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Inequalities of Hardy-type for Multiple Integrals on Time Scales

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Abstract

We extend some inequalities of Hardy-type on time scales for functions depending on more than one parameter. The results are proved by using induction principle, properties of integrals on time scales, chain rules for composition of two functions, Hölder's inequality and Fubini's theorem in time scales settings.

Keywords: Time scales, induction principle, Hardy-type inequalities.
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1. Introduction and Preliminaries

The discrete Hardy inequality was given in 1920 by Hardy [6] to find Hilbert double series sum. He [7] (see also [8]) proved an inequality by using the calculus of variations which is continuous version of his inequality given in [6]. Afterward these inequalities have been studied in more general settings. New proofs, extensions and generalizations of these inequalities have taken place in many papers. For example Chan [4] proved some extensions of Hardy-type inequality in 1979. Pachpatte [10] considered generalizations of inequalities established by chan [4] in 1992. In 2005, P. Rehak proved Hardy's inequality via time scales in [11]. In 2015, S. H. Saker and D. O'Regan [12] extended results of Pachpatte's inequalities [10] in time scales settings. For some other extensions of Hardy inequalities on time scales for function of several variables, the readers are referred to [5, 9, 13, 14].

Let us start to discuss the basic concepts used in the paper from [2, 3]. A time scale $\mathbb{T}$ is nonempty as well as close set in $\mathbb{R}$. So $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}$ are some examples of time scales. In present paper assume that $\sup \mathbb{T} = \infty$ and a time scale interval is denoted by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$ for $t_0 \in \mathbb{T}$.

The operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\sigma(\omega) := \inf \{y \in \mathbb{T}; y > \omega\} \text{ and } \rho(\omega) := \sup \{y \in \mathbb{T}; y < \omega\}$$

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are forward as well as backward jump operators respectively for $\omega \in \mathbb{T}$. The point ω satisfying $\sigma(\omega) > \omega$ is called right-scattered and left-scattered when $\rho(\omega) < \omega$. The points which are left and right-scattered simultaneously are isolated. Also the point ω is right-dense if $\omega < \sup \mathbb{T}$ and $\sigma(\omega) = \omega$ and is left-dense when $\omega > \inf \mathbb{T}$ and $\rho(\omega) = \omega$. The points which are left and right-dense simultaneously are dense points.

Notation: $g^\sigma(\omega) = g(\sigma(\omega))$ for any function $g = \mathbb{T} \to \mathbb{R}$.

A function $\mu : \mathbb{T} \to [0, \infty)$ for a time scale $\mathbb{T}$ is called the graininess function if $\mu(\omega) := \sigma(\omega) - \omega$ for $\omega \in \mathbb{T}$.

Assume $h_1, h_2 : \mathbb{T} \to \mathbb{R}$ are differentiable at $\omega \in \mathbb{T}$. Then

(i) Derivative for product of two functions at $\omega \in \mathbb{T}$ is,

$$(h_1h_2)^\Delta(\omega) = {h_1}^\Delta(\omega)h_2(\omega) + {h_1}^\sigma(\omega){h_2}^\Delta(\omega) = h_1(\omega){h_2}^\Delta(\omega) + {h_1}^\Delta(\omega){h_2}^\sigma(\omega). \tag{1.1}$$

(ii) Quotient rule to find derivative is,

$$\left(\frac{h_1}{h_2}\right)^\Delta(\omega) = \frac{{h_1}^\Delta(\omega)h_2(\omega) - h_1(\omega){h_2}^\Delta(\omega)}{h_2(\omega){h_2}^\sigma(\omega)}.$$

Chain Rules

(i) Let $u : \mathbb{R} \to \mathbb{R}$ be differentiable and $v : \mathbb{T} \to \mathbb{R}$ is delta differentiable, then

$$[u(v(\omega))]^\Delta = \int_0^1 u'[hv^\sigma + (1-h)v]dhv^\Delta(\omega). \tag{1.2}$$

(ii) The fact that $v^\sigma(\omega) = v(\omega) + \mu(\omega)v^\Delta(\omega)$ and $u : \mathbb{T} \to \mathbb{R}$ is delta differentiable, gives

$$[u(v(\omega))]^\Delta = \int_0^1 u'[v + h\mu(\omega)v^\Delta(\omega)]dhv^\Delta(\omega). \tag{1.3}$$

(iii) Assume that $u : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and v is continuous from $\mathbb{R} \to \mathbb{R}$ and $v : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then a real number $c \in [\omega, \sigma(\omega)]$ exists such that

$$[u(v(\omega))]^\Delta = u'(v(c))v^\Delta(\omega). \tag{1.4}$$

If $\xi_1, \xi_2 : \mathbb{T} \to \mathbb{R}$ are delta integrable and $m, \hat{n} \in \mathbb{T}$, then integration by parts formula for delta integrals is

$$\int_m^{\hat{n}} \xi_l(\omega){\xi_2}^\Delta(\omega)\Delta\omega = [\xi_l(\omega)\xi_2(\omega)]_m^{\hat{n}} - \int_m^{\hat{n}} {\xi_l}^\Delta(\omega){\xi_2}^\sigma(\omega)\Delta\omega. \tag{1.5}$$

From chain rule (1.3),

$$(log\omega)^\Delta = \int_0^1 \frac{1}{[\omega + h\mu(\omega)]}dh = \frac{1}{\mu(\omega)}\log\left(\frac{\omega + \mu(\omega)}{\omega}\right),$$

when $\mu(\omega) \neq 0$. Therefore,

$$z(\omega) := (\log\omega)^\Delta = \begin{cases} \frac{1}{\mu(\omega)}\log\left(\frac{\omega+\mu(\omega)}{\omega}\right) & \mu(\omega) \neq 0, \\ \frac{1}{\omega} & \mu(\omega) = 0, \end{cases} \tag{1.6}$$

which gives,

$$\int_{\omega_0}^{\omega} z(s)\Delta s = \log\left(\frac{\omega}{\omega_0}\right), \quad for \quad \omega \in \mathbb{T}.$$

As the generalization of (1.6), we find

$$(\log\theta(\omega))^{\Delta} = Z(\omega) := \begin{cases} \frac{1}{\mu(\omega)}\log\left(1+\mu(\omega)\frac{\theta^{\Delta}(\omega)}{\omega}\right) & \mu(\omega)\neq 0, \\ \frac{\theta^{\Delta}(\omega)}{\Delta(\omega)} & \mu(\omega)=0, \end{cases}$$

provided that $\frac{\theta^{\Delta}(\omega)}{\theta(\omega)}\in\mathbb{R}$. Therefore,

$$\int_{\omega_0}^{\omega} Z(s)\Delta s = \log\left(\frac{\theta(\omega)}{\theta(\omega_0)}\right) \quad for \quad \omega\in\mathbb{T}.$$

Hölder's inequality

Let $\omega_1,\omega_2\in\mathbb{T}$, for rd-continuous functions $\phi,\psi:[\omega_1,\omega_2]\to\mathbb{R}$, we have

$$\int\limits_{\omega_1}^{\omega_2}|\phi(\omega)\,\psi(\omega)|\Delta\omega \le \left[\int\limits_{\omega_1}^{\omega_2}|\phi(\omega)|^{p_2}\Delta\omega\right]^{\frac{1}{p_2}}\left[\int\limits_{\omega_1}^{\omega_2}|\psi(\omega)|^{p_1}\Delta\omega\right]^{\frac{1}{p_1}}, \tag{1.7}$$

where $p_1>1$ and $p_2=p_1/(p_1-1)$.

Fubini's Theorem [1]

Let $(\Gamma,\mathcal{M},\alpha_{\Delta})$ and $(\Sigma,\mathcal{L},\beta_{\Delta})$ be two finite dimensional time scale measure spaces.

We consider the measure space $(\Gamma\times\Sigma,\mathcal{M}\times\mathcal{L},\alpha_{\Delta}\times\beta_{\Delta})$, where $\mathcal{M}\times\mathcal{L}$ is σ-algebra product generated by the family $\{E\times F: E\in\mathcal{M}, F\in\mathcal{L}\}$ and

$$(\alpha_{\Delta}\times\beta_{\Delta})(E\times F)=\alpha_{\Delta}(E)\beta_{\Delta}(F),$$

then Fubini's theorem holds, more precisely if $\phi(\pi)=\int_{\Gamma}\xi(\gamma,\pi)\Delta\gamma$ and $\psi(\gamma)=\int_{\Sigma}\xi(\gamma,\pi)\Delta\pi$ for almost everywhere $\pi\in\Sigma$ and $\gamma\in\Gamma$ respectively, also if $\xi:\Gamma\times\Sigma\to\mathbb{R}$ is a $\alpha_{\Delta}\times\beta_{\Delta}$- integrable function, then ϕ is β_{Δ}- integrable on Σ and ψ is α_{Δ}- integrable on Γ, and

$$\int\limits_{\Gamma}\Delta\gamma\int\limits_{\Sigma}\xi(\gamma,\pi)\Delta\pi = \int\limits_{\Sigma}\Delta\pi\int\limits_{\Gamma}\xi(\gamma,\pi)\Delta\gamma. \tag{1.8}$$

Aim of the paper is to extend the work done in [12] for functions of several variables.

2. Main results

We assume throughout that all the functions are non-negative and the integrals considered exist.

Theorem 2.1. *Let $\mathbb{T}_l$ denote time scales and $h_l(t_l)$ be non-decreasing functions for $t_l\in\mathbb{T}_l$, where $l=1,\ldots,\hat{n}$. Define $\forall(t_1,\ldots,t_{\hat{n}})\in\mathbb{T}_1\times\cdots\times\mathbb{T}_{\hat{n}}$,*

$$\Lambda_{\hat{n}}(t_1,\ldots,t_{\hat{n}}) := \prod_{l=1}^{\hat{n}}\frac{1}{h_l(t_l)}\int_{t_l}^{\infty}\cdots\int_{t_{\hat{n}}}^{\infty}\prod_{l=1}^{\hat{n}}\frac{h_l(s_l)}{s_l}f(s_1,\ldots,s_{\hat{n}})\Delta s_{\hat{n}}\cdots\Delta s_1. \tag{2.1}$$

Suppose $p_1/p_2>1, p_1,p_2>0,$ and for $t_l\in[1,\infty)_{\mathbb{T}_l}$

$$1-\frac{p_1}{p_2}\frac{\log\sigma_l(t_l)g_l^{\Delta_l}(t_l)}{h_l^{\sigma_l}(t_l)z_l(t_l)} \ge M_l,\ \forall M_l>0. \tag{2.2}$$

Then the following inequality holds

$$\int_1^\infty \cdots \int_1^\infty \prod_{l=1}^{\hat{n}} z_l(t_l)[\Lambda_{\hat{n}}(t_1,\ldots,t_{\hat{n}})]^{\frac{p_1}{p_2}} \Delta t_{\hat{n}} \cdots \Delta t_1$$
$$\le \left(\frac{p_1}{p_2}\right)^{\frac{\hat{n}p_1}{p_2}} \left(\prod_{l=1}^{\hat{n}} \frac{1}{M_l}\right)^{\frac{p_1}{p_2}} \int_1^\infty \cdots \int_1^\infty \left\{\prod_{l=1}^{\hat{n}} z_l(t_l)\left(\frac{\log \sigma_l(t_l) g_l(t_l)}{t_l g_l^{\sigma_l}(t_l) z_l(t_l)}\right)^{\frac{p_1}{p_2}}\right\} f^{\frac{p_1}{p_2}}(t_1,\ldots,t_{\hat{n}}) \Delta t_{\hat{n}} \cdots \Delta t_1. \quad (2.3)$$

Proof. We use mathematical induction. For $\hat{n} = 1$, the statement is true by [12, Theorem 2.1]. Consider that (2.3) holds for $1 \le \hat{n} \le \hat{k}$.

To prove the result for $\hat{n} = \hat{k} + 1$, we write the left hand side of (2.3) as follows:

$$\int_1^\infty \cdots \int_1^\infty \prod_{l=1}^{\hat{k}} z_l(t_l)\left\{\int_1^\infty z_{\hat{k}+1}(t_{\hat{k}+1})\Big[\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})\Big]^{\frac{p_1}{p_2}} \Delta t_{\hat{k}+1}\right\}\Delta t_{\hat{k}} \cdots \Delta t_1. \quad (2.4)$$

Denote,

$$I_{\hat{k}+1} = \int_1^\infty z_{\hat{k}+1}(t_{\hat{k}+1})\Big[\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})\Big]^{\frac{p_1}{p_2}} \Delta t_{\hat{k}+1}.$$

By using (1.5) with $\xi_1 = \Big[\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})\Big]^{\frac{p_1}{p_2}}$ and ${\xi_2}^\Delta = z_{\hat{k}+1}(t_{\hat{k}+1})$, we have

$$I_{\hat{k}+1} = \left|\log(t_{\hat{k}+1})\Big[\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})\Big]^{\frac{p_1}{p_2}}\right|_1^\infty$$
$$+ \int_1^\infty \log \sigma_{\hat{k}+1}(t_{\hat{k}+1})\left(-\frac{\partial}{\partial t_{\hat{k}+1}}\Big[\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})\Big]^{\frac{p_1}{p_2}}\right)\Delta t_{\hat{k}+1}. \quad (2.5)$$

Use chain rule (1.3) with $u = t^{\frac{p_1}{p_2}}$ and $v = \Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})$ to obtain

$$-\frac{\partial}{\partial t_{\hat{k}+1}}\Big[\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})\Big]^{\frac{p_1}{p_2}} = \frac{p_1}{p_2}\int_0^1 \left[\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1}) + h\mu(t_{\hat{k}+1})\frac{\partial}{\partial t_{\hat{k}+1}}\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}-1} dh$$
$$\times \left(-\frac{\partial}{\partial t_{\hat{k}+1}}\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})\right) \ge 0. \quad (2.6)$$

Use (2.1) for $\hat{n} = \hat{k}+1$ and differentiate with respect to $t_{\hat{k}+1}$ with the help of (1.1) to get

$$\frac{\partial}{\partial t_{\hat{k}+1}}\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1}) = -\left\{\frac{h_{\hat{k}+1}(t_{\hat{k}+1})}{t_{\hat{k}+1} g_{\hat{k}+1}^{\sigma_{\hat{k}+1}}(t_{\hat{k}+1})}\Lambda_{\hat{k}}(t_1,\ldots,t_{\hat{k}+1}) + \frac{h_{\hat{k}+1}^{\Delta_{\hat{k}+1}}(t_{\hat{k}+1})}{h_{\hat{k}+1}^{\sigma_{\hat{k}+1}}(t_{\hat{k}+1})}\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})\right\} \le 0, \quad (2.7)$$

where for fix $t_{\hat{k}+1} \in T_{\hat{k}+1}$,

$$\Lambda_{\hat{k}}(t_1,\ldots,t_{\hat{k}+1}) := \prod_{l=1}^{\hat{k}} \frac{1}{g_l(t_l)} \int_{t_l}^\infty \cdots \int_{t_{\hat{k}}}^\infty \prod_{l=1}^{\hat{k}} \frac{g_l(s_l)}{s_l} f(s_1,\ldots,s_{\hat{k}},t_{\hat{k}+1})\Delta s_{\hat{k}} \cdots \Delta s_1. \quad (2.8)$$

Combine (2.6) and (2.7), then substitute in (2.5) to obtain

$$I_{\hat{k}+1} \leq \frac{p_1}{p_2}\int_1^{\infty} \log \sigma_{\hat{k}+1}(t_{\hat{k}+1})\left\{\frac{h_{\hat{k}+1}(t_{\hat{k}+1})}{t_{\hat{k}+1}g_{\hat{k}+1}^{\sigma_{\hat{k}+1}}(t_{\hat{k}+1})}\Lambda_{\hat{k}}(t_1,\ldots,t_{\hat{k}+1}) + \frac{h_{\hat{k}+1}^{\Delta_{\hat{k}+1}}(t_{\hat{k}+1})}{h_{\hat{k}+1}^{\sigma_{\hat{k}+1}}(t_{\hat{k}+1})}\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})\right\}$$
$$\times \int_0^1 \left[\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1}) + h\mu(t_{\hat{k}+1})\frac{\partial}{\partial t_{\hat{k}+1}}\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}-1} dh\Delta t_{\hat{k}+1}.$$

Since $\frac{\partial}{\partial t_{\hat{k}+1}}\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1}) \leq 0$, therefore

$$I_{\hat{k}+1}\left\{1 - \frac{p_1}{p_2}\frac{\log \sigma_{\hat{k}+1}(t_{\hat{k}+1})g_{\hat{k}+1}^{\Delta_{\hat{k}+1}}(t_{\hat{k}+1})}{z_{\hat{k}+1}(t_{\hat{k}+1})g_{\hat{k}+1}^{\sigma_{\hat{k}+1}}(t_{\hat{k}+1})}\right\} \leq \frac{p_1}{p_2}\int_1^{\infty} \frac{log\sigma_{\hat{k}+1}(t_{\hat{k}+1})g_{\hat{k}+1}(t_{\hat{k}+1})\Lambda_{\hat{k}}(t_1,\ldots,t_{\hat{k}+1})}{t_{\hat{k}+1}g_{\hat{k}+1}^{\sigma_{\hat{k}+1}}(t_{\hat{k}+1})\left[z_{\hat{k}+1}(t_{\hat{k}+1})\right]^{\frac{p_1-p_2}{p_1}}}$$
$$\times \left[(z_{\hat{k}+1}(t_{\hat{k}+1}))^{\frac{p_1-p_2}{p_1}}[\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})]^{\frac{p_1-p_2}{p_2}}\right]\Delta t_{\hat{k}+1}.$$

Apply Hölder's inequality on right hand side with indices p_1/p_2 and $p_1/(p_1-p_2)$ and use (2.2) for $l = \hat{k}+1$ to get

$$I_{\hat{k}+1} \leq \left(\frac{p_1}{M_{\hat{k}+1}p_2}\right)^{\frac{p_1}{p_2}} \int_1^{\infty} z_{\hat{k}+1}(t_{\hat{k}+1})\left(\frac{log\sigma_{\hat{k}+1}(t_{\hat{k}+1})g_{\hat{k}+1}(t_{\hat{k}+1})\Lambda_{\hat{k}}(t_1,\ldots,t_{\hat{k}+1})}{t_{\hat{k}+1}g_{\hat{k}+1}^{\sigma_{\hat{k}+1}}(t_{\hat{k}+1})z_{\hat{k}+1}(t_{\hat{k}+1})}\right)^{\frac{p_1}{p_2}}\Delta t_{\hat{k}+1}. \tag{2.9}$$

Put (2.9) in (2.4) and exchange integrals k times by using Fubini's theorem on right hand side to find

$$\int_1^{\infty}\cdots\int_1^{\infty}\prod_{l=1}^{\hat{k}+1} z_l(t_l)\left[\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}}\Delta t_{\hat{k}+1}\cdots\Delta t_1$$
$$\leq \left(\frac{p_1}{M_{\hat{k}+1}p_2}\right)^{\frac{p_1}{p_2}}\int_1^{\infty} z_{\hat{k}+1}(t_{\hat{k}+1})\left(\frac{log\sigma_{\hat{k}+1}(t_{\hat{k}+1})g_{\hat{k}+1}(t_{\hat{k}+1})}{t_{\hat{k}+1}h_{\hat{k}+1}{}^{\sigma_{\hat{k}+1}}(t_{\hat{k}+1})z_{\hat{k}+1}(t_{\hat{k}+1})}\right)^{\frac{p_1}{p_2}}$$
$$\times\left(\int_1^{\infty}\cdots\int_1^{\infty}\prod_{l=1}^{\hat{k}} z_l(t_l)[\Lambda_{\hat{k}}(t_1,\ldots,t_{\hat{k}+1})]^{\frac{p_1}{p_2}}\Delta t_{\hat{k}}\cdots\Delta t_l\right)\Delta t_{\hat{k}+1}. \tag{2.10}$$

By using induction hypothesis for $\Lambda_{\hat{k}}(t_1,\ldots,t_{\hat{k}+1})$ (instead of $\Lambda_{\hat{k}}(t_1,\ldots,t_{\hat{k}})$) for fix $t_{\hat{k}+1}$ in (2.10), we have

$$\int_1^{\infty}\cdots\int_1^{\infty}\prod_{l=1}^{\hat{k}+1} z_l(t_l)\left[\Lambda_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}}\Delta t_{\hat{k}+1}\cdots\Delta t_1 \leq \left(\frac{p_1}{p_2}\right)^{\frac{(\hat{k}+1)p_1}{p_2}}\left(\prod_{l=1}^{\hat{k}+1}\frac{1}{M_l}\right)^{\frac{p_1}{p_2}}$$
$$\times\int_1^{\infty}\cdots\int_1^{\infty}\left\{\prod_{l=1}^{\hat{k}+1} z_l(t_l)\left(\frac{\log\sigma_l(t_l)g_l(t_l)}{t_l g_l^{\sigma_l}(t_l)z_l(t_l)}\right)^{\frac{p_1}{p_2}}\right\} f^{\frac{p_1}{p_2}}(t_1,\ldots,t_{\hat{k}+1})\Delta t_{\hat{k}+1}\cdots\Delta t_1.$$

□

Corollary 2.2. *Let $\mathbb{R}_l^+$ be time scales, $h_l(r_l)$ be non-negative, non-decreasing functions on $\mathbb{R}_l^+$ where $l = 1,\ldots,\hat{n}$. Define for any $(r_1,\ldots,r_{\hat{n}}) \in \mathbb{R}_1^+\times\cdots\times\mathbb{R}_{\hat{n}}^+$,*

$$\Lambda_{\hat{n}}(r_1,\ldots,r_{\hat{n}}) := \prod_{l=1}^{\hat{n}}\frac{1}{h_l(r_l)}\int_{r_1}^{\infty}\cdots\int_{r_{\hat{n}}}^{\infty}\prod_{l=1}^{\hat{n}}\frac{h_l(s_l)}{s_l}f(s_1,\ldots,s_{\hat{n}})ds_{\hat{n}}\cdots ds_1.$$

and suppose $p_1, p_2 > 0$ such that $p_1/p_2 > 1$ and for $r_l \in [1,\infty)_{\mathbb{R}_l^+}$,

$$1 - \frac{p_1}{p_2}\frac{r_l {h_l}'(r_l)}{h_l}\log(r_l) \geq M_l; \qquad r_l \in [1,\infty)_{\mathbb{R}_l^+},\ \forall M_l > 0.$$

Then the following inequality holds

$$\int_1^\infty \cdots \int_1^\infty \frac{1}{r_1\cdots r_{\hat{n}}}[\Lambda_{\hat{n}}(r_1,\dots,r_{\hat{n}})]^{\frac{p_1}{p_2}} dr_{\hat{n}}\cdots dr_1$$
$$\leq \left(\frac{p_1}{p_2}\right)^{\frac{\hat{n}p_1}{p_2}} \left(\prod_{l=1}^{\hat{n}} \frac{1}{M_l}\right)^{\frac{p_1}{p_2}} \int_1^\infty \cdots \int_1^\infty \left\{\prod_{l=1}^{\hat{n}} \frac{1}{r_l}(\log r_l)^{\frac{p_1}{p_2}}\right\} f^{\frac{p_1}{p_2}}(r_1,\dots,r_{\hat{n}}) dr_{\hat{n}}\cdots dr_1.$$

Proof. Use $\mathbb{T} = \mathbb{R}_l^+\ \forall\ l \in (1,\cdots,\hat{n})$ in Theorem 2.1 to get the above result. □

Corollary 2.3. *For $l \in (1,\cdots,\hat{n})$, $q_l > 1$ if we fix $\mathbb{T}_l = q_l^{\mathbb{N}_0}$ in Theorem 2.1 and for $(q_1^{m_1},\dots,q_{\hat{n}}^{m_{\hat{n}}}) \in q_1^{\mathbb{N}_0}\times\cdots\times q_{\hat{n}}^{\mathbb{N}_0}$, define*

$$\Lambda_{\hat{n}}(q_1^{m_1},\dots,q_{\hat{n}}^{m_{\hat{n}}}) = \prod_{l=1}^{\hat{n}} \frac{1}{g_l(q_l^{m_l})} \sum_{\hat{k}_1=m_1}^\infty \cdots \sum_{\hat{k}_{\hat{n}}=m_{\hat{n}}}^\infty \prod_{l=1}^{\hat{n}} g_l(q_l^{\hat{k}_l}) f(q_l^{\hat{k}_1},\dots,q_l^{\hat{k}_{\hat{n}}})(q_l-1).$$

Also for $p_1, p_2 > 0$ such that $p_1/p_2 > 1$. suppose that,

$$1 - \frac{p_1}{p_2}\frac{\log(q_l^{m_l+1})\Delta_l g_l(q_l^{m_l})}{g_l(q_l^{m_l+1}) z_l(q_l^{m_l})} \geq M_l,\ \forall M_l > 0$$

Then,

$$\sum_{m_1=1}^\infty \cdots \sum_{m_{\hat{n}}=1}^\infty \left\{\prod_{l=1}^{\hat{n}} q_l^{m_l}(q_l-1)\log\left(\frac{q_l^{m_l}+1}{q_l^{m_l}}\right)\right\}[\Lambda_{\hat{n}}(q_1^{m_1},\dots,q_{\hat{n}}^{m_{\hat{n}}})]^{\frac{p_1}{p_2}}$$
$$\leq \left(\frac{p_1}{p_2}\right)^{\frac{\hat{n}p_1}{p_2}} \left(\prod_{l=1}^{\hat{n}} \frac{1}{M_l}\right)^{\frac{p_1}{p_2}} \sum_{m_1=1}^\infty \cdots \sum_{m_{\hat{n}}=1}^\infty \left\{\prod_{l=1}^{\hat{n}} q_l^{m_l}(q_l-1)\log\left(\frac{q_l^{m_l}+1}{q_l^{m_l}}\right)\right\}$$
$$\times \left(\frac{\log(q_l^{m_l+1}) g_l(q_l^{m_l})}{q_l^{m_l} g_l(q_l^{m_l+1})\log\left(\frac{q_l^{m_l}+1}{q_l^{m_l}}\right)}\right)^{\frac{p_1}{p_2}} f^{\frac{p_1}{p_2}}(q_1^{m_1},\dots,q_{\hat{n}}^{m_{\hat{n}}}).$$

Proof. Use $\mathbb{T}_l = q_l^{N_0}\ q_l > 1,\ l \in (1,\cdots,\hat{n})$ in Theorem 2.1 to get the result. □

Corollary 2.4. *Let $p_1, p_2 > 0$ be such that $p_1/p_2 > 1$ and for $l \in (1,\cdots,\hat{n})$ and $\alpha_l > 0$, $g_l(\alpha_l m_l)$ are non-negative, non-decreasing sequences, where $m_l \in N$. Then for $(\alpha_1 m_1,\dots,\alpha_{\hat{n}} m_{\hat{n}}) \in \alpha_1\mathbb{N}\times\cdots\times\alpha_{\hat{n}}\mathbb{N}$, define*

$$\Lambda_{\hat{n}}(\alpha_1 m_1,\dots,\alpha_{\hat{n}} m_{\hat{n}}) = \prod_{l=1}^{\hat{n}} \frac{1}{g_l(\alpha_l m_l)} \sum_{\hat{k}_1=m_1}^\infty \cdots \sum_{\hat{k}_{\hat{n}}=m_{\hat{n}}}^\infty \prod_{l=1}^{\hat{n}} \frac{g_l(\alpha_l \hat{k}_l)}{\hat{k}_l} f(\alpha_1\hat{k}_1,\dots,\alpha_{\hat{n}}\hat{k}_{\hat{n}}).$$

Also suppose that,

$$1 - \frac{p_1}{p_2}\frac{\log(\alpha_l m_l + 1)\Delta_l g_l(\alpha_l m_l)}{g_l(\alpha_l m_l + 1) z_l(\alpha_l m_l)} \geq M_l,$$

for some constant $M_l > 0$, where $l = 1, \ldots, \hat{n}$. Then,

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_{\hat{n}}=1}^{\infty} \left\{ \prod_{l=1}^{\hat{n}} \alpha_l \log\left(\frac{\alpha_l m_l + 1}{\alpha_l m_l}\right) \right\} [\Lambda_{\hat{n}}(\alpha_1 m_1, \ldots, \alpha_{\hat{n}} m_{\hat{n}})]^{\frac{p_1}{p_2}}$$
$$\leq \left(\frac{p_1}{p_2}\right)^{\frac{\hat{n}p_1}{p_2}} \left(\prod_{l=1}^{\hat{n}} \frac{1}{M_l}\right)^{\frac{p_1}{p_2}} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{\hat{n}}=1}^{\infty} \left\{ \prod_{l=1}^{\hat{n}} \alpha_l \log\left(\frac{\alpha_l m_l + 1}{\alpha_l m_l}\right) \right\}$$
$$\times \left(\frac{\log(\alpha_l m_l + 1) g_l(\alpha_l m_l)}{\alpha_l m_l g_l(\alpha_l m_l + 1) \log\left(\frac{\alpha_l m_l + 1}{\alpha_l m_l}\right)} \right)^{\frac{p_1}{p_2}} f^{\frac{p_1}{p_2}}(\alpha_1 m_1, \ldots, \alpha_{\hat{n}} m_{\hat{n}}).$$

Proof. Use $\mathbb{T}_l = \alpha_l \mathbb{N}\ \alpha_l > 0,\ l \in (1, \cdots, \hat{n})$ in Theorem 2.1 to get the result. □

Corollary 2.5. *Let $p_1, p_2 > 0$ be such that $p_1/p_2 > 1$. Assume that $h_l(t_l)$ are non-decreasing functions on time scales $\mathbb{T}_l$ and $\theta_l(t_l)$ are non-negative functions such that $\frac{\theta_l{}^{\Delta_l}(t_l)}{\theta_l(t_l)} \in \mathbb{R}_l$ where $l = 1, \ldots, \hat{n}$. $\Lambda_{\hat{n}}(t_1, \ldots, t_{\hat{n}})$ is defined in (2.1) and assume for $t_l \in [1, \infty)_{\mathbb{T}_l}$,*

$$1 - \frac{p_1}{p_2} \frac{t_l h_l^{\Delta_l}(t_l) \log \theta_l^{\sigma_l}(t_l)}{h_l^{\sigma_l}(t_l) Z_l(t_l)} \geq M_l,\ \forall M_l > 0.$$

Then

$$\int_1^{\infty} \cdots \int_1^{\infty} \prod_{l=1}^{\hat{n}} Z_l(t_l) [\Lambda_{\hat{n}}(t_1, \ldots, t_{\hat{n}})]^{\frac{p_1}{p_2}} \Delta t_{\hat{n}} \cdots \Delta t_1$$
$$\leq \left(\frac{p_1}{p_2}\right)^{\frac{\hat{n}p_1}{p_2}} \left(\prod_{l=1}^{\hat{n}} \frac{1}{M_l}\right)^{\frac{p_1}{p_2}} \int_1^{\infty} \cdots \int_1^{\infty} \prod_{l=1}^{\hat{n}} Z_l(t_l) \left(\frac{h_l(t_l) \log \theta_l^{\sigma_l}(t_l)}{t_l h_l^{\sigma_l}(t_l) Z_l(t_l)} \right)^{\frac{p_1}{p_2}} f^{\frac{p_1}{p_2}}(t_1, \ldots, t_{\hat{n}}) \Delta t_{\hat{n}} \cdots \Delta t_1.$$

Proof. To obtain the result, use the functions $Z_l(t_l)$ instead of $z_l(t_l)$ in Theorem 2.1 and proceed similarly. □

To prove some further results, apply chain rule (1.2) and the inequality.

$$a_1{}^{\delta} + a_2{}^{\delta} \leq (a_1 + a_2)^{\delta} \leq 2^{\delta - 1}(a_1{}^{\delta} + a_2{}^{\delta}); \quad \text{if} \quad a_1, a_2 \geq 0 \quad \text{and} \quad \delta \geq 1. \tag{2.11}$$

Theorem 2.6. *Let $\mathbb{T}_l$ be time scales and $h_l(t_l)$ are non-decreasing functions on $\mathbb{T}_l$ where $l = 1, \ldots, \hat{n}$. $\Lambda_{\hat{n}}(t_1, \ldots, t_{\hat{n}})$ is defined in (2.1). Suppose $p_1, p_2 > 0$ such that $p_1/p_2 \geq 2$, also for $t_l \in [1, \infty)_{\mathbb{T}_l}$,*

$$1 - \frac{2^{\frac{p_1}{p_2} - 1} \log \sigma_l(t_l) h_l^{\Delta_l}(t_l)}{h_l^{\sigma_l}(t_l) z_l(t_l)} \geq M_l,\ \forall M_l > 0$$

Then the following inequality holds

$$\int_1^{\infty} \cdots \int_1^{\infty} \prod_{l=1}^{\hat{n}} z_l(t_l) [\Lambda_{\hat{n}}(t_1, \ldots, t_{\hat{n}})]^{\frac{p_1}{p_2}} \Delta t_{\hat{n}} \cdots \Delta t_1 \leq 2^{\frac{\hat{n}p_1(p_1 - p_2)}{p_2{}^2}} \left(\prod_{l=1}^{\hat{n}} \frac{1}{M_l}\right)^{\frac{p_1}{p_2}}$$
$$\int_1^{\infty} \cdots \int_1^{\infty} \left\{ \prod_{l=1}^{\hat{n}} z_l(t_l) \left(\frac{\log \sigma_l(t_l) h_l(t_l)}{t_l h_l^{\sigma_l}(t_l) z_l(t_l)} \right)^{\frac{p_1}{p_2}} \right\} f^{\frac{p_1}{p_2}}(t_1, \ldots, t_{\hat{n}}) \Delta t_{\hat{n}} \cdots \Delta t_1.$$

Proof. Use mathematical induction method and proceed as in the proof of Theorem 2.1 for $\hat{n} = \hat{k}+1$ to get,

$$I_{\hat{k}+1} = \left|\log(t_{\hat{k}+1})\left[\Lambda_{\hat{k}+1}(t_1,\dots,t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}}\right|_1^{\infty} + \int_1^{\infty}\log\sigma_{\hat{k}+1}(t_{\hat{k}+1})\left(-\frac{\partial}{\partial t_{\hat{k}+1}}\left[\Lambda_{\hat{k}+1}(t_1,\dots,t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}}\right)\Delta t_{\hat{k}+1}. \tag{2.12}$$

Use chain rule formula (1.2) with $u = t^{\frac{p_1}{p_2}}$ and $v = \Lambda_{\hat{k}+1}(t_1,\dots,t_{\hat{k}+1})$,

$$-\frac{\partial}{\partial t_{\hat{k}+1}}\left[\Lambda_{\hat{k}+1}(t_1,\dots,t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}} = \frac{p_1}{p_2}\int_0^1\left[h\Lambda_{\hat{k}+1}^{\sigma_{\hat{k}+1}}(t_1,\dots,t_{\hat{k}+1}) + (1-h)\Lambda_{\hat{k}+1}(t_1,\dots,t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}-1}dh \times\left(-\frac{\partial}{\partial t_{\hat{k}+1}}\Lambda_{\hat{k}+1}(t_1,\dots,t_{\hat{k}+1})\right) \geq 0.$$

Apply (2.11),

$$-\frac{\partial}{\partial t_{\hat{k}+1}}\left[\Lambda_{\hat{k}+1}(t_1,\dots,t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}} \leq 2^{\frac{p_1}{p_2}-2}\left(-\frac{\partial}{\partial t_{\hat{k}+1}}\Lambda_{\hat{k}+1}(t_1,\dots,t_{\hat{k}+1})\right)\left\{\left[\Lambda_{\hat{k}+1}^{\sigma_{\hat{k}+1}}(t_1,\dots,t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}-1} + \left[\Lambda_{\hat{k}+1}(t_1,\dots,t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}-1}\right\}. \tag{2.13}$$

Also from (2.7) of Theorem 2.1,

$$\frac{\partial}{\partial t_{\hat{k}+1}}\Lambda_{\hat{k}+1}(t_1,\dots,t_{\hat{k}+1}) = -\left\{\frac{h_{\hat{k}+1}(t_{\hat{k}+1})}{t_{\hat{k}+1}h_{\hat{k}+1}^{\sigma_{\hat{k}+1}}(t_{\hat{k}+1})}\Lambda_{\hat{k}}(t_1,\dots,t_{\hat{k}+1}) + \frac{h_{\hat{k}+1}^{\Delta_{\hat{k}+1}}(t_{\hat{k}+1})}{h_{\hat{k}+1}^{\sigma_{\hat{k}+1}}(t_{\hat{k}+1})}\Lambda_{\hat{k}+1}(t_1,\dots,t_{\hat{k}+1})\right\}, \tag{2.14}$$

where $\Lambda_{\hat{k}}(t_1,\dots,t_{\hat{k}+1})$ is defined in (2.8). Since $\frac{\partial}{\partial t_2}\Lambda(t_l,t_2) \leq 0$ and $\sigma_2(t_2) \geq t_2$, therefore (2.13) can be written as,

$$-\frac{\partial}{\partial t_{\hat{k}+1}}\left[\Lambda_{\hat{k}+1}(t_1,\dots,t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}} \leq 2^{\frac{p_1}{p_2}-1}\left(-\frac{\partial}{\partial t_{\hat{k}+1}}\Lambda_{\hat{k}+1}(t_1,\dots,t_{\hat{k}+1})\right)\left[\Lambda_{\hat{k}+1}(t_1,\dots,t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}-1}. \tag{2.15}$$

Combine (2.14) and (2.15) in (2.12) and proceed similar as Theorem 2.1 to reach the proof. □

Consider the case when $\frac{p_1}{p_2} \leq 2$. Also

$$2^{\epsilon-1}({a_1}^{\epsilon} + {a_2}^{\epsilon}) \leq (a_1+a_2)^{\epsilon} \leq ({a_1}^{\epsilon} + {a_2}^{\epsilon}), \tag{2.16}$$

where $a_1, a_2 \geq 0$ and $0 \leq \epsilon \leq 1$.

Theorem 2.7. *Let $p_1, p_2 > 0$ such that $\frac{p_1}{p_2} \leq 2$. Assume that $h_l(t_l)$ are non-decreasing functions on time scales $\mathbb{T}_l$ where $l = 1,\dots,\hat{n}$. Let $\Lambda_{\hat{n}}(t_1,\dots,t_{\hat{n}})$ and $\Lambda_{\hat{k}}(t_1,\dots,t_{\hat{n}})$ as defined in (2.1) and (2.8) respectively and assume that for each $t_l \in [1,\infty)_{\mathbb{T}_l}$*

$$1 - \frac{2\log\sigma_l(t_l){h_l}^{\Delta_l}(t_l)}{{h_l}^{\sigma_l}(t_l)z_l(t_l)} \geq M_l$$

hold for some constants $M_l > 0$. Then

$$\int\limits_1^\infty \cdots \int\limits_1^\infty \prod_{l=1}^{\hat{n}} z_l(t_l)[\Lambda_{\hat{n}}(t_l,\ldots,t_{\hat{n}})]^{\frac{p_1}{p_2}} \Delta t_{\hat{n}} \cdots \Delta t_l$$
$$\leq \left(\prod_{l=1}^{\hat{n}} \frac{2}{M_l}\right)^{\frac{p_1}{p_2}} \int\limits_1^\infty \cdots \int\limits_1^\infty \prod_{l=1}^{\hat{n}} z_l(t_l)\left(\frac{\log \sigma_l(t_l) h_l(t_l)}{t_l {h_l}^{\sigma_l}(t_l) z_l(t_l)}\right)^{\frac{p_1}{p_2}} f^{\frac{p_1}{p_2}}(t_1,\ldots,t_{\hat{n}})\Delta t_{\hat{n}} \cdots \Delta t_l.$$

Proof. Apply (2.16) when $\frac{p_1}{p_2} - 1 < 1$ instead of inequality (2.11), to see

$$\begin{aligned}\left(\frac{p_1}{p_2}\right)\int_0^1 [h\Lambda^\sigma + (1-h)\Lambda]^{(p_1/p_2)-1} dh &\leq (\Lambda^\sigma)^{(p_1/p_2)-1} + (\Lambda)^{(p_1/p_2)-1},\\ &\leq 2(\Lambda)^{(p_1/p_2)-1}; \quad p_1/p_2 \leq 2. \end{aligned} \tag{2.17}$$

Use (2.17) and proceed as in Theorem 2.6 to obtain the required inequality. □

In next results, for any $(t_1,\ldots,t_{\hat{n}}) \in \mathbb{T}_1 \times \cdots \times \mathbb{T}_{\hat{n}}$, use the following operator:

$$\Omega_{\hat{n}}(t_1,\ldots,t_{\hat{n}}) := \prod_{l=1}^{\hat{n}} \frac{1}{h_l(t_l)} \int_0^{t_l} \cdots \int_0^{t_{\hat{n}}} \prod_{l=1}^{\hat{n}} \frac{h_l(s_l)}{s_l} f(s_1,\ldots,s_{\hat{n}})\Delta s_{\hat{n}} \cdots \Delta s_1. \tag{2.18}$$

Theorem 2.8. *Let $\mathbb{T}_l$ be time scales and $h_l(t_l)$ are non-increasing functions on $\mathbb{T}_l$ where $l = 1,\ldots,\hat{n}$ and (2.18) hold for any $(t_1,\ldots,t_{\hat{n}}) \in \mathbb{T}_1 \times \cdots \times \mathbb{T}_{\hat{n}}$. Suppose $p_1, p_2 > 0$ such that $p_1/p_2 \geq 2$, and*

$$1 + 2^{\frac{p_1}{p_2}-1} \frac{|\log(t_l)| h_l^{\Delta_l}(t_l)\Omega_l(\sigma_1(t_1),\ldots,\sigma_{l-1}(t_{l-1}),t_l,\ldots,t_{\hat{n}})}{h_l^{\sigma_l}(t_l){\Omega_l}^{\sigma_l}(\sigma_1(t_1),\ldots,\sigma_{l-1}(t_{l-1}),t_l,\ldots,t_{\hat{n}})z_l(t_l)} \geq M_l, \ \ \forall M_l > 0. \tag{2.19}$$

Then the following inequality holds,

$$\int_0^1 \cdots \int_0^1 \prod_{l=1}^{\hat{n}} z_l(t_l)[\Omega^\sigma_{\hat{n}}(t_1,\ldots,t_{\hat{n}})]^{\frac{p_1}{p_2}} \Delta t_{\hat{n}} \cdots \Delta t_1 \leq 2^{\frac{\hat{n}p_1(p_1-p_2)}{{p_2}^2}} \left(\prod_{l=1}^{\hat{n}} \frac{1}{M_l}\right)^{\frac{p_1}{p_2}}$$
$$\times \int_0^1 \cdots \int_0^1 \left\{\prod_{l=1}^{\hat{n}} z_l(t_l)\left(\frac{|\log(t_l)| h_l(t_l)}{t_l h_l^{\sigma_l}(t_l) z_l(t_l)}\right)^{\frac{p_1}{p_2}}\right\} f^{\frac{p_1}{p_2}}(t_1,\ldots,t_{\hat{n}})\Delta t_{\hat{n}} \cdots \Delta t_1. \tag{2.20}$$

Proof. To prove the required result, we use mathematical induction method. For $\hat{n} = 1$, the statement is true by [12, Theorem 2.9]. Let statement (2.20) holds for $1 \leq \hat{n} \leq \hat{k}$

Now to prove the result for $\hat{n} = \hat{k} + 1$, write the left hand side of (2.20) as:

$$\int_0^1 \cdots \int_0^1 \prod_{l=1}^{\hat{k}} z_l(t_l)\left\{\int_0^1 z_{\hat{k}+1}(t_{\hat{k}+1})\left[\Omega^\sigma_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}} \Delta t_{\hat{k}+1}\right\}\Delta t_{\hat{k}} \cdots \Delta t_1. \tag{2.21}$$

Denote,

$$\begin{aligned} I_{\hat{k}+1} &= \int_0^1 z_{\hat{k}+1}(t_{\hat{k}+1})\left[\Omega^\sigma_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}} \Delta t_{\hat{k}+1}\\ &= \int_0^1 z_{\hat{k}+1}(t_{\hat{k}+1})\left[\Omega^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}} \Delta t_{\hat{k}+1}. \end{aligned}$$

Integrate by using parts formula (1.5) with ${\xi_2}^{\Delta} = z_{\hat{k}+1}(t_{\hat{k}+1})$ and ${\xi_1}^{\sigma} = [\Omega^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(\sigma_1(t_1), \ldots, \sigma_{\hat{k}}(t_{\hat{k}}), t_{\hat{k}+1})]^{\frac{p_1}{p_2}}$ and use chain rule formula (1.2), to obtain

$$I_{\hat{k}+1} = \frac{p_1}{p_2}$$
$$\times \int_0^1 |log(t_{\hat{k}+1})| \int_0^1 \left[h\Omega^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(\sigma_1(t_1), \ldots, \sigma_{\hat{k}}(t_{\hat{k}}), t_{\hat{k}+1}) + (1-h)\Omega_{\hat{k}+1}(\sigma_1(t_1), \ldots, \sigma_{\hat{k}}(t_{\hat{k}}), t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}-1} dh$$
$$\times \left\{\frac{\partial}{\partial t_{\hat{k}+1}}\Omega_{\hat{k}+1}(\sigma_1(t_1), \ldots, \sigma_{\hat{k}}(t_{\hat{k}}), t_{\hat{k}+1})\right\} \Delta t_{\hat{k}+1}. \quad (2.22)$$

Use product rule of derivative (1.1), to obtain

$$\frac{\partial}{\partial t_{\hat{k}+1}}\Omega_{\hat{k}+1}(\sigma_1(t_1), \ldots, \sigma_{\hat{k}}(t_{\hat{k}}), t_{\hat{k}+1}) = \frac{h_{\hat{k}+1}(t_{\hat{k}+1})}{t_{\hat{k}+1} h^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(t_{\hat{k}+1})}\Omega_{\hat{k}}(\sigma_1(t_1), \ldots, \sigma_{\hat{k}}(t_{\hat{k}}), t_{\hat{k}+1})$$
$$- \frac{h^{\Delta_{\hat{k}+1}}_{\hat{k}+1}(t_{\hat{k}+1})}{h^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(t_{\hat{k}+1})}\Omega_{\hat{k}+1}(\sigma_1(t_1), \ldots, \sigma_{\hat{k}}(t_{\hat{k}}), t_{\hat{k}+1}) \geq 0, \quad (2.23)$$

where for fixed $t_{\hat{k}+1} \in \mathbb{T}_{\hat{k}+1}$,

$$\Omega_{\hat{k}}(\sigma_1(t_1), \ldots, \sigma_{\hat{k}}(t_k), t_{\hat{k}+1}) = \prod_{l=1}^{\hat{k}} \frac{1}{g_l(\sigma_l(t_l))} \int_0^{\sigma_1(t_1)} \cdots \int_0^{\sigma_{\hat{k}}(t_k)} \prod_{l=1}^{\hat{k}} \frac{g_l(s_l)}{s_l} f(s_1, \ldots, s_{\hat{k}}, t_{\hat{k}+1}) \Delta s_{\hat{k}} \cdots \Delta s_1.$$

Apply inequality (2.11) on the term

$$\left[h\Omega^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(\sigma_1(t_1), \ldots, \sigma_{\hat{k}}(t_{\hat{k}}), t_{\hat{k}+1}) + (1-h)\Omega_{\hat{k}+1}(\sigma_1(t_1), \ldots, \sigma_{\hat{k}}(t_{\hat{k}}), t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}-1}$$

of (2.22) also note that $\frac{\partial}{\partial t_{\hat{k}+1}}\Omega_{\hat{k}+1}(\sigma_1(t_1), \ldots, \sigma_{\hat{k}}(t_{\hat{k}}), t_{\hat{k}+1}) \geq 0$, write

$$\frac{p_1}{p_2}\int_0^1 \left[h\Omega^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(\sigma_1(t_1), \ldots, \sigma_{\hat{k}}(t_{\hat{k}}), t_{\hat{k}+1}) + (1-h)\Omega_{\hat{k}+1}(\sigma_1(t_1), \ldots, \sigma_{\hat{k}}(t_{\hat{k}}), t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}-1} dh$$
$$\leq 2^{\frac{p_1}{p_2}-2}\left[\Omega^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(\sigma_1(t_1), \ldots, \sigma_{\hat{k}}(t_{\hat{k}}), t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}-1}. \quad (2.24)$$

Substitute (2.24) and (2.23) into (2.22) and use (2.19) for $l = \hat{k}+1$, to obtain

$$I_{\hat{k}+1} \leq \frac{2^{\frac{p_1}{p_2}-1}}{M_{\hat{k}+1}} \int_0^1 \left(\frac{|\log(t_{\hat{k}+1})| h_{\hat{k}+1}(t_{\hat{k}+1})\Omega_{\hat{k}}(\sigma_1(t_1), \ldots, \sigma_{\hat{k}}(t_{\hat{k}}), t_{\hat{k}+1})}{t_{\hat{k}+1} h^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(t_{\hat{k}+1})[z_{\hat{k}+1}(t_{\hat{k}+1})]^{\frac{p_1-p_2}{p_1}}}\right)$$
$$\times \left([z_{\hat{k}+1}(t_{\hat{k}+1})]^{\frac{p_1-p_2}{p_1}}[\Omega^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(\sigma_1(t_1), \ldots, \sigma_{\hat{k}}(t_{\hat{k}}), t_{\hat{k}+1})]^{\frac{p_1-p_2}{p_2}}\right)\Delta t_{\hat{k}+1}.$$

Apply Hölder's inequality on right hand side with indices p_1/p_2 and $p_1/(p_1-p_2)$,

$$I_{\hat{k}+1} \leq 2^{\frac{p_1(p_1-p_2)}{{p_2}^2}}\left(\frac{1}{M_{\hat{k}+1}}\right)^{\frac{p_1}{p_2}} \int_0^1 z_{\hat{k}+1}(t_{\hat{k}+1})\left(\frac{|\log(t_{\hat{k}+1})| h_{\hat{k}+1}(t_{\hat{k}+1})\Omega_{\hat{k}}(\sigma_1(t_1), \ldots, \sigma_{\hat{k}}(t_{\hat{k}}), t_{\hat{k}+1})}{t_{\hat{k}+1} h^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(t_{\hat{k}+1}) z_{\hat{k}+1}(t_{\hat{k}+1})}\right)^{\frac{p_1}{p_2}} \Delta t_{\hat{k}+1}.$$
$$(2.25)$$

Put (2.25) in (2.21) and exchange integrals $\hat{k}$ times by using Fubini's theorem on right hand side,

$$\int_0^1 \cdots \int_0^1 \prod_{l=1}^{\hat{k}+1} z_l(t_l)[\Omega^{\sigma}_{\hat{k}+1}(t_l,\ldots,t_{\hat{k}+1})]^{\frac{p_1}{p_2}} \Delta t_{\hat{k}+1}\cdots\Delta t_l$$
$$\leq 2^{\frac{p_1(p_1-p_2)}{{p_2}^2}} \left(\frac{1}{M_{\hat{k}+1}}\right)^{\frac{p_1}{p_2}} \int_0^1 z_{\hat{k}+1}(t_{\hat{k}+1}) \left(\frac{|\log(t_{\hat{k}+1})| h_{\hat{k}+1}(t_{\hat{k}+1})}{t_{\hat{k}+1} h^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(t_{\hat{k}+1}) z_{\hat{k}+1}(t_{\hat{k}+1})}\right)^{\frac{p_1}{p_2}}$$
$$\left(\int_0^1 \cdots \int_0^1 \prod_{l=1}^{\hat{k}} z_{\hat{k}}(t_{\hat{k}})[\Omega_{\hat{k}}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),t_{\hat{k}+1})]^{\frac{p_1}{p_2}}\Delta t_l\right)\Delta t_{\hat{k}+1}. \tag{2.26}$$

Therefore use (2.20) for $\Omega_{\hat{k}}(t_l,\ldots,t_{\hat{k}+1})$ instead of $\Omega_{\hat{k}}(t_l,\ldots,t_{\hat{k}})$ for fix $t_{\hat{k}+1}$ in (2.26), to get

$$\int_0^1 \cdots \int_0^1 \prod_{l=1}^{\hat{k}+1} z_l(t_l)\left[\Omega^{\sigma}_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}} \Delta t_{\hat{k}+1}\cdots\Delta t_1 \leq 2^{\frac{(\hat{k}+1)p_1(p_1-p_2)}{{p_2}^2}} \left(\prod_{l=1}^{\hat{k}+1}\frac{1}{M_l}\right)^{\frac{p_1}{p_2}}$$
$$\times \int_0^1 \cdots \int_0^1 \left\{\prod_{l=1}^{\hat{k}+1} z_l(t_l)\left(\frac{|\log(t_l)| h_l(t_l)}{t_l h^{\sigma_l}_l(t_l) z_l(t_l)}\right)^{\frac{p_1}{p_2}}\right\} f^{\frac{p_1}{p_2}}(t_1,\ldots,t_{\hat{k}+1})\Delta t_{\hat{k}+1}\cdots\Delta t_1.$$

$\square$

Theorem 2.9. *Let $\mathbb{T}_l$ be time scales and $h_l(t_l)$ are non-increasing functions on $\mathbb{T}_l$ where $l = 1,\ldots,\hat{n}$ and (2.18) holds for any $(t_1,\ldots,t_{\hat{n}}) \in \mathbb{T}_1\times\cdots\times\mathbb{T}_{\hat{n}}$. Suppose $p_1, q_1 > 0$ such that $p_1/p_2 \geq 2$, also for $t_l \in [1,\infty)_{\mathbb{T}_l}$,*

$$1+\frac{p_1}{p_2}\frac{|\log(t_l)| h^{\Delta_l}_l(t_l)\Omega_l(\sigma_1(t_1),\ldots,\sigma_{l-1}(t_{l-1}),t_l,\ldots,t_{\hat{n}})}{h^{\sigma_l}_l(t_l)\Omega^{\sigma_l}_l(\sigma_1(t_1),\ldots,\sigma_{l-1}(t_{l-1}),t_l,\ldots,t_{\hat{n}}) z_l(t_l)} \geq M_l,\ \forall M_l > 0 \tag{2.27}$$

Then the following inequality holds

$$\int_0^1 \cdots \int_0^1 \prod_{l=1}^{\hat{n}} z_l(t_l)[\Omega^{\sigma}_{\hat{n}}(t_1,\ldots,t_{\hat{n}})]^{\frac{p_1}{p_2}} \Delta t_{\hat{n}}\cdots\Delta t_1$$
$$\leq \left(\frac{p_1}{p_2}\right)^{\frac{\hat{n}p_1}{p_2}} \left(\prod_{l=1}^{\hat{n}}\frac{1}{M_l}\right)^{\frac{p_1}{p_2}} \int_0^1 \cdots \int_0^1 \left\{\prod_{l=1}^{\hat{n}} z_l(t_l)\left(\frac{|\log(t_l)| h_l(t_l)}{t_l h^{\sigma_l}_l(t_l) z_l(t_l)}\right)^{\frac{p_1}{p_2}}\right\} f^{\frac{p_1}{p_2}}(t_1,\ldots,t_{\hat{n}})\Delta t_{\hat{n}}\cdots\Delta t_1. \tag{2.28}$$

Proof. To prove the required result, we use mathematical induction method. For $\hat{n} = 1$, the statement is true by [12, Theorem 2.10]. Let (2.28) holds for $1 \leq \hat{n} \leq \hat{k}$.

To prove the result for $\hat{n} = \hat{k}+1$, consider (as in the proof of Theorem 2.8)

$$I_{\hat{k}+1} = \int_0^1 z_{\hat{k}+1}(t_{\hat{k}+1})\left[\Omega^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}}\Delta t_{\hat{k}+1}.$$

Integrate by using parts formula (1.5) with

$${\xi_1}^{\Delta} = z_{\hat{k}+1}(t_{\hat{k}+1}),\ {\xi_2}^{\sigma} = [\Omega^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),t_{\hat{k}+1})]^{\frac{p_1}{p_2}},$$

to obtain

$$I_{\hat{k}+1} \leq \int_0^1 |log(t_{\hat{k}+1})| \left(\frac{\partial}{\partial t_{\hat{k}+1}}[\Omega_{\hat{k}+1}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),t_{\hat{k}+1})]^{\frac{p_1}{p_2}}\right)\Delta t_{\hat{k}+1}. \tag{2.29}$$

Apply the chain rule formula (1.4), see that there exists $c_{\hat{k}+1} \in [t_{\hat{k}+1}, \sigma_{\hat{k}+1}(t_{\hat{k}+1})]$ such that

$$\frac{\partial}{\partial t_{\hat{k}+1}}[\Omega_{\hat{k}+1}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),t_{\hat{k}+1})]^{\frac{p_1}{p_2}} = \left(\frac{p_1}{p_2}\right)[\Omega_{\hat{k}+1}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),c_{\hat{k}+1})]^{\frac{p_1}{p_2}-1}\left(\frac{\partial}{\partial t_{\hat{k}+1}}\Omega_{\hat{k}+1}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),t_{\hat{k}+1})\right). \quad (2.30)$$

Use the fact that $\frac{\partial}{\partial t_{\hat{k}+1}}\Omega_{\hat{k}+1}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),t_{\hat{k}+1}) \geq 0$ from (2.23), also $\sigma_{\hat{k}+1}(t_{\hat{k}+1}) \geq c_{\hat{k}+1}$ implies $\Omega^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),t_{\hat{k}+1}) \geq \Omega_{\hat{k}+1}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),c_{\hat{k}+1})$ in (2.30), to get

$$\frac{\partial}{\partial t_{\hat{k}+1}}[\Omega_{\hat{k}+1}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),t_{\hat{k}+1})]^{\frac{p_1}{p_2}} \leq \left(\frac{p_1}{p_2}\right)\left[\Omega^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}-1}\left(\frac{\partial}{\partial t_{\hat{k}+1}}\Omega_{\hat{k}+1}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),t_{\hat{k}+1})\right). \quad (2.31)$$

Put (2.23) of Theorem 2.8 and (2.31) in (2.29),

$$\begin{aligned} I_{\hat{k}+1} \leq \frac{p_1}{p_2}\int\limits_0^1 (|log t_{\hat{k}+1}|)&\left[\Omega^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}-1} \\ &\left[\frac{h_{\hat{k}+1}(t_{\hat{k}+1})}{t_{\hat{k}+1}h^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(t_{\hat{k}+1})}\Omega_{\hat{k}}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),t_{\hat{k}+1})\right]\Delta t_{\hat{k}+1} \\ &-\frac{p_1}{p_2}\int\limits_0^1 (|log t_{\hat{k}+1}|)\left[\Omega^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}-1} \\ &\left[\frac{h^{\Delta_{\hat{k}+1}}_{\hat{k}+1}(t_{\hat{k}+1})}{h^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(t_{\hat{k}+1})}\Omega_{\hat{k}+1}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),t_{\hat{k}+1})\right]\Delta t_{\hat{k}+1}. \end{aligned}$$

Proceed similar to the proof of Theorem 2.8 and use (2.27) , to get

$$\begin{aligned} &\int\limits_0^1\cdots\int\limits_0^1\prod_{l=1}^{\hat{k}+1} z_l(t_l)[\Omega^{\sigma}_{\hat{k}+1}(t_l,\ldots,t_{\hat{k}+1})]^{\frac{p_1}{p_2}}\Delta t_{\hat{k}+1}\cdots\Delta t_l \\ &\leq \left(\frac{p_1}{M_{\hat{k}+1}p_2}\right)^{\frac{p_1}{p_2}}\int\limits_0^1 z_{\hat{k}+1}(t_{\hat{k}+1})\left(\frac{|\log(t_{\hat{k}+1})|h_{\hat{k}+1}(t_{\hat{k}+1})}{t_{\hat{k}+1}h^{\sigma_{\hat{k}+1}}_{\hat{k}+1}(t_{\hat{k}+1})z_{\hat{k}+1}(t_{\hat{k}+1})}\right)^{\frac{p_1}{p_2}} \\ &\quad\left(\int\limits_0^1\cdots\int\limits_0^1 z_{\hat{k}}(t_{\hat{k}})[\Omega_{\hat{k}}(\sigma_1(t_1),\ldots,\sigma_{\hat{k}}(t_{\hat{k}}),t_{\hat{k}+1})]^{\frac{p_1}{p_2}}\Delta t_l\right)\Delta t_{\hat{k}+1}. \quad (2.32) \end{aligned}$$

Use (2.28) for $\Omega_{\hat{k}}(t_l,\ldots,t_{\hat{k}+1})$ instead of $\Omega_{\hat{k}}(t_l,\ldots,t_{\hat{k}})$ for fix $t_{\hat{k}+1}$ in (2.32), to obtain

$$\begin{aligned} \int_0^1\cdots\int_0^1\prod_{l=1}^{\hat{k}+1} z_l(t_l)\left[\Omega^{\sigma}_{\hat{k}+1}(t_1,\ldots,t_{\hat{k}+1})\right]^{\frac{p_1}{p_2}}\Delta t_{\hat{k}+1}\cdots\Delta t_1 &\leq \left(\frac{p_1}{p_2}\right)^{\frac{(\hat{k}+1)p_1}{p_2}}\left(\prod_{l=1}^{\hat{k}+1}\frac{1}{M_l}\right)^{\frac{p_1}{p_2}} \\ &\times\int_0^1\cdots\int_0^1\left\{\prod_{l=1}^{\hat{k}+1} z_l(t_l)\left(\frac{|\log(t_l)|h_l(t_l)}{t_l h^{\sigma_l}_l(t_l)z_l(t_l)}\right)^{\frac{p_1}{p_2}}\right\} f^{\frac{p_1}{p_2}}(t_1,\ldots,t_{\hat{k}+1})\Delta t_{\hat{k}+1}\cdots\Delta t_1. \end{aligned}$$

□

Theorem 2.10. *Let $p_1, q_1 > 0$ be such that $p_1/p_2 \leq 2$. Assume that $h_l(t_l)$ are non-increasing function on time scales $\mathbb{T}_l$ where $l = 1, \ldots, \hat{n}$. Let $\Omega_{\hat{n}}(t_l, \ldots, t_{\hat{n}})$ be defined in (2.18) and assume that (2.19) holds for some constants $M_j > 0$. Then*

$$\int_0^1 \cdots \int_0^1 \prod_{l=1}^{\hat{n}} z_l(t_l)(\Omega_{\hat{n}}^{\sigma}(t_1, \ldots, t_{\hat{n}}))^{\frac{p_1}{p_2}} \Delta t_{\hat{n}} \cdots \Delta t_1$$
$$\leq (2\hat{n})^{\frac{p_1}{p_2}} \left(\prod_{l=1}^{\hat{n}} \frac{1}{M_l} \right)^{\frac{p_1}{p_2}} \int_0^1 \cdots \int_0^1 \prod_{l=1}^{\hat{n}} z_l(t_l) \left(\frac{|\log(t_l)| h_l(t_l)}{t_l h_l^{\sigma_l}(t_l) z_l(t_l)} \right)^{\frac{p_1}{p_2}} f^{\frac{p_1}{p_2}}(t_1, \ldots, t_{\hat{n}}) \Delta t_{\hat{n}} \cdots \Delta t_1.$$

Proof. Apply the inequality (2.16) on the term $[h\Omega^{\sigma} + (1-h)\Omega]^{\frac{p_1}{p_2}-1}$ when $p_1/p_2 \leq 2$, to have

$$\begin{aligned} \frac{p_1}{p_2} \int_0^1 [h\Omega^{\sigma} + (1-h)\Omega]^{\frac{p_1}{p_2}-1} dh &\leq \frac{p_1}{p_2} \int_0^1 [h^{\frac{p_1}{p_2}-1}(\Omega^{\sigma})^{\frac{p_1}{p_2}-1} + (1-h)^{\frac{p_1}{p_2}-1} \Omega^{\frac{p_1}{p_2}-1}] dh \\ &= \left[(\Omega^{\sigma})^{\frac{p_1}{p_2}-1} + \Omega^{\frac{p_1}{p_2}-1} \right] \leq 2(\Omega^{\sigma})^{\frac{p_1}{p_2}-1}. \end{aligned} \tag{2.33}$$

Proceed as in prove of Theorem 2.8 and further use (2.33) to complete the result. □

3. Conclusion

In this paper several Hardy type inequalities involving several variables are proved by using time scales calculus, The obtained inequalities are also discussed in quantum calculus, differential calculus and α discrete calculus where $\alpha > 0$.

Moreover, similar to inequalities given in Corollary 2.2–Corollary 2.4, which we get by choosing special time scales in Theorem 2.1, we can obtain inequalities by choosing special time scales in Theorem 2.6–Theorem 2.10.

It is also possible to establish inequalities in more general settings by using the functions $Z_l(t_l)$ instead of $z_l(t_l)$ in Theorem 2.6–Theorem 2.10, as we have done in Corollary 2.5.

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