Journal of Prime Research in Mathematics Vol. **16(2)***(2020), 89-108*

IMPROVEMENT OF THE HARDY INEQUALITY INVOLVING k-FRACTIONAL CALCULUS

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Abstract. The major idea of this paper is to establish some new improvements of the Hardy inequality by using k-fractional integral of Riemann-type, Caputo k-fractional derivative, Hilfer k-fractional derivative and Riemann-Liouville (k, r)-fractional integral. We discuss the log-convexity of the related linear functionals. We also deduce some known results from our general results.

Key words: Convex function, kernel, fractional derivative, fractional integrals, means.
MSC: 26D15, 26D10, 26A33.

1. **Introduction**

The study of non integer order derivative and integral operators is known as fractional calculus. It draws increasing attention due to its applications in many fields see e.g. the books [1, 2].

Let us recall the well known definitions of Riemann-Liouville fractional integrals (see [3], [4]).

Definition 1. *Let $[a, b]$ be a finite interval on real axis $\mathbb{R}$. The left-sided and right-sided Riemann-Liouville fractional integrals $I^{\alpha}_{a^+}f$ and $I^{\alpha}_{b^-}f$ of order $\alpha > 0$ respectively are defined by*

$$I^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(y)(x-y)^{\alpha-1}dy, \quad (x > a),$$

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and

$$I_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_x^b f(y)(y-x)^{\alpha-1} dy, \quad (x < b),$$

where $\Gamma(\alpha)$ *is the Gamma function.*

The first result yields that fractional integral operators are bounded in $L_p(a,b)$, $-\infty < a < b < \infty$, $1 \leq p \leq \infty$, that is

$$\|I_{a^+}^{\alpha} f\|_p \leq K\|f\|_p, \qquad \|I_{b^-}^{\alpha} f\|_p \leq K\|f\|_p, \tag{1}$$

where

$$K = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}.$$

G. H. Hardy proved the inequality (1) involving left-sided fractional integral in one of his initial paper (see [5]) and is known as inequality of G. H. Hardy (see [6]).

We initiate some essential notation and recollect a number of basic specifics about convex functions, log-convex functions [7, 8] as well as exponentially convex functions [9, 10, 11]. For More details, see [12, 13, 14, 15, 16] and references therein.

Definition 2. *A positive function* f *is said to be* logarithmically convex *if* $\log f$ *is a convex function on an interval* $I \subseteq \mathcal{R}$*. For such function* f*, we shortly say* f *is* log-*convex. A positive function* f *is* log-convex in the Jensen sense *if for each* $a, b \in I$

$$f^2\left(\frac{a+b}{2}\right) \leq f(a)f(b)$$

holds, i.e., if $\log f$ *is convex in the Jensen sense.*

Definition 3. [17] *Let* $I \subseteq \mathbb{R}$ *be an interval and* $f : I \to \mathbb{R}$ *be convex on* I*. Then for* $s_1, s_2, s_3 \in I$ *such that* $s_1 < s_2 < s_3$*, the following inequality holds:*

$$f(s_1)(s_3 - s_2) + f(s_2)(s_1 - s_3) + f(s_3)(s_2 - s_1) \geq 0.$$

The function f *is* log-*convex on an interval* I*, if and only if for all* $s_1, s_2, s_3 \in I$*,* $s_1 < s_2 < s_3$ *it holds*

$$[f(s_2)]^{s_3 - s_1} \leq [f(s_1)]^{s_3 - s_2}[f(s_3)]^{s_2 - s_1}.$$

Definition 4. [10, p. 373] *A function* $h : (a,b) \to \mathbb{R}$ *is exponentially convex if it is continuous and*

$$\sum_{i,j=1}^{n} a_i a_j h(x_i + x_j) \geq 0,$$

for all $n \in \mathbb{N}$ *and all choices of* $a_i \in \mathbb{R}$*,* $x_i + x_j \in (a,b)$*,* $i, j = 1, ..., n$*.*

Proposition 1. [9] *Let $h : (a,b) \to \mathbb{R}$. The following statements are equivalent.*
(i) h is exponentially convex.
(ii) h is continuous and

$$\sum_{i,j=1}^{n} a_i a_j h\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

for every $n \in \mathbb{N}$, $a_i \in (a,b)$ and $x_i \in (a,b), 1 \leq i \leq n$.

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures and A_k be an integral operator defined by

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y) d\mu_2(y), \tag{2}$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative kernel, f is measurable function on Ω_2, and

$$0 < K(x) := \int_{\Omega_2} k(x,y) d\mu_2(y), \qquad x \in \Omega_1. \tag{3}$$

The following theorem is given in [18] (see also [19]).

Theorem 2. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by (3). Suppose that the function $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$ and that v is defined on Ω_2 by*

$$v(y) := \int_{\Omega_1} \frac{u(x)k(x,y)}{K(x)} d\mu_1(x) < \infty. \tag{4}$$

If Φ is convex on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\Omega_1} u(x) \Phi\left(A_k f(x)\right) d\mu_1(x) \leq \int_{\Omega_2} v(y) \Phi\left(f(y)\right) d\mu_2(y),$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $Imf \subseteq I$, where A_k is defined by (2).

Lemma 3. *Let $s \in \mathbb{R}$, $\varphi_s : \mathbb{R}^+ \to \mathbb{R}$ be a function defined as*

$$\varphi_s(x) := \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0,1, \\ -\log x, & s = 0, \\ x \log x, & s = 1. \end{cases} \tag{5}$$

Then φ_s is strictly convex on $\mathbb{R}^+$ for each $s \in \mathbb{R}$.

Proof. Since $\varphi_s''(x) = x^{s-2} > 0$ for all $x \in \mathbb{R}^+, s \in \mathbb{R}$, therefore φ_s is strictly convex on $\mathbb{R}^+$ for each $s \in \mathbb{R}$. □

The following theorem is given in [19].

Theorem 4. *Let the assumption of the Theorem 2 be satisfied and φ_s be defined by (5). Let f be a positive function. Then the function $\rho : \mathbb{R} \to [0, \infty)$ defined by*

$$\rho(s) = \int_{\Omega_2} v(y)\varphi_s(f(y))d\mu_2(y) - \int_{\Omega_1} u(x)\varphi_s(A_k f(x))d\mu_1(x), \tag{6}$$

is exponentially convex.

The function ρ being exponentially convex is also log-convex function. Then by Definition 3 the following inequality holds true:

$$[\rho(p)]^{q-r} \leq [\rho(q)]^{p-r}[\rho(r)]^{q-p}, \tag{7}$$

for every choice $r, p, q \in \mathbb{R}$, such that $r < p < q$.

Upcoming result is given in [20].

Theorem 5. *Let the assumptions of the Theorem 2 be satisfied. If Φ is convex on the interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int\, I$, then the inequality*

$$\begin{aligned} &\int_{\Omega_2} v(y)\Phi\left(f(y)\right) d\mu_2(y) - \int_{\Omega_1} u(x)\Phi\left(A_k f(x)\right) d\mu_1(x) \\ &\geq \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,y)\, \big|\, |\Phi(f(y)) - \Phi(A_k f(x))| \\ &- |\varphi(A_k f(x))|.|f(y) - A_k f(x)|\, \big|\, d\mu_2(y)\, d\mu_1(x) \end{aligned} \tag{8}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $f(y) \in I$, for all fixed $y \in \Omega_2$ where A_k is defined by (2).

The paper is planned in the following manner. After this introduction, in Section 2 we give the improvement of G. H. Hardy's inequality for Riemnn-Liouville fractional integral and deduce some known results from our general results. Section 3 deals with improvement of an inequality of G. H. Hardy via Caputo k-fractional derivative. As special case we shall derive the result of Caputo fractional derivative given in [21]. Section 4 is dedicated to Hilfer k-fractional derivative and the log-convexity of the related functionals. Last section is devoted to (k, r)-fractional integral along the log-convexity of the functionals.

2. Main Results

Definition 5. [22] *If $f \in L_1[a,b]$ and $k > 0$, then the Riemann-Liouville k-fractional integral $I^{\alpha}_{k,a}$ of order α is defined by*

$$I^{\alpha}_{k,a} f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t (t-\tau)^{\frac{\alpha}{k}-1} f(\tau) d\tau, \alpha > 0, t \in [a,b], \tag{9}$$

where Γ_k is the Gamma k-function which is defined as:

$$\Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-\frac{t^k}{k}} dt,$$

and

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha).$$

Now we give our first main result.

Theorem 6. *Let $s > 1$, $I^{\alpha}_{k,a} f$ denotes the left-sided Riemann-Liouville k-fractional integral of order α and $\rho_1 : \mathbb{R} \to [0, \infty)$. Then the following inequality holds true:*

$$\rho_1(s) \leq \sigma_1(s), \tag{10}$$

where

$$\rho_1(s) = \frac{1}{s(s-1)} \left[\int_a^b (b-y)^{\frac{\alpha}{k}} f^s(y) dy - \int_a^b (x-a)^{\frac{\alpha}{k}} \left(\frac{\Gamma_k(\alpha+k)}{(x-a)^{\frac{\alpha}{k}}} I^{\alpha}_{k,a} f(x) \right)^s dx \right],$$

$$\sigma_1(s) = \frac{(b-a)^{\alpha(1-s)}}{s(s-1)} \left[(b-a)^{\frac{\alpha}{k}s} \int_a^b f^s(y) dy - (\Gamma_k(\alpha+k))^s \int_a^b (I^{\alpha}_{k,a} f(x))^s dx \right]. \tag{11}$$

Proof. Applying Theorem 2 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$\check{k}(x,t) = \begin{cases} \frac{1}{k\Gamma_k(\alpha)} (x-t)^{\frac{\alpha}{k}-1}, & a \leq t \leq x\,; \\ 0, & x < t \leq b\,, \end{cases} \tag{12}$$

we get

$$\check{K}(x) = \frac{1}{\alpha \Gamma_k(\alpha)} (x-a)^{\frac{\alpha}{k}}, \tag{13}$$

and the integral operator $A_k f(x)$ take the form

$$\check{A}_k f(x) = \frac{\alpha}{k(x-a)^{\frac{\alpha}{k}}} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt. \tag{14}$$

Then the equation (6)

$$\rho_1(s) = \int_a^b \check{v}(y)\varphi_s(f(y))dy - \int_a^b u(x)\varphi_s\left(\frac{\Gamma_k(\alpha+k)}{(x-a)^{\frac{\alpha}{k}}} I^{\alpha}_{k,a} f(x)\right) dx. \tag{15}$$

For particular weight function $u(x) = (x-a)^{\frac{\alpha}{k}}$, we obtain $\check{v}(y) = (b-y)^{\frac{\alpha}{k}}$ and we take $\varphi_s(x) = \frac{x^s}{s(s-1)},\ x \in \mathbb{R}^+$. So (15) becomes

$$\begin{aligned}
\rho_1(s) &= \frac{1}{s(s-1)}\left[\int_a^b (b-y)^{\frac{\alpha}{k}} f^s(y)dy - \int_a^b (x-a)^{\frac{\alpha}{k}}\left(\frac{\Gamma_k(\alpha+k)}{(x-a)^{\frac{\alpha}{k}}} I^{\alpha}_{k,a} f(x)\right)^s dx\right] \\
&\leq \frac{1}{s(s-1)}\left[(b-a)^{\frac{\alpha}{k}}\int_a^b f^s(y)dy - (b-a)^{\frac{\alpha}{k}(1-s)}(\Gamma_k(\alpha+k))^s \int_a^b (I^{\alpha}_{k,a} f(x))^s dx\right] \\
&= \frac{(b-a)^{\alpha(1-s)}}{s(s-1)}\left[(b-a)^{\frac{\alpha}{k}s}\int_a^b f^s(y)dy - (\Gamma_k(\alpha+k))^s \int_a^b (I^{\alpha}_{k,a} f(x))^s dx\right] \\
&= \sigma_1(s).
\end{aligned}$$

It follows (10). □

Remark 1. *For particular value of* $k = 1$, *we get the improvement of G. H. Hardy's inequality given in* [21, Theorem 2.3].

Now we shall provide the refinement of the Theorem 6. For this let us continue by taking a positive difference between the the left hand-side and the right-hand side of refined Hardy-type inequality given in (8).

$$\begin{aligned}
\Psi(\Phi) = &\int_{\Omega_2} v(y)\Phi\left(f(y)\right) d\mu_2(y) - \int_{\Omega_1} u(x)\Phi\left(A_k f(x)\right) d\mu_1(x) \\
&- \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,y)\,\big|\,|\Phi(f(y)) - \Phi(A_k f(x))| \\
&- |\varphi(A_k f(x))|.|f(y) - A_k f(x)|\,\big|\, d\mu_2(y)\, d\mu_1(x) \geq 0.
\end{aligned} \tag{16}$$

Theorem 7. *Let the assumptions of the Theorem 6 be satisfied and* $\psi_1 : \mathbb{R} \to [0,\infty)$. *Then the following inequality holds:*

$$0 \leq \Psi_1(s) \leq \sigma_1(s) - \lambda_1(s) \leq \sigma_1(s), \tag{17}$$

where

$$\Psi_1(s) = \frac{1}{s(s-1)}\left[\int_a^b (b-y)^{\frac{\alpha}{k}} f^s(y)dy - \int_a^b (x-a)^{\frac{\alpha}{k}}\left(\frac{\Gamma_k(\alpha+k)}{(x-a)^{\frac{\alpha}{k}}} I^{\alpha}_{k,a} f(x)\right)^s dx\right] - \lambda_1(s), \tag{18}$$

$$\lambda_1(s) = \frac{\alpha}{ks(s-1)}\int_a^b\int_a^x (x-y)^{\frac{\alpha}{k}-1}\left|\left|f^s(y) - \left(\frac{\Gamma_k(\alpha+k)}{(x-a)^{\frac{\alpha}{k}}} I^{\alpha}_{k,a} f(x)\right)^s\right| - s\left|\frac{\Gamma_k(\alpha+k)}{(x-a)^{\frac{\alpha}{k}}} I^{\alpha}_{k,a} f(x)\right|^{s-1} \cdot \left|f(y) - \frac{\Gamma_k(\alpha+k)}{(x-a)^{\frac{\alpha}{k}}} I^{\alpha}_{k,a} f(x)\right|\right| dy\, dx, \tag{19}$$

and $\sigma_1(s)$ *is defined by* (11).

Proof. Rewrite equation (16) with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$ and $\check{k}$, $\check{K}$, $\check{A}_k f(x)$ is defined by (12), (13), (14) respectively. For particular weight function $u(x) = (x-a)^{\frac{\alpha}{k}}$, we obtain $\check{v}(y) = (b-y)^{\frac{\alpha}{k}}$. If we take $\Phi(x) = x^s/s(s-1)$, $x \in \mathbb{R}^+$, after some calculations (16) becomes

$$\Psi_1(s) = \frac{1}{s(s-1)}\left[\int_a^b (b-y)^{\frac{\alpha}{k}} f^s(y)dy - \int_a^b (x-a)^{\frac{\alpha}{k}}\left(\frac{\Gamma_k(\alpha+k)}{(x-a)^{\frac{\alpha}{k}}} I^{\alpha}_{k,a} f(x)\right)^s dx\right]$$
$$- \frac{\alpha}{ks(s-1)}\int_a^b\int_a^x (x-t)^{\frac{\alpha}{k}-1}\left|\left|f^s(y) - \left(\frac{\Gamma_k(\alpha+k)}{(x-a)^{\frac{\alpha}{k}}} I^{\alpha}_{k,a} f(x)\right)^s\right| - s\left|\frac{\Gamma_k(\alpha+k)}{(x-a)^{\frac{\alpha}{k}}} I^{\alpha}_{k,a} f(x)\right|^{s-1} \cdot \left|f(y) - \frac{\Gamma_k(\alpha+k)}{(x-a)^{\frac{\alpha}{k}}} I^{\alpha}_{k,a} f(x)\right|\right| dy\, dx \geq 0.$$

Since $\frac{\alpha}{k}(1-s) \leq 0$ for $s > 1$ and $\lambda_1(s) \geq 0$, we obtain that

$$0 \leq \Psi_1(s) \leq \rho_1(s) - \lambda_1(s) \leq \rho_1(s).$$

This complete the proof. □

Remark 2. *For particular value of* $k = 1$ *we get the improvement of G. H. Hardy's inequality given in* [23, Corollary 2.1].

3. Results for Caputo k-fractional derivative

Following definition of Caputo k-fractional derivatives is given in [24].

Definition 6. *Let $\alpha > 0, k \geq 1$ and $\alpha \notin \{1,2,3,...\}$, $n = [\alpha]+1$, $f \in AC^n[a,b]$. The left and right sided Caputo k-fractional derivatives of order α are defined as follows:*

$$ {}^{C}D^{\alpha,k}_{a+}f(x) = \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}}dt, x > a \tag{20} $$

and

$$ {}^{C}D^{\alpha,k}_{b-}f(x) = \frac{(-1)^n}{k\Gamma_k(n-\frac{\alpha}{k})}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}}dt, x < b \tag{21} $$

where $\Gamma_k(\alpha)$ is the k-Gamma function defined as

$$ \Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1}e^{\frac{-t^k}{k}}dt, $$

also

$$ \Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha) $$

If $\alpha = n \in \{1,2,3,...\}$ and usual derivative of order n exists, then Caputo k-fractional derivative $({}^{C}D^{\alpha,1}_{a+}f)(x)$ coincides with $f^{(n)}(x)$.

Theorem 8. *Let $s > 1$, $\left({}^{C}D^{\alpha,k}_{a+}f\right)(x)$ denotes the Caputo k-fractional derivative defined by (20) and $\rho_2 : \mathbb{R} \to [0,\infty)$. Then the following inequality holds true:*

$$ \rho_2(s) \leq \sigma_2(s), \tag{22} $$

where

$$ \begin{aligned} \rho_2(s) \quad = \quad & \frac{1}{s(s-1)}\left[\int_a^b (b-y)^{\frac{n-\alpha}{k}}\left(f^{(n)}(y)\right)^s dy \right. \\ & \left. - \int_a^b (x-a)^{\frac{n-\alpha}{k}}\left(\frac{k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{(x-a)^{\left(n-\frac{\alpha}{k}\right)}}\left({}^{C}D^{\alpha,k}_{a+}f\right)(x)\right)^s dx\right] \end{aligned} \tag{23} $$

and

$$ \begin{aligned} \sigma_2(s) \quad = \quad & \frac{(b-a)^{\left(\frac{n-\alpha}{k}\right)(1-s)}}{s(s-1)}\left[(b-a)^{\left(\frac{n-\alpha}{k}\right)s}\int_a^b \left(f^{(n)}(y)\right)^s dy \right. \\ & \left. - \left(k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)\right)^s \int_a^b \left(\left({}^{C}D^{\alpha,k}_{a+}f\right)(x)\right)^s dx\right]. \end{aligned} \tag{24} $$

Proof. Applying Theorem 2 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$\hat{k}(x,y) = \begin{cases} \frac{1}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}(x-y)^{n-\frac{\alpha}{k}-1}, & a \le y \le x; \\ 0, & x < y \le b, \end{cases} \tag{25}$$

$$\hat{K}(x) = \frac{1}{k\left(n-\frac{\alpha}{k}\right)\Gamma_k\left(n-\frac{\alpha}{k}\right)}(x-a)^{n-\frac{\alpha}{k}}, \tag{26}$$

and

$$\hat{A}_k f(x) = \frac{k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{(x-a)^{\left(n-\frac{\alpha}{k}\right)}}\left({}^C D_{a+}^{\alpha,k} f\right)(x) \tag{27}$$

Then the equation (6) take the form

$$\rho_2(s) = \int_a^b \hat{v}(y)\varphi_s\left(f^{(n)}(y)\right)dy - \int_a^b u(x)\varphi_s\left(\frac{k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{(x-a)^{\left(n-\frac{\alpha}{k}\right)}}\left({}^C D_{a+}^{\alpha,k} f\right)(x)\right)dx. \tag{28}$$

For particular weight function $u(x) = (x-a)^{n-\frac{\alpha}{k}}$, $x \in (a,b)$ we get $\hat{v}(y) = (b-y)^{\frac{n-\alpha}{k}}$ and we take $\varphi_s(x) = \frac{x^s}{s(s-1)}$, $x \in \mathbb{R}^+$. So (28) becomes

$$\begin{aligned} \rho_2(s) &= \frac{1}{s(s-1)}\left[\int_a^b (b-y)^{\frac{n-\alpha}{k}}\left(f^{(n)}(y)\right)^s dy \right. \\ &\quad \left. - \int_a^b (x-a)^{\frac{n-\alpha}{k}}\left(\frac{k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{(x-a)^{\left(n-\frac{\alpha}{k}\right)}}\left({}^C D_{a+}^{\alpha,k} f\right)(x)\right)^s dx\right] \\ &\le \frac{(b-a)^{\left(\frac{n-\alpha}{k}\right)(1-s)}}{s(s-1)}\left[(b-a)^{\left(\frac{n-\alpha}{k}\right)s}\int_a^b \left(f^{(n)}(y)\right)^s dy \right. \\ &\quad \left. - \left(k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)\right)^s \int_a^b \left(\left({}^C D_{a+}^{\alpha,k} f\right)(x)\right)^s dx\right] \\ &= \sigma_2(s). \end{aligned}$$

It follows (22) . □

The improved version of Theorem 8 is given in next theorem.

Theorem 9. *Let the assumptions of the Theorem 8 be satisfied and* $\Psi_2 : \mathbb{R} \to [0,\infty)$. *Then the following inequality holds:*

$$0 \le \Psi_2(s) \le \sigma_2(s) - \lambda_2(s) \le \sigma_2(s), \tag{29}$$

where

$$\Psi_2(s) = \frac{1}{s(s-1)}\left[\int_a^b (b-y)^{\frac{n-\alpha}{k}}\left(f^{(n)}(y)\right)^s dy - \int_a^b (x-a)^{\frac{n-\alpha}{k}}\left(\frac{k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{(x-a)^{\left(n-\frac{\alpha}{k}\right)}}\left({}^C D_{a+}^{\alpha,k} f\right)(x)\right)^s dx\right] - \lambda_2(s),$$

$$\lambda_2(s) = \frac{\left(n-\frac{\alpha}{k}\right)}{s(s-1)}\int_a^b\int_a^x (x-y)^{\frac{\nu(n-\mu)}{k}-1}\left|\left|\left(f^{(n)}(y)\right)^s - \frac{k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{(x-a)^{\left(n-\frac{\alpha}{k}\right)}}\left({}^C D_{a+}^{\alpha,k} f\right)(x)^s\right| - s\left|\frac{k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{(x-a)^{\left(n-\frac{\alpha}{k}\right)}}\left({}^C D_{a+}^{\alpha,k} f\right)(x)\right|^{s-1} \cdot \left|f^{(n)}(y) - \frac{k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{(x-a)^{\left(n-\frac{\alpha}{k}\right)}}\left({}^C D_{a+}^{\alpha,k} f\right)(x)\right|\right| dy\,dx, \tag{30}$$

and $\sigma_2(s)$ *is defined by* (24).

Proof. Rewrite equation (16) with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$ with $\hat{k}$, $\hat{K}$ and $\hat{A}_k f(x)$ is defined by (25), (26), (27) respectively. For particular weight function $u(x) = (x-a)^{\left(n-\frac{\alpha}{k}\right)}$, we obtain $v(y) = (b-y)^{\left(n-\frac{\alpha}{k}\right)}$. If we take $\Phi(x) = x^s/s(s-1),\ x \in \mathbb{R}^+$, after some calculations we get

$$\psi_2(s) = \frac{1}{s(s-1)}\left[\int_a^b (b-y)^{\frac{n-\alpha}{k}}\left(f^{(n)}(y)\right)^s dy - \int_a^b (x-a)^{\frac{n-\alpha}{k}}\left(\frac{k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{(x-a)^{\left(n-\frac{\alpha}{k}\right)}}\left({}^C D_{a+}^{\alpha,k} f\right)(x)\right)^s dx\right] - \lambda_2(s).$$

Since $\left(n-\frac{\alpha}{k}\right)(1-s) < 0$ for $s > 1$ and $\lambda_2(s) \geq 0$. Then

$$\Psi_2(s) \leq \rho_2(s) - \lambda_2(s) \leq \rho_2(s). \tag{31}$$

This complete the proof. □

Remark 3. *Choose* $k = 1$ *in Theorem 8 we get the result for Caputo factional derivative, see* [21, Theorem 2.9].

4. **Results Involving Hilfer k-Fractional Derivative**

Let us recall the definition of Hilfer k-fractional derivative presented in [25].

Definition 7. *Let $f \in L^1[a,b], f * K_{(1-\nu)(1-\mu)} \in AC^1[a,b]$. The k-fractional derivative operator ${}^kD^{\mu,\nu}_{a+}$ of order $0<\nu\le 1$ with respect to $x\in[a,b]$ is defined by*

$$\left({}^kD^{\mu,\nu}_{a+}f\right)(x) := I^{\nu(1-\mu)}_{a+,k}\frac{d}{dx}\left(I^{(1-\nu)(1-\mu)}_{a+,k}f(x)\right), \tag{32}$$

whenever the right hand side exists. The derivative (32) is called Hilfer k-fractional derivative.

The more general integral representation of equation (32) is defined by:
Let $f\in L^1[a,b], f*K_{(1-\nu)(n-\mu)}\in AC^n[a,b], n-1<\mu<n, 0<\nu\le 1, n\in \mathbb{N}$. Then

$$\left({}^kD^{\mu,\nu}_{a+}f\right)(x) = \left(I^{\nu(n-\mu)}_{a+,k}\frac{d^n}{dx^n}\left(I^{(1-\nu)(n-\mu)}_{a+,k}f(x)\right)\right), \tag{33}$$

which coincide with (32) for $n=1$.
Specially for $k=1$, $\nu=0$, $D^{\mu,0}_{a+}f = D^{\mu}_{a+}f$ is a Riemann-Liouville fractional derivative of order μ, and for $\nu=1$ it is a Caputo fractional derivative $D^{\mu,1}_{a+}f = {}^CD^{\mu}_{a+}f$ of order μ. Applying the properties of Riemann-Liouville integral the relation (33) can be rewritten in the form:

$$\begin{aligned}\left({}^kD^{\mu,\nu}_{a+}f\right)(x) &= \left(I^{\nu(n-\mu)}_{a+}\left(\left(D^{n-(1-\nu)(n-\mu)}_{a+}f\right)(x)\right)\right)\\ &= \frac{1}{k\Gamma_k\left(\nu\left(n-\mu\right)\right)}\int_a^x (x-y)^{\frac{\nu(n-\mu)}{k}-1}\left(\left(D^{\mu+\nu(n-\mu)}_{a+}f\right)(t)\right)dt.\end{aligned} \tag{34}$$

Now we shall establish the improvement for Hilfer k-fractional derivative.

Theorem 10. *Let $s>1$, $\left({}^kD^{\mu,\nu}_{a+}f\right)(x)$ denotes the Hilfer k-fractional derivative defined by (33) and $\rho_3:\mathbb{R}\to[0,\infty)$. Then the following inequality holds true:*

$$\rho_3(s)\le\sigma_3(s), \tag{35}$$

where

$$\begin{aligned}\rho_3(s) = \frac{1}{s(s-1)}\Bigg[&\int_a^b (b-y)^{\frac{\nu(n-\mu)}{k}} f^s(y)dy\\ &-\int_a^b (x-a)^{\frac{\nu(n-\mu)}{k}}\left(\frac{\Gamma_k\left(\nu\left(n-\mu\right)+k\right)}{(x-a)^{\frac{\nu(n-\mu)}{k}}}\left({}^kD^{\mu,\nu}_{a+}f\right)(x)\right)^s dx\Bigg]\end{aligned} \tag{36}$$

and

$$\sigma_3(s) = \frac{(b-a)^{\frac{\nu(n-\mu)}{k}(1-s)}}{s(s-1)}\left[(b-a)^{\frac{\nu(n-\mu)}{k}s}\int_a^b f^s(y)dy - (\Gamma_k(\nu(n-\mu)+k))^s\int_a^b\left(\left({}^kD_{a+}^{\mu,\nu}f\right)(x)\right)^s dx\right]. \quad (37)$$

Proof. Applying Theorem 2 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx, d\mu_2(y)=dy$,

$$\hat{k}(x,y)=\begin{cases}\frac{1}{k\Gamma_k(\nu(n-\mu))}(x-y)^{\frac{\nu(n-\mu)}{k}-1}, & a\le y\le x;\\ 0, & x<y\le b,\end{cases} \quad (38)$$

$$\hat{K}(x)=\frac{1}{(\nu(n-\mu))\Gamma_k(\nu(n-\mu))}(x-a)^{\frac{\nu(n-\mu)}{k}}. \quad (39)$$

and

$$\hat{A}_kf(x)=\frac{\Gamma_k(\nu(n-\mu)+k)}{(x-a)^{\frac{\nu(n-\mu)}{k}}}\left({}^kD_{a+}^{\mu,\nu}f\right)(x) \quad (40)$$

Then the equation (6) becomes

$$\rho_3(s)=\int_a^b \hat{v}(y)\varphi_s\left(\left(D_{a+}^{\mu+\nu(n-\mu)}f\right)(t)\right)dy - \int_a^b u(x)\varphi_s\left(\frac{\Gamma_k(\nu(n-\mu)+k)}{(x-a)^{\frac{\nu(n-\mu)}{k}}}\left({}^kD_{a+}^{\mu,\nu}f\right)(x)\right)dx. \quad (41)$$

For particular weight function $u(x)=(x-a)^{\frac{\nu(n-\mu)}{k}}$, $x\in(a,b)$ we get $\hat{v}(y)=(b-y)^{\frac{\nu(n-\mu)}{k}}$ and we take $\varphi_s(x)=\frac{x^s}{s(s-1)}$, $x\in\mathbb{R}^+$. So (41) becomes

$$\rho_3(s) = \frac{1}{s(s-1)}\left[\int_a^b (b-y)^{\frac{\nu(n-\mu)}{k}} f^s(y)dy - \int_a^b (x-a)^{\frac{\nu(n-\mu)}{k}}\left(\frac{\Gamma_k(\nu(n-\mu)+k)}{(x-a)^{\frac{\nu(n-\mu)}{k}}}\left({}^kD_{a+}^{\mu,\nu}f\right)(x)\right)^s dx\right]$$

$$
\begin{aligned}
&\leq \frac{(b-a)^{\frac{\nu(n-\mu)}{k}(1-s)}}{s(s-1)}\left[(b-a)^{\frac{\nu(n-\mu)}{k}s}\int_a^b f^s(y)dy\right. \\
&\left.-(\Gamma_k(\nu(n-\mu)+k))^s\int_a^b\left(\left({}^kD_{a+}^{\mu,\nu}f\right)(x)\right)^s dx\right] \\
&= \sigma_4(s).
\end{aligned}
$$

It follows (35) . □

The improvement of Theorem 10 is given in next theorem.

Theorem 11. *Let the assumptions of the Theorem 10 be satisfied and* $\Psi_3 : \mathbb{R}\to[0,\infty)$. *Then the following inequality holds:*

$$
0\leq\Psi_3(s)\leq\sigma_3(s)-\lambda_3(s)\leq\sigma_3(s), \tag{42}
$$

where

$$
\begin{aligned}
\Psi_3(s)=&\frac{1}{s(s-1)}\left[\int_a^b(b-y)^{\frac{\nu(n-\mu)}{k}}f^s(y)dy\right. \\
&\left.-\int_a^b(x-a)^{\frac{\nu(n-\mu)}{k}}\left(\frac{\Gamma_k(\nu(n-\mu)+k)}{(x-a)^{\frac{\nu(n-\mu)}{k}}}\left({}^kD_{a+}^{\mu,\nu}f\right)(x)\right)^s dx\right]-\lambda_3(s),
\end{aligned}\tag{43}
$$

$$
\begin{aligned}
\lambda_3(s)=&\frac{\nu(n-\mu)}{ks(s-1)}\int_a^b\int_a^x(x-y)^{\frac{\nu(n-\mu)}{k}-1}\left|\left|f^s(y)\right.\right. \\
&\left.-\left(\frac{\Gamma_k(\nu(n-\mu)+k)}{(x-a)^{\frac{\nu(n-\mu)}{k}}}\left({}^kD_{a+}^{\mu,\nu}f\right)(x)\right)^s\right| \\
&-s\left|\left(\frac{\Gamma_k(\nu(n-\mu)+k)}{(x-a)^{\frac{\nu(n-\mu)}{k}}}\left({}^kD_{a+}^{\mu,\nu}f\right)(x)\right)\right|^{s-1} \\
&\left.\cdot\left|f(y)-\left(\frac{\Gamma_k(\nu(n-\mu)+k)}{(x-a)^{\frac{\nu(n-\mu)}{k}}}\left({}^kD_{a+}^{\mu,\nu}f\right)(x)\right)\right|\right|dy\,dx,
\end{aligned}\tag{44}
$$

and $\sigma_3(s)$ *is defined by* (37).

Proof. Rewrite equation (16) with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx$, $d\mu_2(y)=dy$ with $\hat{k}$, $\hat{K}$ and $\hat{A}_kf(x)$ is defined by (38), (39), (40) respectively. For particular weight function $u(x) = (x-a)^{\frac{\nu(n-\mu)}{k}}$, we obtain $v(y) = (b-y)^{\frac{\nu(n-\mu)}{k}}$. If we take $\Phi(x) = x^s/s(s-1),\ x \in \mathbb{R}^+$, after some calculations we get

$$\psi_3(s) = \frac{1}{s(s-1)}\left[\int_a^b (b-y)^{\frac{\nu(n-\mu)}{k}} f^s(y)dy \right.$$
$$\left. - \int_a^b (x-a)^{\frac{\nu(n-\mu)}{k}} \left(\frac{\Gamma_k(\nu(n-\mu)+k)}{(x-a)^{\frac{\nu(n-\mu)}{k}}}\left({}^kD^{\mu,\nu}_{a+}f\right)(x)\right)^s dx\right] - \lambda_3(s).$$

Since $\frac{\nu(n-\mu)}{k}(1-s) < 0$ for $s > 1$ and $\lambda_3(s) \geq 0$. Then

$$\Psi_3(s) \leq \sigma_3(s) - \lambda_3(s) \leq \sigma_3(s).$$

This complete the proof. □

Remark 4. *Specially for $k=1$, $\nu=0$, $D^{\mu,0}_{a+}f = D^{\mu}_{a+}f$ in Theorem 10 we get the result for Riemann-Liouville fractional derivative of order μ, and for $\nu=1$ it is a Caputo fractional derivative $D^{\mu,1}_{a+}f = {}^CD^{\mu}_{a+}f$ of order μ.*

5. Results for the Riemann-Liouvile (k,r)-fractional Integral

Definition 8. [26] *Let f be a continuous function on on a the finite real interval $[a,b]$. Then Riemann-Liouville (k,r)-fractional integral of f of order $\alpha > 0$ is defined by:*

$$I^{\alpha,r}_{k,a}f(t) = \frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t (t^{r+1} - \tau^{r+1})^{\frac{\alpha}{k}-1}\tau^r f(\tau)d\tau, \tag{45}$$

where $k>0$, $r \in \mathbb{R}\backslash\{1\}$ and Γ_k is the generalization of the classical Γ function defined by Diaz et al. in [27] and is given as follows:

$$\Gamma_k(t) = \lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{t}{k}-1}}{(t)_{n,k}},\ k>0,\ \Re(t)>0, \tag{46}$$

where $(t)_{n,k} = t(t+k)(t+2k)\ldots,(t+(n-1)k)$, $n \geq 1$, is called Pochhammer k symbol. The integral representation is given by

$$\Gamma_k(t) = \int_0^\infty x^{t-1}e^{\frac{-x^k}{k}}dx,\ \Re(t)>0. \tag{47}$$

Theorem 12. *Let $s>1$, $r \geq 0$, $I^{\alpha,r}_{k,a}$ denotes the Riemann-Liouville (k,r)-fractional integral of order $\alpha > 0$ and $\rho_4 : \mathbb{R} \to [0,\infty)$. Then the following inequality holds true:*

$$\rho_4(s) \leq \sigma_4(s), \tag{48}$$

where

$$\rho_4(s) = \frac{1}{s(s-1)}\left[\int_a^b y^r(b^{r+1}-y^{r+1})^{\frac{\alpha}{k}} f^s(y)dy - \int_a^b x^r(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}\left(\frac{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}} I_{k,a}^{\alpha,r}f(x)\right)^s dx\right], \quad (49)$$

$$\sigma_4(s) = \frac{(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}(1-s)}}{s(s-1)}\left[(b^{r+1}-a^{r+1})^{\frac{\alpha s}{k}}\int_a^b y^r f^s(y)dy - \left(\Gamma_k(\alpha+k)(r+1)^{\frac{\alpha}{k}}\right)^s\int_a^b (I_{k,a}^{\alpha,r}f(x))^s dx\right].$$

Proof. Applying Theorem 2 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx, d\mu_2(y)=dy$,

$$\tilde{k}(x,t) = \begin{cases} \frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\left(x^{r+1}-y^{r+1}\right)^{\frac{\alpha}{k}-1}y^r, & a\le y\le x\,; \\ 0, & x<t\le b\,, \end{cases} \quad (50)$$

$$\tilde{K}(x) = \frac{(r+1)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\left(x^{r+1}-a^{r+1}\right)^{\frac{\alpha}{k}} \quad (51)$$

and

$$\tilde{A}_k f(x) = \frac{\alpha(r+1)}{k(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}\int_a^x \left(x^{r+1}-a^{r+1}\right)^{\frac{\alpha}{k}-1}y^r f(y)dy. \quad (52)$$

For particular weight function $u(x)=x^r(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}$, we obtain $\tilde{v}(y)=y^r(b^{r+1}-y^{r+1})^{\frac{\alpha}{k}}$ and we take $\varphi_s(x)=\frac{x^s}{s(s-1)}$, $x\in\mathbb{R}^+$. So (15) becomes

$$\begin{aligned}
\rho_4(s) &= \frac{1}{s(s-1)}\left[\int_a^b y^r(b^{r+1}-y^{r+1})^{\frac{\alpha}{k}} f^s(y)dy\right.\\
&\quad \left. -\int_a^b x^r(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}\left(\frac{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}} I_{k,a}^{\alpha,r} f(x)\right)^s dx\right]\\
&\le \frac{1}{s(s-1)}\left[\int_a^b y^r(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}} f^s(y)dy\right.\\
&\quad \left. -\left((r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)\right)^s (b^{r+1}-a^{r+1})^{\frac{\alpha}{k}(1-s)}\int_a^b x^r\left(I_{k,a}^{\alpha,r} f(x)\right)^s dx\right]\\
&= \frac{(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}(1-s)}}{s(s-1)}\left[(b^{r+1}-a^{r+1})^{\frac{\alpha s}{k}}\int_a^b y^r f^s(y)dy\right.\\
&\quad \left. -\left(\Gamma_k(\alpha+k)(r+1)^{\frac{\alpha}{k}}\right)^s\int_a^b (I_{k,a}^{\alpha,r} f(x))^s dx\right]\\
&= \sigma_4(s).
\end{aligned}$$

It follows (48). □

Remark 5. *For particular value of* $r=0$*, we get Theorem 6.*

Remark 6. *For particular value of* $k=1$, $r=0$ *we get the improvement of G. H. Hardy's inequality given in* [21, Theorem 2.3].

Theorem 13. *Let the assumption of the Theorem 12 be satisfied and* $\Psi_4 : \Re \to [0,\infty]$. *Then the following inequality holds:*

$$0 \le \Psi_4(s) \le \sigma_4(s) - \lambda_4(s) \le \sigma_4(s),$$

where

$$\begin{aligned}
\Psi_4(s) &= \frac{1}{s(s-1)}\left[\int_a^b \left(b^{r+1}-y^{r+1}\right)^{\frac{\alpha}{k}} y^r f^s(y)dy\right.\\
&\quad \left. -\int_a^b \left(x^{r+1}-a^{r+1}\right)^{\frac{\alpha}{k}} x^r\left(\frac{\alpha\Gamma_k(\alpha)(r+1)^{\frac{\alpha}{k}}}{(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}(I_{a,k}^{\alpha,r}) f(x)\right)^s dx\right] - \lambda_4(s)
\end{aligned} \tag{53}$$

$$\lambda_4(s) = \frac{\alpha(r+1)}{ks(s-1)} \int_a^b x^r \int_a^x \left(x^{r+1} - y^{r+1}\right)^{\frac{\alpha}{k}-1} y^r \, || f^s(y)$$

$$- \left(\frac{\alpha\Gamma_k(\alpha)(r+1)^{\frac{\alpha}{k}}}{(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}} (I^{\alpha,r}_{a,k}) f(x) \right)^s \Bigg| - s \left| \left(\frac{\alpha\Gamma_k(\alpha)(r+1)^{\frac{\alpha}{k}}}{(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}} (I_{a,k}) f(x) \right)^s \right|^{s-1}$$

$$|f(y) - \left(\frac{\alpha\Gamma_k(\alpha)(r+1)^{\frac{\alpha}{k}}}{(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}} (I^{\alpha,r}_{a,k}) f(x) \right) | \Bigg| dydx \tag{54}$$

and $\sigma_4(s)$ is defined in defined by (52).

Proof. Rewrite equation with $\Omega_1 = \Omega_2 = (a,b), d\mu_1(x) = dx, d\mu_2(y) = dy$, $\check{k}(x,t)$, $\check{K}(x)$, $\check{A}_k f(x)$ are given by (50), (51), (52) respectively. For particular weight function $u(x) = x^r(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}$, we obtain $\check{v}(y) = y^r(b^{r+1}-y^{r+1})^{\frac{\alpha}{k}}$. If we take $\varphi(x) = \frac{x^s}{s(s-1)}$, for $x \in \Re^+$, after some calculations we get

$$\begin{aligned}
\Psi_4(s) \quad = \quad & \frac{1}{s(s-1)} \left[\int_a^b \left(b^{r+1} - y^{r+1}\right)^{\frac{\alpha}{k}} y^r f^s(y) dy - \int_a^b \left(x^{r+1} - a^{r+1}\right)^{\frac{\alpha}{k}} \right. \\
& \left. \times x^r \left(\frac{\alpha\Gamma_k(\alpha)(r+1)^{\frac{\alpha}{k}}}{(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}} (I^{\alpha,r}_{a,k}) f(x) \right)^s dx \right] \\
& - \frac{\alpha(r+1)}{ks(s-1)} \int_a^b x^r \int_a^x \left(x^{r+1} - y^{r+1}\right)^{\frac{\alpha}{k}-1} y^r || f^s(y) \\
& - \left(\frac{\alpha\Gamma_k(\alpha)(r+1)^{\frac{\alpha}{k}}}{(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}} (I^{\alpha,r}_{a,k}) f(x) \right)^s | \\
& - s| \left(\frac{\alpha\Gamma_k(\alpha)(r+1)^{\frac{\alpha}{k}}}{(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}} (I_{a,k}) f(x) \right)^s |^{s-1} |f(y) \\
& \quad - \left(\frac{\alpha\Gamma_k(\alpha)(r+1)^{\frac{\alpha}{k}}}{(x^{r+1}-a^{r+1})^{\frac{\alpha}{k}}} (I^{\alpha,r}_{a,k}) f(x) \right) || dydx \geq 0.
\end{aligned} \tag{55}$$

Since $\frac{\alpha(1-s)}{k} \leq 0$ for $s > 1$ and $\lambda_5(s) \geq 0$, we obtain that

$$0 \leq \Psi_4(s) \leq \sigma_5(s) - \lambda_4(s) \leq \sigma_4(s)$$

In the following theorem we prove the three different cases for the above results related to log-convexity.

Theorem 14. *For $i = 1, 2, 3, 4$ the following inequalities hold true:*

$$(i). \qquad [\rho_i(p)]^{\frac{q-r}{q-p}} [\rho_i(q)]^{\frac{r-p}{q-p}} \leq \sigma_i(r) \tag{56}$$

$$(ii). \qquad [\rho_i(r)]^{\frac{p-q}{p-r}}[\rho_i(p)]^{\frac{q-r}{p-r}} \leq \sigma_i(q) \tag{57}$$

$$(iii). \qquad \rho_i(p) \leq [\sigma_i(r)]^{\frac{q-p}{q-r}}[\sigma_i(q)]^{\frac{p-r}{q-r}} \tag{58}$$

for every choice $p, q, r \in \mathbb{R}$, *such that* $1 < r < p < q$.

Proof. We will prove this theorem just in case $i = 1$, since all other case are proved analogous.
(i). Since the function ρ_1 is exponentially convex, it is also log-convex. Then for $1 < r < p < q, \quad r, p, q \in \mathbb{R}$, (7) can be written as

$$[\rho_1(p)]^{q-r}[\rho_1(q)]^{r-p} \leq [\rho_1(r)]^{q-p}.$$

This implies that

$$\begin{aligned}
&[\rho_1(p)]^{\frac{q-r}{q-p}}[\rho_1(q)]^{\frac{r-p}{q-p}} \\
&\leq \frac{(b-a)^{\alpha(1-s)}}{s(s-1)}\left[(b-a)^{\frac{\alpha}{k}s}\int_a^b f^s(y)dy - (\Gamma_k(\alpha+k))^s\int_a^b (I^{\alpha}_{k,a}f(x))^s dx\right] \\
&= \sigma_1(r).
\end{aligned}$$

It follows (56).

(ii). Now (7) can be written as,

$$[\rho_1(r)]^{p-q}[\rho_1(p)]^{q-r} \leq [\rho_1(q)]^{p-r}.$$

This implies that

$$\begin{aligned}
&[\rho_1(r)]^{\frac{p-q}{p-r}}[\rho_1(p)]^{\frac{q-r}{p-r}} \\
&\leq \frac{(b-a)^{\alpha(1-s)}}{s(s-1)}\left[(b-a)^{\frac{\alpha}{k}s}\int_a^b f^q(y)dy - (\Gamma_k(\alpha+k))^s\int_a^b (I^{\alpha}_{k,a}f(x))^q dx\right] \\
&= \sigma_1(q).
\end{aligned}$$

It follows (57).

(iii). The (7) can be written as,

$$[\rho_1(p)]^{\frac{q-r}{p-r}} \leq [\rho_1(r)]^{\frac{q-p}{p-r}}[\rho_1(q)],$$

$$[\rho_1(p)]^{\frac{q-r}{p-r}} \leq [\rho_1(r)]^{\frac{q-p}{p-r}}\sigma_1(q).$$

This implies that

$$\rho_1(p) \leq [\sigma_1(r)]^{\frac{q-p}{q-r}}[\sigma_1(q)]^{\frac{p-r}{q-r}}.$$

It follows (58). □

Likewise we can give the results for Hilfer k-fractional derivative, Caputo k-fractional derivative and Riemann-Liouville (k, r)-fractional integral but we omit the details.

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