_1 \vee Q_2) \leq 4ZC_1^*(Q_1)ZC_1^*(Q_2) + 2[M_1(Q_2) - 2e_2]ZC_1^*(Q_1) + 2[M_1(Q_1) -2e_1]ZC_1^*(Q_2)+8ZC_2(Q_1)ZC_2(Q_2)+4[ZC_1(Q_1)ZC_2(Q_2)+ZC_1(Q_2)ZC_2(Q_1)] + (n_2 + \delta_2 + 2e_2)M_2(Q_1) + (n_1 + \delta_1 + 2e_1)M_2(Q_2) + (n_2 + \delta_2 + 5e_2)M_1(Q_1) + (n_1 + \delta_1 + 5e_1)M_1(Q_2) + (n_1 + \delta_1)e_2 + (n_2 + \delta_2)e_1 + 10e_1e_2 + e_2[\bar{M}_1(Q_1) +\bar{M}_2(Q_1)]+e_1[\bar{M}_1(Q_2)+\bar{M}_2(Q_2)]+2[M_1(Q_1)-2e_1] \sum_{b_1b_2\in E(Q_2)}[d_{Q_2}(b_1)\tau_{Q_2}(b_2) + d_{Q_2}(b_2)\tau_{Q_2}(b_1)] + 2[M_1(Q_2) - 2e_2] \sum_{a_1a_2\in E(Q_1)}[d_{Q_1}(a_1)\tau_{Q_1}(a_2) + d_{Q_1}(a_2) \times\tau_{Q_1}(a_1)]+4 \sum_{a_1a_2\in E(Q_1)} \sum_{b_1b_2\in E(Q_2)}[d_{Q_1}(a_1)\tau_{Q_1}(a_2)\tau_{Q_2}(b_2)+d_{Q_1}(a_2)\tau_{Q_1}(a_1) \times \tau_{Q_2}(b_1)] + 2 \sum_{a_1a_2\in E(Q_1)} \sum_{b_1b_2\in E(Q_2)}[d_{Q_1}(a_1)d_{Q_2}(b_2) + d_{Q_1}(a_2)d_{Q_2}(b_1)] + 4 \sum_{a_1a_2\in E(Q_1)} \sum_{b_1b_2\in E(Q_2)}[d_{Q_2}(b_1)\tau_{Q_2}(b_2)\tau_{Q_1}(a_2) + d_{Q_2}(b_2)\tau_{Q_2}(b_1)\tau_{Q_1}(a_1)] +4 \sum_{a_1a_2\in E(Q_1)} \sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_1) + \tau_{Q_1}(a_2)\tau_{Q_2}(b_2)] + 2 \sum_{a_1a_2\in E(Q_1)} \sum_{b_1b_2\notin E(Q_2)}[d_{Q_1}(u_1)\tau_{Q_1}(u_2)\tau_{Q_2}(v_2) + d_{Q_1}(u_2)\tau_{Q_1}(u_1)\tau_{Q_2}(v_1)] + \sum_{a_1a_2\in E(Q_1)} \sum_{b_1b_2\notin E(Q_2)}[d_{Q_1}(u_1)d_{Q_2}(v_2)+d_{Q_1}(u_2)d_{Q_2}(v_1)]+4ZC_2(Q_1) \sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_2}(v_1) \times\tau_{Q_2}(v_2)]+2 \sum_{a_1a_2\in E(Q_1)} \sum_{b_1b_2\notin E(Q_2)}[d_{Q_2}(b_1)\tau_{Q_2}(b_2)\tau_{Q_1}(a_2)+d_{Q_2}(b_2)\tau_{Q_2}(b_1) \times \tau_{Q_1}(a_1)] + 2 \sum_{a_1a_2\in E(Q_1)} \sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_1) + \tau_{Q_1}(a_2)\tau_{Q_2}(b_2)] + 2 \sum_{b_1b_2\in E(Q_2)} \sum_{a_1a_2\notin E(Q_1)}[d_{Q_1}(a_1)\tau_{Q_1}(a_2)\tau_{Q_2}(b_2) + d_{Q_1}(a_2)\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)] + \sum_{b_1b_2\in E(Q_2)} \sum_{a_1a_2\notin E(Q_1)}[d_{Q_1}(a_1)d_{Q_2}(b_2) + d_{Q_1}(a_2)d_{Q_2}(b_1)] + 4ZC_2(Q_2)$

$\times \sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1}(a_1)\tau_{Q_1}(a_2)] + 2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[d_{Q_2}(b_1)\tau_{Q_2}(b_2)$
$\times\tau_{Q_1}(a_2)+d_{Q_2}(b_2)\tau_{Q_2}(b_1)\tau_{Q_1}(a_1)]+2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)$
$+ \tau_{Q_1}(a_2)\tau_{Q_2}(b_2)]$.

Proof. $ZC_2(Q_1\vee Q_2) = \sum_{(a_1,b_1)(a_2,b_2)\in E(Q_1\vee Q_2)}[\tau_{Q_1\vee Q_2}(a_1,b_1)\times\tau_{Q_1\vee Q_2}(a_2,b_2)]$
$= \sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\vee Q_2}(a,b_1)\times\tau_{Q_1\vee Q_2}(a,b_2)]$
$+ \sum_{b\in V(Q_2)}\sum_{a_1a_2\in E(Q_1)}[\tau_{Q_1\vee Q_2}(a_1,b)\times\tau_{Q_1\vee Q_2}(a_2,b)]$
$+ \sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\vee Q_2}(a_1,b_1)\times\tau_{Q_1\vee Q_2}(a_2,b_2)]$
$+ \sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1\vee Q_2}(a_1,b_1)\times\tau_{Q_1\vee Q_2}(a_2,b_2)]$
$+ \sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1\vee Q_2}(a_1,b_1)\times\tau_{Q_1\vee Q_2}(a_2,b_2)]$.

Taking

$$\sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\vee Q_2}(a,b_1)\times\tau_{Q_1\vee Q_2}(a,b_2)] \leq \sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}\times[\{d_{Q_1}(a)+2\tau_{Q_1}(a)\tau_{Q_2}(b_1)+d_{Q_2}(b_1)+1\}\times\{d_{Q_1}(a)+ 2\tau_{Q_1}(a)\tau_{Q_2}(b_2)+d_{Q_2}(b_2)+1\}] = \sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[d^2_{Q_1}(a)+ 2d_{Q_1}(a)\tau_{Q_1}(a)\{\tau_{Q_2}(b_1)+\tau_{Q_2}(b_2)\}+d_{Q_1}(a)\times\{d_{Q_2}(b_1)+d_{Q_2}(b_2)\}+2d_{Q_1}(a)+ 4\tau^2_{Q_1}(a)\tau_{Q_2}(b_1)\tau_{Q_2}(b_2)+2\tau_{Q_1}(a)\{d_{Q_2}(b_1)\tau_{Q_2}(b_2)+d_{Q_2}(b_2)\tau_{Q_2}(b_1)\}+ 2\tau_{Q_1}(a)\{\tau_{Q_2}(b_1)+\tau_{Q_2}(b_2)\}+\{d_{Q_2}(b_1)d_{Q_2}(b_2)\}+\{d_{Q_2}(b_1)+d_{Q_2}(b_2)\}+1] = e_2M_1(Q_1)+2ZC^*_1(Q_1)ZC^*_1(Q_2)+2e_1M_1(Q_2)+4e_1e_2+4ZC_1(Q_1)ZC_2(Q_2)+ 2[M_1(Q_1)-2e_1]ZC^*_1(Q_2)+n_1M_2(Q_2)+n_1M_1(Q_2)+n_1e_2+2[M_1(Q_1)- 2e_1]\times\sum_{b_1b_2\in E(Q_2)}[d_{Q_2}(b_1)\tau_{Q_2}(b_2)+d_{Q_2}(b_2)\tau_{Q_2}(b_1)].$$

Similarly

$$\sum_{b\in V(Q_2)}\sum_{a_1a_2\in E(Q_1)}[\tau_{Q_1\vee Q_2}(a_1,b)\times\tau_{Q_1\vee Q_2}(a_2,b)] = n_2M_2(Q_1)+2e_2M_1(Q_1)+n_2M_1(Q_1)+4ZC_2(Q_1)ZC_1(Q_2)+ 2ZC^*_1(Q_2)ZC^*_1(Q_1)+2[M_1(Q_2)-2e_2]ZC^*_1(Q_1)+e_1M_1(Q_2)+4e_1e_2+n_2e_1+ 2[M_1(Q_2)-2e_2]\sum_{a_1a_2\in E(Q_1)}[d_{Q_1}(a_1)\tau_{Q_1}(a_2)+d_{Q_1}(a_2)\tau_{Q_1}(a_1)].$$

Also taking

$$\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1\vee Q_2}(a_1,b_1)\times\tau_{Q_1\vee Q_2}(a_2,b_2)] \leq \sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\{d_{Q_1}(a_1)+2\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)+d_{Q_2}(b_1)+1\}\times \{d_{Q_1}(a_2)+2\tau_{Q_1}(a_2)\tau_{Q_2}(b_2)+d_{Q_2}(b_2)+1\}]$$

Since

$$\sum_{b_1b_2\notin E(Q_2)} = n_2(n_2-1)-2e_2 = \delta_2 = \delta_2M_2(Q_1)+2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[d_{Q_1}(a_1)\tau_{Q_1}(a_2)\tau_{Q_2}(b_2)+ d_{Q_1}(a_2)\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)]+\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[d_{Q_1}(a_1)d_{Q_2}(b_2)+ d_{Q_1}(a_2)d_{Q_2}(b_1)]+\delta_2M_1(Q_1)+4ZC_2(G_1)\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_2}(b_1)\tau_{Q_2}(b_2)]+ 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[d_{Q_2}(b_1)\tau_{Q_2}(b_2)\tau_{Q_1}(a_2)+d_{Q_2}(b_2)\tau_{Q_2}(b_1)\tau_{Q_1}(a_1)]+$$

$$2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)+\tau_{Q_1}(a_2)\tau_{Q_2}(b_2)]+e_1\bar{M}_2(Q_2)+ e_1\bar{M}_1(Q_2)+\delta_2 e_1.$$

Similarly

$$\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1\vee Q_2}(a_1,b_1)\times\tau_{Q_1\vee Q_2}(a_2,b_2)]\le \sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\{d_{Q_1}(a_1)+2\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)+d_{Q_2}(b_1)+1\}\times \{d_{Q_1}(a_2)+2\tau_{Q_1}(a_2)\tau_{Q_2}(b_2)+d_{Q_2}(b_2)+1\}]$$

Since

$$\sum_{a_1a_2\notin E(Q_1)}=n_1(n_1-1)-2e_1=\delta_1=$$

$$e_2\bar{M}_2(Q_1)+2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[d_{Q_1}(a_1)\tau_{Q_1}(a_2)\tau_{Q_2}(b_2)+ d_{Q_1}(a_2)\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)]+\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[d_{Q_1}(a_1)d_{Q_2}(b_2)+ d_{Q_1}(a_2)d_{Q_2}(b_1)]+e_2\bar{M}_1(Q_1)+4ZC_2(Q_2)\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1}(a_1)\tau_{Q_1}(a_2)]+ 2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[d_{Q_2}(b_1)\tau_{Q_2}(b_2)\tau_{Q_1}(a_2)+d_{Q_2}(b_2)\tau_{Q_2}(b_1)\tau_{Q_1}(a_1)]+ 2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)+\tau_{Q_1}(a_2)\tau_{Q_2}(b_2)]+\delta_1 M_2(Q_2)+ \delta_1 M_1(Q_2)+\delta_1 e_2.$$

and

$$\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\vee Q_2}(a_1,b_1)\times\tau_{Q_1\vee Q_2}(a_2,b_2)]=2e_2M_2(Q_1)+ 4\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[d_{Q_1}(a_1)\tau_{Q_1}(a_2)\tau_{Q_2}(b_2)+d_{Q_1}(a_2)\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)]+ 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[d_{Q_1}(a_1)d_{Q_2}(b_2)+d_{Q_1}(a_2)d_{Q_2}(b_1)]+2e_2M_1(Q_1)+ 8ZC_2(Q_1)ZC_2(Q_2)+4\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[d_{Q_2}(b_1)\tau_{Q_2}(b_2)\tau_{Q_1}(a_2)+ d_{Q_2}(b_2)\tau_{Q_2}(b_1)\tau_{Q_1}(a_1)]+4\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)+ \tau_{Q_1}(a_2)\tau_{Q_2}(b_2)]+2e_1M_2(Q_2)+2e_1M_1(Q_2)+2e_1e_2.$$

Consequently, we achieve our desired result. □

Theorem 6. *Let Q_1 and Q_2 be two connected graphs. Then, ZC_1^* of the disjunction (co-normal product) of Q_1 and Q_2 are as follows:*

$$ZC_1^*(Q_1\vee Q_2)\le 2[M_1(Q_2)-2e_2]ZC_1^*(Q_1)+2[M_1(Q_1)-2e_1]ZC_1^*(Q_2)+ (n_2+\delta_2+2e_2)M_1(Q_1)+(n_1+\delta_1+2e_1)M_1(Q_2)+2e_2(n_1+\delta_1)+2e_1(n_2+\delta_2)+ 12e_1e_2+e_1\bar{M}_1(Q_2)+e_2\bar{M}_1(Q_1)+4\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)+ \tau_{Q_1}(a_2)\tau_{Q_2}(b_2)]+2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)+ \tau_{Q_1}(a_2)\tau_{Q_2}(b_2)]+2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)+\tau_{Q_1}(a_2)\tau_{Q_2}(b_2)].$$

Proof. Take

$$ZC_1^*(Q_1\vee Q_2)=\sum_{(a_1,b_1)(a_2,b_2)\in E(Q_1\vee Q_2)}[\tau_{Q_1\vee Q_2}(a_1,b_1)+\tau_{Q_1\vee Q_2}(a_2,b_2)]= \sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\vee Q_2}(a,b_1)+\tau_{Q_1\vee Q_2}(a,b_2)]+ \sum_{b\in V(Q_2)}\sum_{a_1a_2\in E(Q_1)}[\tau_{Q_1\vee Q_2}(a_1,b)+\tau_{Q_1\vee Q_2}(a_2,b)]+ \sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\vee Q_2}(a_1,b_1)+\tau_{Q_1\vee Q_2}(a_2,b_2)]+$$

$$\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1\vee Q_2}(a_1,b_1)+\tau_{Q_1\vee Q_2}(a_2,b_2)]+\\ \sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1\vee Q_2}(a_1,b_1)+\tau_{Q_1\vee Q_2}(a_2,b_2)].$$

Taking

$$\sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\vee Q_2}(a,b_1)+\tau_{Q_1\vee Q_2}(a,b_2)]\leq\\ \sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\{d_{Q_1}(a)+2\tau_{Q_1}(a)\tau_{Q_2}(b_1)+d_{Q_2}(b_1)+1\}+\{d_{Q_1}(a)+\\ 2\tau_{Q_1}(a)\tau_{Q_2}(b_2)+d_{Q_2}(b_2)+1\}]=\\ 4e_1e_2+2[M_1(Q_1)-2e_1]ZC_1^*(Q_2)+n_1M_1(Q_2)+2n_1e_2.$$

Similarly

$$\sum_{b\in V(Q_2)}\sum_{a_1a_2\in E(Q_1)}[\tau_{Q_1\vee Q_2}(a_1,b)+\tau_{Q_1\vee Q_2}(a_2,b)]\leq\\ n_2M_1(Q_1)+2[M_1(Q_2)-2e_2]ZC_1^*(Q_1)+4e_1e_2+2n_2e_1.$$

Also taking

$$\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1\vee Q_2}(a_1,b_1)+\tau_{Q_1\vee Q_2}(a_2,b_2)]\leq\\ \sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\{d_{Q_1}(a_1)+2\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)+d_{Q_2}(b_1)+1\}+\\ \{d_{Q_1}(a_2)+2\tau_{Q_1}(a_2)\tau_{Q_2}(b_2)+d_{Q_2}(b_2)+1\}]=\\ \delta_2M_1(G_1)+2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)+\tau_{Q_1}(a_2)\tau_{Q_2}(b_2)]+\\ e_1\bar{M}_1(Q_2)+2\delta_2e_1.$$

Similarly

$$\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1\vee Q_2}(a_1,b_1)+\tau_{Q_1\vee Q_2}(a_2,b_2)]\leq\\ e_2\bar{M}_1(Q_1)+2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)+\tau_{Q_1}(a_2)\tau_{Q_2}(b_2)]+\\ \delta_1M_1(G_2)+2\delta_1e_2.$$

And

$$\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\vee Q_2}(a_1,b_1)+\tau_{Q_1\vee Q_2}(a_2,b_2)]\leq\\ 2e_2M_1(Q_1)+4\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)+\tau_{Q_1}(a_2)\tau_{Q_2}(b_2)]+\\ 2e_1M_1(Q_2)+4e_1e_2.$$

Consequently, we arrive at our desired result. □

Theorem 7. *Let Q_1 and Q_2 be two connected graphs. Then, ZC_1 of the symmetric difference of Q_1 and Q_2 are as follows:*

$$ZC_1(Q_1\oplus Q_2)\leq n_2M_1(Q_1)+ZC_1(Q_1)ZC_1(Q_2)+n_1M_1(Q_2)+n_1n_2+\\ 4n_1e_2+2ZC_1^*(Q_1)[M_1(Q_2)-2e_2]+2ZC_1^*(Q_2)[M_1(Q_1)-2e_1]+2[M_1(Q_1)-\\ 2e_1][M_1(Q_2)-2e_2]+8e_1e_2+4n_2e_1.$$

Proof. Since

$$\tau_{Q_1\oplus Q_2}(a,b)\leq d_{Q_1}(a)+\tau_{Q_1}(a)\tau_{Q_2}(b)+d_{Q_2}(b)+1,$$

where $a\in V(Q_1)$, $b\in V(Q_2)$ and $(a,b)\in Q_1\oplus Q_2$.

$$ZC_1(Q_1 \oplus Q_2) = \sum_{(a,b)\in V(Q_1\oplus Q_2)}[\tau_{Q_1\oplus Q_2}(a,b)]^2 \leq \sum_{a\in V(Q_1)}\sum_{b\in V(Q_2)}[d_{Q_1}(a) + \tau_{Q_1}(a)\tau_{Q_2}(b) + \{d_{Q_2}(b)+1\}]^2 = \sum_{a\in V(Q_1)}[n_2 d_{Q_1}^2(a) + \tau_{Q_1}^2(a)ZC_1(Q_2) + M_1(Q_2) + n_2 + 4e_2 + 2d_{Q_1}(a)\tau_{Q_1}(a)\{M_1(Q_2) - 2e_2\} + 2\tau_{Q_1}(a)ZC_1^*(Q_2) + 2\tau_{Q_1}(a)\{M_1(Q_2) - 2e_2\} + 4e_2 d_{Q_1}(a) + 2n_2 d_{Q_1}(a)] = n_2 M_1(Q_1) + ZC_1(Q_1)ZC_1(Q_2) + n_1 M_1(Q_2) + n_1 n_2 + 4n_1 e_2 + 2ZC_1^*(Q_1)[M_1(Q_2) - 2e_2] + 2ZC_1^*(Q_2)[M_1(Q_1) - 2e_1] + 2[M_1(Q_1) - 2e_1][M_1(Q_2) - 2e_2] + 8e_1e_2 + 4n_2e_1.$$

□

Theorem 8. *Let Q_1 and Q_2 be two connected graphs. Then, ZC_2 of the symmetric difference of Q_1 and Q_2 are as follows:*

$$ZC_2(Q_1 \oplus Q_2) \leq 2ZC_1^*(Q_1)ZC_1^*(Q_2) + [M_1(Q_2) - 2e_2]ZC_1^*(Q_1) + [M_1(Q_1) - 2e_1]ZC_1^*(Q_2) + ZC_1(Q_1)ZC_2(Q_2) + ZC_1(Q_2)ZC_2(Q_1) + (n_1 + 2\delta_1)M_2(Q_2) + (n_2 + 2\delta_2)M_2(Q_1) + (n_2 + 2\delta_2 + 3e_2)M_1(Q_1) + (n_1 + 2\delta_1 + 3e_1)M_1(Q_2) + (n_1 + 2\delta_1)e_2 + (n_2 + 2\delta_2)e_1 + 8e_1e_2 + 2e_1[\bar{M}_1(Q_2) + \bar{M}_2(Q_2)] + 2e_2[\bar{M}_1(Q_1) + \bar{M}_2(Q_1)] + [M_1(Q_1) - 2e_1]\sum_{b_1b_2\in E(Q_2)}[d_{Q_2}(b_1)\tau_{Q_2}(b_2) + d_{Q_2}(b_2)\tau_{Q_2}(b_1)] + [M_1(Q_2) - 2e_2]\sum_{u_1u_2\in E(Q_1)}[d_{Q_1}(a_1)\tau_{Q_1}(a_2) + d_{Q_1}(a_2)\tau_{Q_1}(a_1)] + 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[d_{Q_1}(a_1)\tau_{Q_1}(a_2)\tau_{Q_2}(b_2) + d_{Q_1}(a_2)\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)] + 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[d_{Q_1}(a_1)d_{Q_2}(b_2) + d_{Q_1}(a_2)d_{Q_2}(b_1)] + 2ZC_2(Q_1)\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_2}(b_1)\tau_{Q_2}(b_2)] + 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[d_{Q_2}(b_2)\tau_{Q_2}(b_1)\tau_{Q_1}(a_1) + d_{Q_2}(b_1)\tau_{Q_2}(b_2)\tau_{Q_1}(a_2)] + 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_1) + \tau_{Q_1}(a_2)\tau_{Q_2}(b_2)] + 2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[d_{Q_1}(a_1)\tau_{Q_1}(a_2)\tau_{Q_2}(b_2) + d_{Q_1}(a_2)\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)] + 2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[d_{Q_1}(a_1)d_{Q_2}(b_2) + d_{Q_1}(a_2)d_{Q_2}(b_1)] + 2ZC_2(Q_2)\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1}(a_1)\tau_{Q_1}(a_2)] + 2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[d_{Q_2}(b_2)\tau_{Q_2}(b_1)\tau_{Q_1}(a_1) + d_{Q_2}(b_1)\tau_{Q_2}(b_2)\tau_{Q_1}(a_2)] + 2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_1) + \tau_{Q_1}(a_2)\tau_{Q_2}(b_2)].$$

Proof. Since

$$ZC_2(Q_1 \oplus Q_2) = \sum_{(a_1,b_1)(a_2,b_2)\in E(Q_1\oplus Q_2)}[\tau_{Q_1\oplus Q_2}(a_1,b_1) \times \tau_{Q_1\oplus Q_2}(a_2,b_2)] = \sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\oplus Q_2}(a,b_1) \times \tau_{Q_1\oplus Q_2}(a,b_2)] + \sum_{b\in V(Q_2)}\sum_{a_1a_2\in E(Q_1)}[\tau_{Q_1\oplus Q_2}(a_1,b) \times \tau_{Q_1\oplus Q_2}(a_2,b)] + \sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1\oplus Q_2}(a_1,b_1) \times \tau_{Q_1\oplus Q_2}(a_2,b_2)] + \sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1\oplus Q_2}(a_1,b_1) \times \tau_{Q_1\oplus Q_2}(a_2,b_2)].$$

Taking

$$\sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\oplus Q_2}(a,b_1) \times \tau_{Q_1\oplus Q_2}(a,b_2)] \leq \sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\{d_{Q_1}(a) + \tau_{Q_1}(a)\tau_{Q_2}(b_1) + d_{Q_2}(b_1) + 1\} \times \{d_{Q_1}(a) + \tau_{Q_1}(a)\tau_{Q_2}(b_2) + d_{Q_2}(b_2) + 1\}] = e_2M_1(Q_1) + ZC_1^*(Q_1)ZC_1^*(Q_2) + 2e_1M_1(Q_2) +$$

$$4e_1e_2 + ZC_1(Q_1)ZC_2(Q_2) + [M_1(Q_1) - 2e_1]\sum_{b_1b_2\in E(Q_2)}[d_{Q_2}(b_1)\tau_{Q_2}(b_2) + d_{Q_2}(b_2)\tau_{Q_2}(b_1)] + [M_1(Q_1) - 2e_1]ZC_1^*(Q_2) + n_1M_2(Q_2) + n_1M_1(Q_2) + n_1e_2.$$

Similarly

$$\sum_{b\in V(Q_2)}\sum_{a_1a_2\in E(Q_1)}[\tau_{Q_1\oplus Q_2}(a_1,b)\times\tau_{Q_1\oplus Q_2}(a_2,b)] leq n_2M_2(Q_1) + [M_1(Q_2) - 2e_2]\sum_{a_1a_2\in E(Q_1)}[d_{Q_1}(a_1)\tau_{Q_1}(a_2) + d_{Q_1}(a_2)\tau_{Q_1}(a_1)] + 2e_2M_1(Q_1) + n_2M_1(Q_1) + ZC_1(Q_2)ZC_2(Q_1) + ZC_1^*(Q_1)ZC_1^*(G_2) + [M_1(Q_2) - 2e_2]ZC_1^*(Q_1) + e_1M_1(Q_2) + 4e_1e_2 + n_2e_1.$$

Also taking

$$\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1\oplus Q_2}(a_1,b_1)\times\tau_{Q_1\oplus Q_2}(a_2,b_2)] \leq 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\{d_{Q_1}(a_1) + \tau_{Q_1}(a_1)\tau_{Q_2}(b_1) + d_{Q_2}(b_1) + 1\}\times\{d_{Q_1}(a_2) + \tau_{Q_1}(a_2)\tau_{Q_2}(b_2) + d_{Q_2}(b_2) + 1\}] = 2\delta_2M_2(Q_1) + 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[d_{Q_1}(u_1)\tau_{Q_1}(u_2)\tau_{Q_2}(v_2) + d_{Q_1}(u_2)\tau_{Q_1}(u_1)\tau_{Q_2}(v_1)] + 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[d_{Q_1}(u_1)d_{Q_2}(v_2) + d_{Q_1}(u_2)d_{Q_2}(v_1)] + 2\delta_2M_1(Q_1) + 2ZC_2(Q_1)\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_2}(v_1)\tau_{Q_2}(v_2)] + 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[d_{Q_2}(v_2)\tau_{Q_2}(v_1)\tau_{Q_1}(u_1) + d_{Q_2}(v_1)\tau_{Q_2}(v_2)\tau_{Q_1}(u_2)] + 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1}(u_1)\tau_{Q_2}(v_1) + \tau_{Q_1}(u_2)\tau_{Q_2}(v_2)] + 2e_1[\delta_2 + \bar{M}_1(Q_2) + \bar{M}_2(Q_2)].$$

Similarly

$$\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1\oplus Q_2}(a_1,b_1)\times\tau_{Q_1\oplus Q_2}(a_2,b_2)] \leq 2e_2\bar{M}_2(Q_1) + 2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[d_{Q_1}(u_1)\tau_{Q_1}(u_2)\tau_{Q_2}(v_2) + d_{Q_1}(u_2)\tau_{Q_1}(u_1)\tau_{Q_2}(v_1)] + 2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[d_{Q_1}(u_1)d_{Q_2}(v_2) + d_{Q_1}(u_2)d_{Q_2}(v_1)] + 2e_2\bar{M}_1(Q_1) + 2ZC_2(Q_2)\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1}(u_1)\tau_{Q_1}(u_2)] + 2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[d_{Q_2}(v_2)\tau_{Q_2}(v_1)\tau_{Q_1}(u_1) + d_{Q_2}(v_1)\tau_{Q_2}(v_2)\tau_{Q_1}(u_2)] + 2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1}(u_1)\tau_{Q_2}(v_1) + \tau_{Q_1}(u_2)\tau_{Q_2}(v_2)] + 2\delta_1[e_2 + M_1(Q_2) + M_2(Q_2)].$$

Consequently,

$$ZC_2(Q_1\oplus Q_2) \leq 2ZC_1^*(Q_1)ZC_1^*(Q_2) + [M_1(Q_2) - 2e_2]ZC_1^*(Q_1) + [M_1(Q_1) - 2e_1]ZC_1^*(Q_2) + ZC_1(Q_1)ZC_2(Q_2) + ZC_1(Q_2)ZC_2(Q_1) + (n_1 + 2\delta_1)M_2(Q_2) + (n_2 + 2\delta_2)M_2(Q_1) + (n_2 + 2\delta_2 + 3e_2)M_1(Q_1) + (n_1 + 2\delta_1 + 3e_1)M_1(Q_2) + (n_1 + 2\delta_1)e_2 + (n_2 + 2\delta_2)e_1 + 8e_1e_2 + 2e_1[\bar{M}_1(Q_2) + \bar{M}_2(Q_2)] + 2e_2[\bar{M}_1(Q_1) + \bar{M}_2(Q_1)] + [M_1(Q_1) - 2e_1]\sum_{b_1b_2\in E(Q_2)}[d_{Q_2}(b_1)\tau_{Q_2}(b_2) + d_{Q_2}(b_2)\tau_{Q_2}(b_1)] + [M_1(Q_2) - 2e_2]\sum_{u_1u_2\in E(Q_1)}[d_{Q_1}(a_1)\tau_{Q_1}(a_2) + d_{Q_1}(a_2)\tau_{Q_1}(a_1)] + 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[d_{Q_1}(a_1)\tau_{Q_1}(a_2)\tau_{Q_2}(b_2) + d_{Q_1}(a_2)\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)] + 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[d_{Q_1}(a_1)d_{Q_2}(b_2) + d_{Q_1}(a_2)d_{Q_2}(b_1)] + 2ZC_2(Q_1)\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_2}(b_1)\tau_{Q_2}(b_2)] + 2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[d_{Q_2}(b_2)\tau_{Q_2}(b_1)\tau_{Q_1}(a_1) + d_{Q_2}(b_1)\tau_{Q_2}(b_2)\tau_{Q_1}(a_2)] +$$

$$2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)+\tau_{Q_1}(a_2)\tau_{Q_2}(b_2)]+$$
$$2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[d_{Q_1}(a_1)\tau_{Q_1}(a_2)\tau_{Q_2}(b_2)+d_{Q_1}(a_2)\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)]+$$
$$2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[d_{Q_1}(a_1)d_{Q_2}(b_2)+d_{Q_1}(a_2)d_{Q_2}(b_1)]+$$
$$2ZC_2(Q_2)\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1}(a_1)\tau_{Q_1}(a_2)]+$$
$$2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[d_{Q_2}(b_2)\tau_{Q_2}(b_1)\tau_{Q_1}(a_1)+d_{Q_2}(b_1)\tau_{Q_2}(b_2)\tau_{Q_1}(a_2)]+$$
$$2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)+\tau_{Q_1}(a_2)\tau_{Q_2}(b_2)].$$

□

Theorem 9. *Let Q_1 and Q_2 be two connected graphs. Then, ZC_1^* of the symmetric difference of Q_1 and Q_2 are as follows:*

$$ZC_1^*(Q_1\oplus Q_2)\leq[M_1(Q_2)-2e_2]ZC_1^*(Q_1)+[M_1(Q_1)-2e_1]ZC_1^*(Q_2)+(n_2+2\delta_2)M_1(Q_1)+(n_1+2\delta_1)M_1(Q_2)+2(n_1+2\delta_1)e_2+2(n_2+2\delta_2)e_1+8e_1e_2+2[e_1\bar{M}_1(Q_2)+e_2\bar{M}_1(Q_1)]+2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1}(u_1)\tau_{G_2}(v_1)+\tau_{Q_1}(u_2)\tau_{G_2}(v_2)]+2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1}(u_1)\tau_{Q_2}(v_1)+\tau_{Q_1}(u_2)\tau_{Q_2}(v_2)].$$

Proof. Since

$$ZC_1^*(Q_1\oplus Q_2)=\sum_{(a_1,b_1)(a_2,b_2)\in E(Q_1\oplus Q_2)}[\tau_{Q_1\oplus Q_2}(a_1,b_1)+\tau_{Q_1\oplus Q_2}(a_2,b_2)]=$$
$$\sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\oplus Q_2}(a,b_1)+\tau_{Q_1\oplus Q_2}(a,b_2)]+$$
$$\sum_{b\in V(Q_2)}\sum_{a_1a_2\in E(Q_1)}[\tau_{Q_1\oplus Q_2}(a_1,b)+\tau_{Q_1\oplus Q_2}(a_2,b)]+$$
$$\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1\oplus Q_2}(a_1,b_1)+\tau_{Q_1\oplus Q_2}(a_2,b_2)]+$$
$$\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1\oplus Q_2}(a_1,b_1)+\tau_{Q_1\oplus Q_2}(a_2,b_2)].$$

Taking

$$\sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\tau_{Q_1\oplus Q_2}(a,b_1)+\tau_{Q_1\oplus Q_2}(a,b_2)]\leq$$
$$\sum_{a\in V(Q_1)}\sum_{b_1b_2\in E(Q_2)}[\{d_{Q_1}(a)+\tau_{Q_1}(a)\tau_{Q_2}(b_1)+d_{Q_2}(b_1)+1\}+\{d_{Q_1}(a)+\tau_{Q_1}(a)\tau_{Q_2}(b_2)+d_{Q_2}(b_2)+1\}]=$$
$$4e_1e_2+[M_1(Q_1)-2e_1]ZC_1^*(Q_2)+n_1M_1(Q_2)+2n_1e_2.$$

Similarly

$$\sum_{b\in V(Q_2)}\sum_{a_1a_2\in E(Q_1)}[\tau_{Q_1\oplus Q_2}(a_1,b)+\tau_{Q_1\oplus Q_2}(a_2,b)]\leq$$
$$n_2M_1(Q_1)+[M_1(Q_2)-2e_2]ZC_1^*(Q_1)+4e_1e_2+2n_2e_1.$$

Also taking

$$\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1\oplus Q_2}(a_1,b_1)+\tau_{Q_1\oplus Q_2}(a_2,b_2)]\leq$$
$$2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\{d_{Q_1}(a_1)+\tau_{Q_1}(a_1)\tau_{Q_2}(b_1)+d_{Q_2}(b_1)+1\}+\{d_{Q_1}(a_2)+\tau_{Q_1}(a_2)\tau_{Q_2}(b_2)+d_{Q_2}(b_2)+1\}]=$$
$$2\delta_2M_1(Q_1)+2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1}(u_1)\tau_{Q_2}(v_1)+\tau_{Q_1}(u_2)\tau_{Q_2}(v_2)]+2e_1\bar{M}_1(Q_2)+4\delta_2e_1.$$

Similarly

$$\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1\oplus Q_2}(a_1,b_1)+\tau_{Q_1\oplus Q_2}(a_2,b_2)]\leq 2e_2\bar{M}_1(Q_1)+2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1}(u_1)\tau_{Q_2}(v_1)+\tau_{Q_1}(u_2)\tau_{Q_2}(v_2)]+2N_1M_1(Q_2)+4N_1e_2.$$

Consequently,

$$ZC_1^*(Q_1\oplus Q_2)\leq[M_1(Q_2)-2e_2]ZC_1^*(Q_1)+[M_1(Q_1)-2e_1]ZC_1^*(Q_2)+(n_2+2\delta_2)M_1(Q_1)+(n_1+2\delta_1)M_1(Q_2)+2(n_1+2\delta_1)e_2+2(n_2+2\delta_2)e_1+8e_1e_2+2[e_1\bar{M}_1(Q_2)+e_2\bar{M}_1(Q_1)]+2\sum_{a_1a_2\in E(Q_1)}\sum_{b_1b_2\notin E(Q_2)}[\tau_{Q_1}(u_1)\tau_{Q_2}(v_1)+\tau_{Q_1}(u_2)\tau_{Q_2}(v_2)]+2\sum_{b_1b_2\in E(Q_2)}\sum_{a_1a_2\notin E(Q_1)}[\tau_{Q_1}(u_1)\tau_{Q_2}(v_1)+\tau_{Q_1}(u_2)\tau_{Q_2}(v_2)].$$

□

4. Applications and Conclusion

Assume that $Q_1\cong P_3$ & $Q_2\cong P_3$ be two particular alkanes called by paths. So, We can compute relation between exact and computed values of these product-related operations (disjunction & symmetric difference) on these alkanes are as follows:

Disjunction:

- Exact value of $ZC_1(P_3\vee P_3)=40$,
- Computed value of $ZC_1(P_3\vee P_3)\leq 189$,
- Exact value of $ZC_2(P_3\vee P_3)=52$,
- Computed value of $ZC_2(P_3\vee P_3)\leq 488$,
- Exact value of $ZC_1^*(P_3\vee P_3)=88$,
- Computed value of $ZC_1^*(P_3\vee P_3)\leq 236$.

Symmetric Difference:

- Exact value of $ZC_1(P_3\oplus P_3)=116$,
- Computed value of $ZC_1(P_3\oplus P_3)\leq 153$,
- Exact value of $ZC_2(P_3\oplus P_3)=240$,
- Computed value of $ZC_2(P_3\oplus P_3)\leq 384$,
- Exact value of $ZC_1^*(P_3\oplus P_3)=140$,
- Computed value of $ZC_1^*(P_3\oplus P_3)\leq 204$.

In this paper, we have computed the general results in the form of upper bounds related to first Zagreb connection index, second Zagreb connection index and modified first Zagreb connection index of the resultant graphs which are obtained with the help of operations of product on graphs such as disjunction (co-normal product) and symmetric difference. However, the problem is still open to obtain other Zagreb connection indices for the resultant graphs based on various operations on graphs.

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References

[1] Nikolic, S., Trinajstic, N., & Tolic, I. M. (2000). Complexity of molecules. *Journal of chemical information and computer sciences, 40*(4), 920-926.

[2] Gonzalez-Diaz, H., Vilar, S., Santana, L., & Uriarte, E. (2007). Medicinal chemistry and bioinformatics-current trends in drugs discovery with networks topological indices. *Current Topics in Medicinal Chemistry, 7*(10), 1015-1029.

[3] Asif, F., Zahid, Z. & Zafar, S. (2020). Leap Zagreb and leap hyper-Zagreb indices of Jahangir and Jahangir derived graphs. *Engineering and Applied Science Letter, 3*(2), 1–8.

[4] Ali, U., Ahmad, Y. & Sardar, M. S.. (2020). On 3–total edge product cordial labeling of tadpole, book and flower graphs. *Open Journal of Mathematical Scineces, 4*(1), 48–55.

[5] Tabassum, A., Umar, M. A., Perveen, M. & Raheem, A. (2020). Antimagicness of subdivided fans. *Open Journal of Mathematical Scineces, 4*(1), 18–22.

[6] Shah, A. & Bokhary, S. A. U. B. (2019). On chromatic polynomial of certain families of dendrimer graphs. *Open Journal of Mathematical Scineces, 3*(1), 404–416.

[7] Zhu, E. & Liu, C. (2019). On smarandachely adjacent vertex total coloring of subcubic graphs. *Open Journal of Mathematical Scineces, 3*(1), 390–397.

[8] Umar, M. A., Ali, N., Tabassum, A., & Ali, B. R. (2019). Book graphs are cycle antimagic. *Open Journal of Mathematical Scineces, 3*(1), 184-190.

[9] Kier, L. B., & Hall, L. H. (1976). *Medicinal Chemistry: Molecular Connectivity In Chemistry And Drug Research. Academic Press.*

[10] Gonzalez-Diaz, H., Vilar, S., Santana, L., & Uriarte, E. (2007). Medicinal chemistry and bioinformatics-current trends in drugs discovery with networks topological indices. *Current Topics in Medicinal Chemistry, 7*(10), 1015-1029.

[11] Devillers, J., & Balaban, A. T. (Eds.). (2000). *Topological indices and related descriptors in QSAR and QSPAR*. CRC Press.

[12] Diudea, M. V. (2001). *QSPR/QSAR studies by molecular descriptors.* Nova Science Publishers.

[13] Varmuza, K., Dehmer, M., & Bonchev, D. (2012). *Statistical modelling of molecular descriptors in QSAR/QSPR*. Wiley-Blackwell, Weinheim.

[14] Todeschini, R., & Consonni, V. (2009). *Molecular descriptors for chemoinformatics: volume I: alphabetical listing/volume II: appendices, references* (Vol. 41). John Wiley & Sons.

[15] Gao, W., Wang, W., & Farahani, M. R. (2016). Topological indices study of molecular structure in anticancer drugs. *Journal of Chemistry, 2016*.

[16] Gutman, I., & Trinajstic, N. (1972). Graph theory and molecular orbitals. Total f-electron energy of alternant hydrocarbons. *Chemical Physics Letters, 17*(4), 535-538.

[17] Randic, M. (1975). Characterization of molecular branching. *Journal of the American Chemical Society, 97*(23), 6609-6615.

[18] Yamaguchi, S. (2008). Estimating the Zagreb indices and the spectral radius of triangle-and quadrangle-free connected graphs. *Chemical Physics Letters, 458*(4-6), 396-398.

[19] Das, K. C., & Gutman, I. (2004). Some properties of the second Zagreb index. *MATCH Commun. Math. Comput. Chem, 52*(1), 3-1.

[20] Furtula, B., & Gutman, I. (2015). A forgotten topological index. *Journal of Mathematical Chemistry, 53*(4), 1184-1190.

[21] Nadeem, M. F., Zafar, S., & Zahid, Z. (2016). On topological properties of the line graphs of subdivision graphs of certain nanostructures. *Applied mathematics and computation, 273*, 125-130.

[22] Ali, A., & Trinajstic, N. (2018). A novel/old modification of the first Zagreb index. *Molecular informatics, 37*(6-7), 1800008.

[23] Ducoffe, G., Marinescu-Ghemeci, R., Obreja, C., Popa, A., & Tache, R. M. (2018, September). Extremal graphs with respect to the modified first Zagreb connection index. *In 2018 20th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC)* (pp. 141-148). IEEE.

[24] Shao, Z., Gutman, I., Li, Z., Wang, S., & Wu, P. (2018). Leap Zagreb indices of trees and unicyclic graphs. *Communications in Combinatorics and Optimization, 3*(2), 179-194.

[25] Du, Z., Ali, A., & Trinajstic, N. (2019). Alkanes with the first three maximal/minimal modified first Zagreb connection indices. *Molecular informatics, 38*(4), 1800116.

[26] Tang, J. H., Ali, U., Javaid, M., & Shabbir, K. (2019). Zagreb Connection Indices of Subdivision and Semi-Total Point Operations on Graphs. *Journal of Chemistry, 2019*.

[27] Das, K. C., Yurttas, A., Togan, M., Cevik, A. S., & Cangul, I. N. (2013). The multiplicative Zagreb indices of graph operations. *Journal of Inequalities and Applications, 2013*(1), 90.

[28] Eskender, B., & Vumar, E. (2013). Eccentric connectivity index and eccentric distance sum of some graph operations. *Transactions on Combinatorics, 2*(1), 103-111.

[29] De, N., Pal, A., Nayeem, S., & Abu, M. (2014). On some bounds and exact formulae for connective eccentric indices of graphs under some graph operations. *International Journal of Combinatorics, 2014*.

[30] Tavakoli, M., Rahbarnia, F., & Ashrafi, A. R. (2014). Some new results on irregularity of graphs. *J. Appl. Math. Inform, 32*, 675-685.

[31] Veylaki, M., Nikmehr, M. J., & Tavallaee, H. A. (2016). The third and hyper-Zagreb coindices of some graph operations. *Journal of Applied Mathematics and Computing, 50*(1-2), 315-325.

[32] Naji, A. M., & Soner, N. D. (2018). The first leap Zagreb index of some graph operations. *Int. J. Appl. Graph Theory, 2*(1), 7-18.

[33] Liu, J. B., Javed, S., Javaid, M., & Shabbir, K. (2019). Computing first general Zagreb index of operations on graphs. *IEEE Access, 7*, 47494-47502.

[34] Liu, J. B., Javaid, M., & Awais, H. M. (2019). Computing Zagreb indices of the subdivision-related generalized operations of graphs. *IEEE Access, 7*, 105479-105488.

[35] Ahmad, M., Saeed, M., Javaid, M., & Hussain, M. (2019). Exact Formula and Improved Bounds for General Sum-Connectivity Index of Graph-Operations. *IEEE Access, 7*, 167290-167299.

[36] AWAIS, H., JAVAID, M., & JAMAL, M. (2019). FORGOTTEN INDEX OF GENERALIZED F-SUM GRAPHS. *Journal of Prime Research in Mathematics, 15*, 115-128.

[37] Zhang, X., Awais, H. M., Javaid, M., & Siddiqui, M. K. (2019). Multiplicative Zagreb Indices of Molecular Graphs. *Journal of Chemistry, 2019.*

[38] Todeschini, R., & Consonni, V. (2008). *Handbook of molecular descriptors* (Vol. 11). John Wiley & Sons.

[39] Gutman, I., Rudic, B., Trinajstic, N., & Wilcox Jr, C. F. (1975). Graph theory and molecular orbitals. XII. Acyclic polyenes. *The Journal of Chemical Physics, 62*(9), 3399-3405.

[40] Ashrafi, A. R., Dolic, T., & Hamzeh, A. (2010). The Zagreb coindices of graph operations. *Discrete applied mathematics, 158*(15), 1571-1578.

[41] Gutman, I., Naji, A. M., & Soner, N. D. (2017). On leap Zagreb indices of graphs. *Communication in Combinatorics and Optimization, 2*(2), 99-117.

[42] West, D. B. (2001). *Introduction to graph theory* (Vol. 2). Upper Saddle River: Prentice hall.