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THE NEW AUXILIARY METHOD IN THE SOLUTION OF THE GENERALIZED BURGERS-HUXLEY EQUATION

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Abstract. A recently developed direct method known as the new auxiliary method shown significant improvement for solving non-leaner partial differential equations (PDEs) and gives more exact solutions compared to the traditional direct method. In this paper, we used the new auxiliary method for solitary wave solutions of the generalized Burgers Huxley equation (B-HE). The new auxiliary method is a very powerful, felicitous, effective method to get solitary wave solutions of PDEs.

Key words: The new auxiliary method, the generalized Burgers Huxley equation, exact solution.
MSC: Primary 34A08, 35R11.

1. Introduction

Zabusky and Kruskal are the two physicists who first introduce soliton in 1965, and they showed that numerous phenomena of our daily life from different fields like Chemistry, Biology, plasma physics, Fluid mechanics mathematical physics, etc can be represented by linear and non-linear partial differential equations. But, mostly it is difficult to solve these partial differential equations due to its complexity, so researcher are always looking for the numerical and analytical methods to find the solution of these types of differential equations [1–13]. In particular, there are many methods exist in the literature to find the analytical solutions of non-linear partial differential equations, for example The Tanh method [14], modified extended direct algebraic method [15],

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The Sub-equation method [16], Exp-function method [17**?**], $G^{'}/G$-Expansion Method [18], Extended Jacobi elliptic function expansion method [19] and many others.

Especially, the non-linear Burgers Huxley equation has been solved by many researchers, for example Wang et al. [20] solved for exact solutions using the Adomian decomposition method, Ismail et al. [21, 27] get the analytical solution in the series form, Deng [23] used the first-integral method and get traveling solitary wave solutions.

However, most of the exact solutions obtain by using the above methods are not explicit in general, therefore it is important to develop a new reliable method for the solution of non-linear partial differential equations. The new auxiliary method is a direct method, which is an effective and reliable method, and gives more exact solution for the non-linear partial differential equations. Therefore, in this article, we present the formulation of the new auxiliary method [24, 25] in the solutions of non-linear generalized B-HE.

The generalized Burgers Huxley equation is [26]

$$u_t + \alpha u^n u_x - u_{xx} = \beta u \left(1 - u^n\right)\left(u^n - \gamma\right), \ 0 \leq x \leq 1, t \geq 0, \tag{1}$$

where n is a positive integer, $\alpha, \beta \geq 0$ are real constants and $\gamma \in (0, 1)$.

The Burgers-Huxley equation describes the interaction between convection effects, diffusion transfer and reaction mechanisms [27], also it is a particular case of many famous equations for example when $n = 1, \alpha = 0$ then it became FitzHugh-Nagumo equation, when $n = 1, \beta = 0$ then it is burger equation, when $n = 1, \alpha = 0, \gamma = -1$ then it becomes Newell–Whitehead equation [28]. The paper is presented as follows: In Section 2, we describe the proposed method. In Section 3, we present the application of the proposed method for the generalized B-HE, and finally the conclusion in Section 4.

2. The Proposed method

In this section, we explain the steps for the new auxiliary method [24]. The non-linear differential equation in form

$$F(w, w_t, w_x, w_y, w_{xy}, w_{xt}, ...) = 0, \tag{2}$$

where F is polynomial of $w(x, y, t)$. The proposed method involve the following steps.

step 1. Apply the transformation:
$F(x, t) = F(\upsilon)$, $\upsilon = x - ct$. where c is constant for (2), by applying this transformation we get the Ordinary Differential Equation (ODE) in the form

$$T(w, w', w'', ...) = 0, \tag{3}$$

where T depicted polynomial of $w(\upsilon)$ and its derivatives.
Step 2. Let (3) has solutions in the form

$$w(\upsilon) = \sum_{i=0}^{N} a_i a^{if(\upsilon))}, \tag{4}$$

where $a_i's$ are constants, and $f(\upsilon)$ in (4) will satisfy

$$f'(\upsilon) = \frac{1}{ln(a)}(p_0 a^{-f(\upsilon)} + q_0 + r_0 a^{-f(\upsilon)}). \tag{5}$$

Step 3. In (4) the value of N will be computed using the balancing of the nonlinear term and higher order derivative.
Step 4. Substituting (4) and (5) into (1) and then equating to zero, we get the system of linear equations which gives the values of a_i, p_0, q_0, r_0.
Step 5. In the last step, substituting the values of $a^{f(\upsilon)}$ into (3), we will get the solitary wave solutions for (1)

3. Application of the proposed method

To show the effectiveness of the proposed method, we solve the generalized B-HE using the proposed method.

First, make the following transformation

$$w = u^n \tag{6}$$

and we obtain the following from (1)

$$ww_t + \alpha w^2 w_x + (1 - \frac{1}{n})w_x^2 - ww_{xx} - \beta n w^2(1-w)(w-\gamma) = 0. \tag{7}$$

Let the traveling wave transformation in the form

$$w(x,t) = w(\upsilon), \tag{8}$$

where $\upsilon = x - ct$. Substituting Eqs. (8) and (6) into (1), we get

$$-cww' + \alpha ww' + 2(1 - \frac{1}{n})(w')^2 - ww'' - \beta n w^2(1-w)(1-\gamma) = 0, \tag{9}$$

where

$$w' = w'(\upsilon) = \frac{\partial w(\upsilon)}{\partial x}.$$

By balancing of non-linear part and highest order derivative, we get the solution in the form

$$w(\upsilon) = a_0 + a_1 a^{f(\upsilon)}. \tag{10}$$

By substituting (10) and its derivatives into (9), and then equating the coefficients for the different powers of $a^{if(\upsilon)}$ to zero, we get algebraic equations. Solving these algebraic equations using Mathematics software, we will get four

sets of solutions;
Set 1:

$$a_0 = \frac{1}{2} - \frac{q_0}{\sqrt{2}\sqrt{\beta}n},\ a_1 = -\frac{\sqrt{2}r_0}{\sqrt{\beta}n},\ c = \alpha + \frac{\sqrt{\beta}(2\gamma + n - 2)}{\sqrt{2}},\ p_0 = \frac{2q_0^2 - \beta n^2}{8r_0}. \tag{11}$$

Set 2:

$$a_0 = \frac{q_0}{\sqrt{2}\sqrt{\beta}n} + \frac{1}{2},\ a_1 = \frac{\sqrt{2}r_0}{\sqrt{\beta}n},\ c = \alpha - \frac{\sqrt{\beta}(2\gamma + n - 2)}{\sqrt{2}},\ p_0 = \frac{2q_0^2 - \beta n^2}{8r_0}. \tag{12}$$

Set 3:

$$a_0 = \frac{1}{2}\left(\gamma - \frac{\sqrt{2}q_0}{\sqrt{\beta}n}\right),\ a_1 = -\frac{\sqrt{2}r_0}{\sqrt{\beta}n},\ c = \alpha + \frac{\sqrt{\beta}(\gamma(n-2)+2)}{\sqrt{2}},\ p_0 = \frac{2q_0^2 - \beta\gamma^2 n^2}{8r_0}. \tag{13}$$

Set 4:

$$a_0 = \frac{1}{2}\left(\gamma + \frac{\sqrt{2}q_0}{\sqrt{\beta}n}\right),\ a_1 = \frac{\sqrt{2}r_0}{\sqrt{\beta}n},\ c = \alpha - \frac{\sqrt{\beta}(\gamma(n-2)+2)}{\sqrt{2}},\ p_0 = \frac{2q_0^2 - \beta\gamma^2 n^2}{8r_0}. \tag{14}$$

Substituting (14) into (13), we get the solitary wave solution of the generalized B-HEs are

$$w(x,t) = (\frac{1}{2} - \frac{q_0}{\sqrt{2}\sqrt{\beta}n}) - \frac{\sqrt{2}r_0}{\sqrt{\beta}n}a^{f(\upsilon)}. \tag{15}$$

Using (11), we get the solitary wave solutions for the Set 1

$$u(x,t) = \left((\frac{1}{2} - \frac{q_0}{\sqrt{2}\sqrt{\beta}n}) - \frac{\sqrt{2}r_0}{\sqrt{\beta}n}a^{f(\upsilon)}\right)^{\frac{1}{n}}, \tag{16}$$

If $q_0^2 - 4p_0r_0 < 0$ and $r_0 \neq 0$,

$$\begin{aligned} &a^{f(\xi)} = \tfrac{-q_0}{2r_0} + \tfrac{\sqrt{4p_0r_0 - q_0^2}}{2r_0}\tan\left(\tfrac{\sqrt{4p_0r_0-q_0^2}}{2}\xi\right) \\ &\text{or} \\ &a^{f(\xi)} = \tfrac{-q_0}{2r_0} - \tfrac{\sqrt{4p_0r_0 - q_0^2}}{2r_0}\cot\left(\tfrac{\sqrt{4p_0r_0-q_0^2}}{2}\xi\right). \end{aligned} \tag{17}$$

If $q_0^2 - 4p_0r_0 > 0$ and $r_0 \neq 0$,

$$a^{f(\xi)} = \frac{-q_0}{2r_0} - \frac{\sqrt{q_0^2 - 4p_0r_0}}{2r_0}\tanh\left(\frac{\sqrt{q_0^2-4p_0r_0}}{2}\xi\right) \tag{18}$$

or

$$a^{f(\xi)} = \frac{-q_0}{2r_0} - \frac{\sqrt{q_0^2 - 4p_0r_0}}{2r_0} \coth\left(\frac{\sqrt{q_0^2 - 4p_0r_0}}{2}\xi\right). \tag{19}$$

If $q_0^2 + 4p_0^2 < 0$, $r_0 \neq 0$ and $r_0 = -p_0$,

$$a^{f(\xi)} = \frac{q_0}{2p_0} - \frac{\sqrt{-\left(q_0^2 + 4p_0^2\right)}}{2p_0} \tan\left(\frac{\sqrt{-\left(q_0^2 + 4p_0^2\right)}}{2}\xi\right) \tag{20}$$

or

$$a^{f(\xi)} = \frac{q_0}{2p_0} + \frac{\sqrt{-\left(q_0^2 + 4p_0^2\right)}}{2p_0} \cot\left(\frac{\sqrt{-\left(q_0^2 + 4p_0^2\right)}}{2}\xi\right). \tag{21}$$

If $q_0^2 + 4p_0^2 > 0$, $r_0 \neq 0$ and $r_0 = -p_0$,

$$a^{f(\xi)} = \frac{q_0}{2p_0} + \frac{\sqrt{\left(q_0^2 + 4p_0^2\right)}}{2p_0} \tanh\left(\frac{\sqrt{\left(q_0^2 + 4p_0^2\right)}}{2}\xi\right). \tag{22}$$

or

$$a^{f(\xi)} = \frac{q_0}{2p_0} + \frac{\sqrt{\left(q_0^2 + 4p_0^2\right)}}{2p_0} \coth\left(\frac{\sqrt{\left(q_0^2 + 4p_0^2\right)}}{2}\xi\right). \tag{23}$$

If $q_0^2 - 4p_0^2 < 0$ and $r_0 = p_0$,

$$a^{f(\xi)} = \frac{-q_0}{2p_0} + \frac{\sqrt{-\left(q_0^2 - 4p_0^2\right)}}{2p_0} \tan\left(\frac{\sqrt{-\left(q_0^2 - 4p_0^2\right)}}{2}\xi\right) \tag{24}$$

or

$$a^{f(\xi)} = \frac{-q_0}{2p_0} - \frac{\sqrt{-\left(q_0^2 - 4p_0^2\right)}}{2p_0} \cot\left(\frac{\sqrt{-\left(q_0^2 - 4p_0^2\right)}}{2}\xi\right). \tag{25}$$

If $q_0^2 - 4p_0^2 > 0$ and $r_0 = p_0$,

$$a^{f(\xi)} = \frac{-q_0}{2p_0} - \frac{\sqrt{\left(q_0^2 - 4p_0^2\right)}}{2p_0} \tanh\left(\frac{\sqrt{\left(q_0^2 - 4p_0^2\right)}}{2}\xi\right) \tag{26}$$

or

$$a^{f(\xi)} = \frac{-q_0}{2p_0} - \frac{\sqrt{\left(q_0^2 - 4p_0^2\right)}}{2p_0} \coth\left(\frac{\sqrt{\left(q_0^2 - 4p_0^2\right)}}{2}\xi\right). \tag{27}$$

If $q_0^2 = 4p_0r_0$,

$$a^{f(\xi)} = -\frac{2 + q_0\xi}{2r_0\xi} \tag{28}$$

If $r_0p_0 < 0$, $q_0 = 0$ and $r_0 \neq 0$,

$$a^{f(\xi)} = -\sqrt{\frac{-p_0}{r_0}}\tanh(\sqrt{-r_0p_0}\xi) \tag{29}$$

or

$$a^{f(\xi)} = -\sqrt{\frac{-p_0}{r_0}}\coth(\sqrt{-r_0p_0}\xi). \tag{30}$$

If $q_0 = 0$ and $p_0 = -r_0$,

$$a^{f(\xi)} = \frac{1 + e^{(-2r_0\xi)}}{-1 + e^{(-2r_0\xi)}}. \tag{31}$$

If $p_0 = r_0 = 0$,

$$a^{f(\xi)} = \cosh(q_0\xi) + \sinh(q_0\xi). \tag{32}$$

If $p_0 = q_0 = k$ and $r_0 = 0$,

$$a^{f(\xi)} = e^{k\xi} - 1. \tag{33}$$

If $q_0 = r_0 = k$ and $p_0 = 0$,

$$a^{f(\xi)} = \frac{e^{k\xi}}{1 - e^{k\xi}}. \tag{34}$$

If $q_0 = p_0 + r_0$,

$$a^{f(\xi)} = -\frac{1 - p_0e^{(p_0-r_0)\xi}}{1 - r_0e^{(p_0-r_0)\xi}}. \tag{35}$$

If $q_0 = -(p_0 + r_0)$,

$$a^{f(\xi)} = \frac{p_0 - e^{(p_0-r_0)\xi}}{r_0 - e^{(p_0-r_0)\xi}}. \tag{36}$$

If $p_0 = 0$,

$$a^{f(\xi)} = \frac{q_0e^{q_0\xi}}{1 - r_0e^{q_0\xi}}. \tag{37}$$

If $r_0 = q_0 = p_0 \neq 0$,

$$a^{f(\xi)} = \frac{1}{2}\left\{\sqrt{3}\tan\left(\frac{\sqrt{3}}{2}p_0\xi\right) - 1\right\}. \tag{38}$$

If $r_0 = q_0 = 0$,

$$a^{f(\xi)} = p_0\xi. \tag{39}$$

If $p_0 = q_0 = 0$,

$$a^{f(\xi)} = \frac{-1}{r_0\xi}. \tag{40}$$

If $r_0 = p_0$ and $q_0 = 0$,

$$a^{f(\xi)} = \tan(p_0\xi). \tag{41}$$

If $r_0 = 0$,

$$a^{f(\xi)} = e^{q_0 \xi} - \frac{\alpha}{\beta}. \tag{42}$$

Similarly, we can find the above solutions for the remaining three sets.

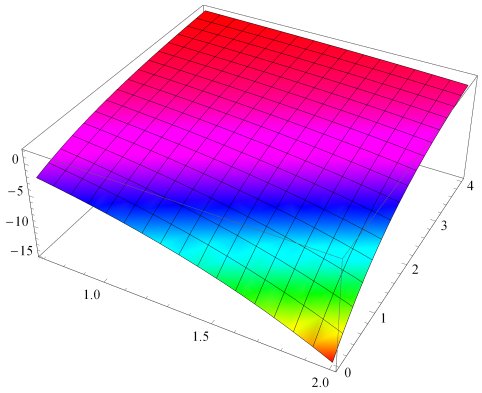

FIGURE 1. Graph of (42) when $\alpha = \beta = 1.5$, $\gamma = 0.001$ and $n = 1$

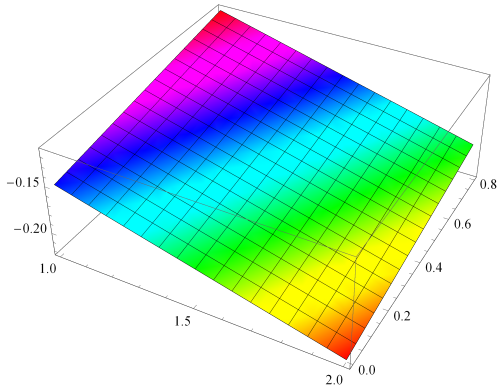

FIGURE 2. Graph of (41) when $\alpha = \beta = 1.5$, $\gamma = 0.001$ and $n = 1$

From the above, it can be seen that we get twenty-six exact solitary wave solutions for (1) from Eqs. (16) to (42) for Set 1 only and similarly, we find these solutions for Set 2 to Set 4, which show that we get more solutions as compared to [26]. Figure 1 to Figure 5 are the 3D graphs of different solutions for (1), which shows the efficiency and reliability of the proposed method. The new auxiliary method is more efficient tool for the solitary wave solutions of linear and non-linear partial differential equations.

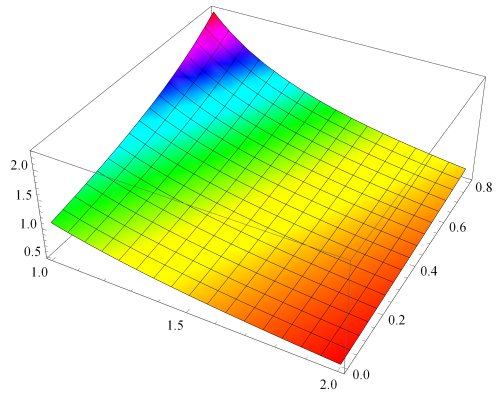

FIGURE 3. Graph of (40) when $\alpha = \beta = 1.5$, $\gamma = 0.01$ and $n = 1$

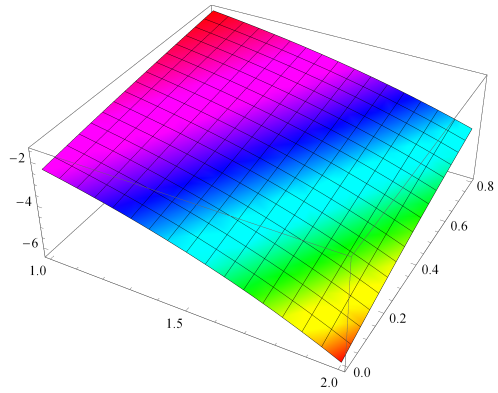

FIGURE 4. Graph of (33) when $\alpha = \beta = 1.5$, $\gamma = 0.001$ and $n = 1$

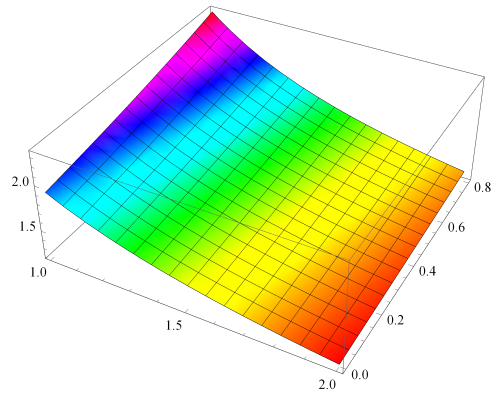

FIGURE 5. Graph of (28) when $\alpha = \beta = 1$, $\gamma = 0.001$ and $n = 1$

4. Conclusion

In this article, we apply the new auxiliary method to get the exact solitary wave solutions for the B-HE. By applying the new auxiliary method, we get twenty-six exact solutions and comparing these solutions with [21, 23, 27]; it is evident that the proposed method gives more solution obtain by the existent method in literature use for the solution of B-HE. The new auxiliary method has many advantages such as it is simple, straight forward, minimize the computational work; therefore it has wide-range applicability and a powerful tool for the exact solitary wave solutions of different type of non-linear PDEs.

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