



Partition dimension of generalized Peterson and Harary graphs

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Abstract

The distance of a connected, simple graph $\mathbb{P}$ is denoted by $d(\alpha_1, \alpha_2)$, which is the length of a shortest path between the vertices $\alpha_1, \alpha_2 \in V(\mathbb{P})$, where $V(\mathbb{P})$ is the vertex set of $\mathbb{P}$. The l -ordered partition of $V(\mathbb{P})$ is $K = \{K_1, K_2, \dots, K_l\}$. A vertex $\alpha \in V(\mathbb{P})$, and $r(\alpha|K) = \{d(\alpha, K_1), d(\alpha, K_2), \dots, d(\alpha, K_l)\}$ be a l -tuple distances, where $r(\alpha|K)$ is the representation of a vertex α with respect to set K . If $r(\alpha|K)$ of α is unique, for every pair of vertices, then K is the resolving partition set of $V(\mathbb{P})$. The minimum number l in the resolving partition set K is known as partition dimension ($pd(\mathbb{P})$). In this paper, we studied the generalized families of Peterson graph, $P_{\lambda, \chi}$ and proved that these families have bounded partition dimension.

Keywords: Generalized Peterson graph, Harary Graph, partition dimension, partition resolving set, sharp bounds of partition dimension.

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1. Introduction

Let $\mathbb{G} = (\mathbb{P}_k)_{k \geq 1}$ be a family with connected graphs $\mathbb{P}_k$ depending on the number of graphs in the family which is denoted by k as follows: $|V(\mathbb{P}_k)| = O(k)$ is the order with condition $\lim_{k \rightarrow \infty} O(k) = \infty$. If $\exists$ a constant number $\alpha \geq 0$ such that $pd(\mathbb{P}_k) \leq \alpha$ for each possible value $k \geq 1$, then we can deduce the result that $\mathbb{G}$ has bounded partition dimension and the upper bound is the constant number α , otherwise the $\mathbb{G}$ does

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not have bounded partition dimension. If every graph in the family $\mathbb{G}$ have the equal partition dimension (which mean that the partition dimension do not depends on the number k), $\mathbb{G}$ is said to be a family with constant partition dimension [16].

In 1975 the idea delivered by Slater had a background in networking, usually referred to as locating set or beacons set. The entire network or a graph is controlled by specifically chosen vertices from the vertex set in this concept. These vertices have to choose with a specific condition that each vertex of a graph has a unique position in terms of representations, we refer to the Definition 1.2 for this concept. Later Melter and Harary rename this concept as resolving set [12]. In the graph's theoretical study, this concept is called a metric basis or basis set of a graph. The count of vertices in a resolving set or metric basis is referred as the metric dimension of a graph [27]. Instead of choosing particular nodes into a subset with the defined condition, it is possible to arrange the entire vertex set into subsets keeping the defined condition of $r(\gamma|F)$, which is actually came from the idea of the unique position of each vertex in a graph. This concept is called as the partition resolving set, and the least number of subsets is called the partition dimension, introduced by Chartrand et al. in 2000 [5]. To better understand this concept, we refer to the mathematical Definition 1.3 and 1.4 and for the latest ideas related to this concept, see [10].

Representing a graph with each of its vertex has unique position is falling in different real-world applications, such as for the strategies, coding, and decoding of mastermind games brief in [9], the popular relation which is named as Djokovic-Winkler linked to this concept [4], the piloting or the guidance of a robot also associated with this unique idea [17], the procedure of verifying and discovering a network related to this concept [3]. There are many applications to explore those, we refer to see [12, 21]. Finding of a resolving set is NP-hard problem [13, 18] and the partition resolving set is the generalization of resolving set it also falls in the category of NP-hard [5].

The concept of resolving partition set and partition dimension extensively appeared in the literature. For example, the graph with partition dimension $|V| - 3$ discussed [2], the graph obtained by few graph operations and its corresponding partition dimension studied in [28], bounds on the partition dimension for convex polytopes in [6, 8, 14], bounds of partition on the circulant and multipartite discussed in [11, 19], chemical structure partitioning discussed in [20], on the bounded partition dimension of the Cartesian product of graphs are studied in [30], [1] gave bounds for the subdivision of different graphs, [25] provide the bounds on tree graph, [26] discussed bounds of unicyclic graphs in the form of subgraphs. For the resolving set and metric dimension of Peterson and generalized Peterson graph, we refer to the articles [23, 15]. For more recent literature and results, we refer to see [25, 22, 24].

Following are basic mathematical definitions of the concepts used in this research work.

Definition 1.1. Suppose P be an undirected, simple graph with the set of vertices named as $V(P)$ and edge set $E(P)$, the distance which also known as geodesics, between $\alpha_1, \alpha_2 \in V(P)$ two vertices is the count of minimum edges between $\alpha_1 - \alpha_2$ path. It is denoted by $d(\alpha_1, \alpha_2)$.

Definition 1.2. Suppose an ordered set of vertices from $V(P)$ labeled as $R = \{\alpha_1, \alpha_2, \dots, \alpha_s\}$ and $\alpha \in V(P)$. The representations $r(\alpha|R)$ of α -vertex with respect to an ordered subset R is the s -tuple distances $(d(\alpha, \alpha_1), d(\alpha, \alpha_2), \dots, d(\alpha, \alpha_s))$. If each vertex from $V(P)$ have unique representations according to R , then R is called a resolving set of graph P , and minimum count of the elements in R is called the metric dimension of graph P and it is represented by $dim(P)$.

Definition 1.3. Let χ is the s -ordered partition set and $r(\alpha|\chi) = \{d(\alpha, \chi_1), d(\alpha, \chi_2), \dots, d(\alpha, \chi_s)\}$, is the s -tuple distance representations of a vertex α with respect to χ . If the representations of α with respect to χ are unique, then B is the partition resolving set of the vertex set of a graph P .

Definition 1.4. The minimum count of subsets in the partition resolving set of $V(P)$ is defined as the partition dimension ($pd(P)$) of P .

Following theorems are very helpful in finding the partition dimension of a graph.

Theorem 1.5. [5] *Let χ be a partition resolving set of $V(P)$ and $\alpha_1, \alpha_2 \in V(P)$. If $d(\alpha_1, \alpha) = d(\alpha_2, \alpha)$ for all vertices $\alpha \in V(P) \setminus (\alpha_1, \alpha_2)$, then α_1, α_2 belongs to different subsets of χ .*

$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4
a_0	1	4	3	0	a_1	2	5	4	0
a_2	3	5	5	0	a_3	4	4	5	0
a_4	5	3	4	0	a_5	5	2	3	0
a_6	4	1	2	0	a_7	3	2	1	0
a_9	2	4	1	0	a_{10}	3	5	2	0
a_{11}	4	5	3	0	a_{12}	4	2	5	0
a_{13}	5	3	5	0	a_{14}	4	2	5	0
a_{15}	3	2	4	0	a_{16}	2	3	3	0

Table 1: Representations of outer vertices

$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4
b_1	2	6	3	0	b_2	4	6	4	0
b_3	5	5	5	0	b_4	6	4	5	0
b_5	6	2	4	0	b_7	3	2	2	0
b_8	1	4	1	0	b_9	1	5	2	0
b_{10}	3	6	3	0	b_{11}	5	6	4	0
b_{12}	6	5	5	0	b_{13}	6	3	5	0
b_{14}	5	1	4	0	b_{15}	4	1	3	0
b_{16}	2	3	2	0					

Table 2: Representations of inner vertices

2. Generalized Petersen Graph $P_{\alpha,\beta}$

For each odd integer $\alpha = 2\beta + 1 \geq 3$, the generalized Petersen graph $P_{\alpha,\beta}$ is a graph with vertex set $a \cup b$ where $\{a_\eta, b_\eta : \eta = 0, 1, \dots, \alpha - 1\}$ and edge set $E = \{a_\eta a_{\eta+1}, b_\eta b_{\eta+\beta}, a_\eta b_\eta : \eta = 0, 1, \dots, \alpha - 1\}$. For our purpose, we call the vertices $a_0, a_1, \dots, a_{\alpha-1}$ outer vertices and $b_0, b_1, \dots, b_{\alpha-1}$ inner vertices. Here and throughout the paper, the subscripts are to be taken as integers modulo α . Following are some bounds on the partition dimension of generalized Petersen graph $P_{\alpha,\beta}$.

Theorem 2.1. *Let $P_{\alpha,\beta}$ be generalized Petersen graphs with $\alpha = 2\beta + 1$ and $\beta \equiv 0 \pmod{4}$, then $pd(P_{\alpha,\beta}) \leq 4$.*

Proof. For $\beta = 4$, it is easy to see that $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{b_0\}$, $\chi_2 = \{b_4\}$, $\chi_3 = \{a_5\}$, $\chi_4 = V(P_{\alpha,\beta}) \setminus \{b_0, b_4, a_5\}$ is a resolving partitioning for $V(P_{9,4})$.

For $\beta \geq 8$, $\beta \equiv 0 \pmod{4}$ and for the chosen index η such that $0 \leq \eta \leq \alpha - 1$, we shall show that $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{b_\eta\}$, $\chi_2 = \{b_{\eta+2\xi+2}\}$, $\chi_3 = \{a_{\eta+\beta}\}$, $\chi_4 = V(P_{\alpha,\beta}) \setminus \{b_\eta, b_{\eta+2\xi+2}, a_{\eta+\beta}\}$ is a resolving partitioning for $V(P_{\alpha,\beta})$, where $\xi = \frac{\beta}{4}$.

For $\beta = 8$, the representations of the vertices of $V(P_{17,8})$, are in the Tables 1 and 2.

It can be seen that all the vertices in $V(P_{17,8})$, have distinct representations with respect to χ . Now for $\beta \geq 8$ the representations of the vertices of $V(P_{\alpha,\beta})$, are: $r(b_{\eta+1}|\chi) = (2, 2\xi + 2, 3, 0)$, $r(b_{\eta+2\beta}|\chi) = (2, 2\xi, 2, 0)$, and remaining in Tables 3 and 4.

From these tables, one can see that all the vertices of $P_{\alpha,\beta}$ lying in column 1 of Table 3 and Table 4 have distinct representations with respect to χ . Thus χ is resolving partitioning for $P_{\alpha,\beta}$. Hence, for all $\alpha = 2\beta + 1$, $\beta \geq 8$ and $\beta \equiv 0 \pmod{4}$, $pd(P_{\alpha,\beta}) \leq 4$. $\square$

Theorem 2.2. *Let $P_{\alpha,\beta}$ be generalized Petersen graphs with $\alpha = 2\beta + 1 \geq 5$ and $\beta \equiv 1 \pmod{4}$, then $pd(P_{\alpha,\beta}) \leq 4$.*

$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	
$a_{\eta+\zeta}$	$\zeta + 1$	$2\xi + \zeta$	$\zeta + 3$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\zeta+2}$	$\zeta + 3$	$2\xi - \zeta + 1$	$\zeta + 5$	0	$0 \leq \zeta \leq 2\xi - 4$
$a_{\eta+2\xi+\zeta-1}$	$2\xi + \zeta$	$4 - \zeta$	$2\xi - \zeta + 1$	0	$0 \leq \zeta \leq 1$
$a_{\eta+2\xi+\zeta+1}$	$2\xi - \zeta + 1$	$2 - \zeta$	$2\xi - \zeta - 1$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\beta-\zeta-1}$	$\zeta + 3$	$2\xi - \zeta - 2$	$\zeta + 1$	0	$0 \leq \zeta \leq 2\xi - 4$
$a_{\eta+\beta+\zeta+1}$	$\zeta + 2$	$2\xi + \zeta$	$\zeta + 1$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\beta+\zeta+3}$	$\zeta + 4$	$2\xi - \zeta + 1$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 3$
$a_{\eta+2\xi+\zeta+\beta+1}$	$2\xi - \zeta + 1$	$3 - \zeta$	$2\xi + 1$	0	$0 \leq \zeta \leq 1$
$a_{\eta+2\beta-\zeta}$	$\zeta + 2$	$2\xi - \zeta - 1$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 3$

Table 3: Representations of outer vertices

$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	
$b_{\eta+\zeta+2}$	$\zeta + 4$	$2\xi - \zeta + 2$	$\zeta + 4$	0	$0 \leq \zeta \leq 2\xi - 3$
$b_{\eta+\zeta+2\xi}$	$2\xi + 2$	$4 - 2\zeta$	$2\xi - \zeta + 1$	0	$0 \leq \zeta \leq 1$
$b_{\eta+2\xi+\zeta+3}$	$2\xi - \zeta$	$2\zeta + 2$	$2\xi - \zeta - 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+2\xi+\zeta+5}$	$2\xi - \zeta - 2$	$\zeta + 5$	$2\xi - \zeta - 4$	0	$0 \leq \zeta \leq 2\xi - 7$
$b_{\eta+\beta-\zeta}$	$2\zeta + 1$	$2\xi + \zeta + 1$	$\zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+\zeta+1}$	$2\zeta + 1$	$2\xi + \zeta + 1$	$\zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+\zeta+3}$	$\zeta + 5$	$2\xi - \zeta + 2$	$\zeta + 4$	0	$0 \leq \zeta \leq 2\xi - 3$
$b_{\eta+2\xi+\zeta+\beta+1}$	$2\xi - \zeta + 2$	$3 - 2\zeta$	$2\xi - \zeta + 1$	0	$0 \leq \zeta \leq 1$
$b_{\eta+2\xi+\zeta+\beta+3}$	$2\xi - \zeta$	$2\zeta + 1$	$2\xi - \zeta - 1$	0	$0 \leq \zeta \leq 1$
$b_{\eta+2\beta-\zeta-1}$	$\zeta + 4$	$2\xi - \zeta - 1$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 6$

Table 4: Representations of inner vertices

$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	
$a_{\eta+\zeta}$	$\zeta + 3$	$2\xi + \zeta + 1$	$\zeta + 3$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\zeta+2}$	$\zeta + 1$	$2\xi - \zeta + 1$	$\zeta + 5$	0	$0 \leq \zeta \leq 2\xi - 3$
$a_{\eta+2\xi+\zeta}$	$2\xi + \zeta - 1$	$3 - \zeta$	$2\xi - \zeta + 1$	0	$0 \leq \zeta \leq 2$
$a_{\eta+\beta-\zeta-1}$	$\zeta + 5$	$2\xi - \zeta - 1$	$\zeta + 1$	0	$0 \leq \zeta \leq 2\xi - 3$
$a_{\eta+\beta+\zeta+1}$	$3 - \zeta$	$2\xi + \zeta + 1$	$\zeta + 1$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\beta+\zeta+3}$	$\zeta + 2$	$2\xi - \zeta + 1$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 1$
$a_{\eta+2\beta-\zeta}$	$\zeta + 4$	$2\xi - \zeta$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 2$

Table 5: Representations of outer vertices

$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	
$b_{\eta+\zeta}$	$4 - 2\zeta$	$2\xi + \zeta + 2$	$\zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\zeta+4}$	$\zeta + 4$	$2\xi - \zeta$	$\zeta + 6$	0	$0 \leq \zeta \leq 2\xi - 4$
$b_{\eta-\beta+\zeta}$	$\zeta + 5$	$2\xi - \zeta + 1$	$\zeta + 1$	0	$0 \leq \zeta \leq 2\xi - 3$
$b_{\eta+\beta+\zeta+1}$	$3 - 2\zeta$	$2\xi + \zeta + 2$	$\zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+\zeta+3}$	$2\zeta + 1$	$2\xi - \zeta + 2$	$\zeta + 4$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+\zeta+5}$	$\zeta + 5$	$2\xi - \zeta$	$\zeta + 6$	0	$0 \leq \zeta \leq 2\xi - 5$
$b_{\eta+\beta+2\xi+\zeta+1}$	$2\xi + \zeta + 1$	$3 - 2\zeta$	$2\xi - \zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+2\xi+\zeta+3}$	$2\xi - \zeta + 3$	$2\zeta + 1$	$2\xi - \zeta$	0	$0 \leq \zeta \leq 1$
$b_{\eta+2\beta-\zeta}$	$\zeta + 5$	$2\xi + \zeta + 1$	$\zeta + 2$	0	$0 \leq \zeta \leq 2\xi - 4$

Table 6: Representations of inner vertices

Proof. For $\beta = 5$, it is easy to see that $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{b_2\}$, $\chi_2 = \{b_4\}$, $\chi_3 = \{a_5\}$, $\chi_4 = V(P_{\alpha,\beta}) \setminus \{b_2, b_4, a_5\}$ is a resolving partitioning for $V(P_{11,5})$.

For $\beta \geq 5$, $\beta \equiv 1(\text{mod } 4)$ and for the chosen index η such that $0 \leq \eta \leq \alpha - 1$, we shall show that $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$, where $\chi_1 = \{b_{\eta+2}\}$, $\chi_2 = \{b_{\eta+2\xi+2}\}$, $\chi_3 = \{a_{\eta+\beta}\}$, $\chi_4 = V(P_{\alpha,\beta}) \setminus \{b_{\eta+2}, b_{\eta+2\xi+2}, a_{\eta+\beta}\}$ is a resolving partitioning for $V(P_{\alpha,\beta})$.

The representations of $V(P_{\alpha,\beta})$, are: $r(b_{\eta+3}|\chi) = (2, 2\xi+1, 5, 0)$, $r(b_{\eta+2\xi+1}|\chi) = (2\xi+1, 2, 2\xi+1, 0)$, $r(b_{\eta+2\xi+3}|\chi) = (2\xi+3, 2, 2\xi-1, 0)$ and in the Tables 5 and 6.

From these tables, one can see that all the vertices of $P_{\alpha,\beta}$ lying in column 1 of Table 5 and Table 6 have distinct representations with respect to χ . Thus χ is resolving partitioning for $P_{\alpha,\beta}$. Hence, for all $\alpha = 2\beta + 1$, $\beta \geq 5$ and $\beta \equiv 1(\text{mod } 4)$, $pd(P_{\alpha,\beta}) \leq 4$. $\square$

Theorem 2.3. Let $P_{\alpha,\beta}$ be generalized Petersen graphs with $\alpha = 2\beta + 1 \geq 2$ and $\beta \equiv 2(\text{mod } 4)$, then $pd(P_{\alpha,\beta}) \leq 4$.

Proof. For $\beta = 6$, it is easy to see from the Tables 7 and 7 that $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{b_3\}$, $\chi_2 = \{b_4\}$, $\chi_3 = \{a_7\}$, $\chi_4 = V(P_{\alpha,\beta}) \setminus \{b_3, b_4, a_7\}$ is a resolving partitioning for $V(P_{13,6})$.

For $\beta = 10$, it is easy to see from the Tables 9 and 10 that $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{b_3\}$, $\chi_2 = \{b_6\}$, $\chi_3 = \{a_{11}\}$, $\chi_4 = V(P_{\alpha,\beta}) \setminus \{b_3, b_6, a_{11}\}$ is a resolving partitioning for $V(P_{21,10})$.

For $\beta \geq 10$, $\beta \equiv 2(\text{mod } 4)$ and for the chosen index η such that $0 \leq \eta \leq \alpha - 1$, we shall show that $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{b_{\eta+3}\}$, $\chi_2 = \{b_{\eta+2\xi+2}\}$, $\chi_3 = \{a_{\eta+\beta+1}\}$, $\chi_4 = V(P_{\alpha,\beta}) \setminus \{b_{\eta+3}, b_{\eta+2\xi+2}, a_{\eta+\beta+1}\}$ is a resolving partitioning for $V(P_{\alpha,\beta})$, where $\xi = \frac{\beta-2}{4}$.

For $\beta \geq 11$, $\beta \equiv 2(\text{mod } 4)$ the representations of the vertices of $V(P_{\alpha,\beta})$, are: $r(a_{\eta}|\chi) = (4, 2\xi+2, 3, 0)$, $r(b_{\eta}|\chi) = (5, 2\xi+3, 2, 0)$, $r(b_{\eta+4}|\chi) = (2, 2\xi, 5, 0)$, $r(b_{\eta+2\xi+1}|\chi) = (2\xi, 2, 2\xi+2, 0)$, $r(b_{\eta+2\xi+3}|\chi) = (2\xi+2, 2, 2\xi+1, 0)$ and in the Tables 11 and 12.

From these tables, one can see that all the vertices of $P_{\alpha,\beta}$ lying in column 1 of Table 1 and Table 12 have distinct representations with respect to χ . Thus χ is resolving partitioning for $P_{\alpha,\beta}$. Hence, for all

$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4
a_0	4	4	3	0	a_1	3	4	3	0	a_2	2	3	4	0
a_3	1	2	4	0	a_4	2	1	3	0	a_5	3	2	2	0
a_6	4	3	1	0	a_8	3	4	1	0	a_9	2	3	2	0
a_{10}	2	2	3	0	a_{11}	3	2	4	0	a_{12}	4	3	4	0

Table 7: Representations of outer vertices

$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4
b_0	5	5	2	0	b_1	4	5	2	0	b_2	2	4	3	0
b_5	4	2	3	0	b_6	5	4	2	0	b_7	5	5	1	0
b_8	3	5	2	0	b_9	1	3	3	0	b_{10}	1	1	4	0
b_{11}	3	1	4	0	b_{12}	5	3	3	0					

Table 8: Representations of inner vertices

$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4
a_0	4	6	3	0	a_1	3	6	3	0
a_2	2	5	4	0	a_3	1	4	5	0
a_4	2	3	6	0	a_5	3	2	6	0
a_6	4	1	5	0	a_7	5	2	4	0
a_8	6	3	3	0	a_9	6	4	2	0
a_{10}	5	5	1	0	a_{12}	3	6	1	0
a_{13}	2	5	2	0	a_{14}	2	4	3	0
a_{15}	3	3	4	0	a_{16}	4	2	5	0
a_{17}	5	2	6	0	a_{18}	6	3	6	0
a_{19}	6	4	5	0	a_{20}	5	5	4	0

Table 9: Representations of outer vertices

$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4
b_0	5	7	2	0	b_1	4	7	2	0
b_2	2	6	3	0	b_4	2	4	5	0
b_5	4	2	6	0	b_7	6	2	5	0
b_8	7	4	4	0	b_9	7	5	3	0
b_{10}	6	6	2	0	b_{11}	5	7	1	0
b_{12}	3	7	2	0	b_{13}	1	6	3	0
b_{14}	1	5	4	0	b_{15}	3	3	5	0
b_{16}	5	1	6	0	b_{17}	6	1	6	0
b_{18}	7	3	5	0	b_{19}	7	5	4	0
b_{20}	6	6	3	0					

Table 10: Representations of inner vertices

$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	
$a_{\eta+\zeta+1}$	$\zeta + 3$	$2\xi - \zeta + 2$	$\zeta + 3$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\zeta+3}$	$\zeta + 1$	$2\xi - \zeta$	$\zeta + 5$	0	$0 \leq \zeta \leq 2\xi - 3$
$a_{\eta+2\xi+\zeta+1}$	$2\xi + \zeta - 1$	$2 - \zeta$	$2\xi - \zeta + 2$	0	$0 \leq \zeta \leq 1$
$a_{\eta+2\xi+\zeta+3}$	$2\xi + \zeta + 1$	$\zeta + 2$	$2\xi - \zeta$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\beta-\zeta}$	$\zeta + 5$	$2\xi - \zeta + 1$	$\zeta + 1$	0	$0 \leq \zeta \leq 2\xi - 3$
$a_{\eta+\beta+\zeta+2}$	$3 - \zeta$	$2\xi - \zeta + 2$	$\zeta + 1$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\beta+\zeta+4}$	$\zeta + 2$	$2\xi - \zeta$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 2$
$a_{\eta+\beta+\zeta+2\xi+3}$	$2\xi + \zeta + 1$	$\zeta + 2$	$2\xi + 2$	0	$0 \leq \zeta \leq 1$
$a_{\eta+2\beta-\zeta}$	$\zeta + 5$	$2\xi - \zeta + 1$	$\zeta + 4$	0	$0 \leq \zeta \leq 2\xi - 3$

Table 11: Representations of outer vertices

$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	
$b_{\eta+\zeta+1}$	$4 - 2\zeta$	$2\xi - \zeta + 3$	$\zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\zeta+5}$	$\zeta + 4$	$2\xi - \zeta - 1$	$\zeta + 6$	0	$0 \leq \zeta \leq 2\xi - 5$
$b_{\eta+2\xi+\zeta+4}$	$2\xi + 3$	$\zeta + 4$	$2\xi - \zeta$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta-\zeta+1}$	$\zeta + 5$	$2\xi - \zeta + 3$	$\zeta + 1$	0	$0 \leq \zeta \leq 2\xi - 3$
$b_{\eta+\beta+\zeta+2}$	$3 - 2\zeta$	$2\xi - \zeta + 3$	$\zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+\zeta+4}$	$2\zeta + 15$	$2\xi - \zeta + 1$	$\zeta + 4$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+6}$	$\zeta + 5$	$2\xi - \zeta - 1$	$\zeta + 6$	0	$0 \leq \zeta \leq 2\xi - 6$
$b_{\eta+\beta+\zeta+1+2\xi}$	$2\xi + \zeta$	$3 - 2\zeta$	$2\xi + \zeta + 1$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+\zeta+3+2\xi}$	$2\xi + \zeta + 2$	$2\zeta + 1$	$2\xi - \zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+2\beta-\zeta}$	$\zeta + 6$	$2\xi - \zeta + 2$	$2\xi - \zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 3$

Table 12: Representations of inner vertices

$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4
a_0	2	5	4	0	a_1	3	4	5	0	a_2	4	3	4	0
a_3	5	2	3	0	a_4	4	1	2	0	a_5	3	2	1	0
a_7	2	4	1	0	a_8	3	5	2	0	a_9	4	1	3	0
a_{10}	5	3	4	0	a_{11}	4	2	5	0	a_{12}	3	2	4	0
a_{13}	2	3	3	0	a_{14}	1	4	3	0					

Table 13: Representations of outer vertices

$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4
b_0	2	6	3	0	b_1	4	5	4	0	b_2	5	4	5	0
b_3	6	2	4	0	b_5	3	2	2	0	b_6	1	4	1	0
b_7	1	5	2	0	b_8	3	6	3	0	b_9	5	5	4	0
b_{10}	6	3	5	0	b_{11}	5	1	4	0	b_{12}	4	1	3	0
b_{13}	2	3	4	0										

Table 14: Representations of inner vertices

$\alpha = 2\beta + 1$, $\beta \geq 2$ and $\beta \equiv 2(\text{mod } 4)$, $pd(P_{\alpha,\beta}) \leq 4$. □

Theorem 2.4. Let $P_{\alpha,\beta}$ be generalized Petersen graphs with $\alpha = 2\beta + 1 \geq 3$ and $\beta \equiv 3(\text{mod } 4)$, then $pd(P_{\alpha,\beta}) \leq 4$.

Proof. For $\beta = 6$, it is easy to see that $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{b_2\}$, $\chi_2 = \{b_6\}$, $\chi_3 = \{a_2\}$, $\chi_4 = V(P_{\alpha,\beta}) \setminus \{b_2, b_6, a_2\}$ is a resolving partitioning for $V(P_{7,3})$.

For $\beta \geq 7$, $\beta \equiv 3(\text{mod } 4)$ and for the chosen index η such that $0 \leq \eta \leq \alpha - 1$, we shall show that $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{b_{\eta+2\beta}\}$, $\chi_2 = \{b_{\eta+2\xi+2}\}$, $\chi_3 = \{a_{\eta+\beta-1}\}$, $\chi_4 = V(P_{\alpha,\beta}) \setminus \{b_{\eta+2\beta}, b_{\eta+2\xi+2}, a_{\eta+\beta-1}\}$ is a resolving partitioning for $V(P_{\alpha,\beta})$, where $\xi = \frac{\beta-3}{4}$.

For $\beta = 7$ the representations of the vertices of $V(P_{15,7})$, are in the Tables 13 and 14. For $\beta > 7$, $\beta \equiv 3(\text{mod } 4)$ the representations of the vertices of $V(P_{\alpha,\beta})$, are: $r(a_{\eta+\beta+2\xi+2}|\chi) = (2\xi + 2, 2, 2\xi + 3, 0)$, $r(a_{\eta+2\beta}|\chi) = (1, 2\xi + 2, 3, 0)$, $r(b_{\eta}|\chi) = (2, 2\xi + 4, 3, 0)$, $r(b_{\eta+2\xi+1}|\chi) = (2\xi + 4, 2, 2\xi + 2, 0)$, $r(b_{\eta+2\xi+3}|\chi) = (2\xi + 2, 2, 2\xi, 0)$, $r(b_{\eta+2\beta-1}|\chi) = (2, 2\xi + 2, 2, 0)$ and in the Tables 15 and 16.

From these tables, one can see that all the vertices of $P_{\alpha,\beta}$ lying in column 1 of Table 15 and Table 16 have distinct representations with respect to χ . Thus χ is resolving partitioning for $P_{\alpha,\beta}$. Hence, for all $\alpha = 2\beta + 1$, $\beta \geq 7$ and $\beta \equiv 3(\text{mod } 4)$, $pd(P_{\alpha,\beta}) \leq 4$. □

$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	
$a_{\eta+\zeta}$	$\zeta + 2$	$2\xi - \zeta + 3$	$\zeta + 4$	0	$0 \leq \zeta \leq 2\xi - 1$
$a_{\eta+2\xi+\zeta}$	$2\xi + \zeta + 2$	$3 - \zeta$	$2\xi - \zeta + 4$	0	$0 \leq \zeta \leq 1$
$a_{\eta+2\xi+\zeta+2}$	$2\xi - \zeta + 2$	$\zeta + 1$	$2\xi - \zeta$	0	$0 \leq \zeta \leq 2\xi - 1$
$a_{\eta+\beta+\zeta}$	$\zeta + 2$	$2\xi + \zeta + 2$	$\zeta + 1$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\beta+\zeta+2}$	$\zeta + 4$	$2\xi - \zeta + 2$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 1$
$a_{\eta+2\beta-\zeta-1}$	$\zeta + 2$	$2\xi - \zeta + 1$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 1$

Table 15: Representations of outer vertices

$r(\cdot \chi)$	χ_1	χ_2	χ_3	χ_4	
$b_{\eta+\zeta+1}$	$\zeta + 1$	$2\xi - \zeta + 3$	$\zeta + 4$	0	$0 \leq \zeta \leq 2\xi - 1$
$b_{\eta+2\xi+\zeta+4}$	$2\xi - \zeta + 1$	$\zeta + 4$	$2\xi - \zeta - 1$	0	$0 \leq \zeta \leq 2\xi - 4$
$b_{\eta+\beta-\zeta-1}$	$2\zeta + 1$	$2\xi - \zeta + 2$	$\zeta + 1$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+\zeta}$	$2\zeta + 1$	$2\xi + \zeta + 3$	$\zeta + 2$	0	$0 \leq \zeta \leq 1$
$b_{\eta+\beta+\zeta+2}$	$\zeta + 5$	$2\xi - \zeta + 3$	$\zeta + 4$	0	$0 \leq \zeta \leq 2\xi - 2$
$b_{\eta+2\xi+\beta+\zeta+1}$	$2\xi - \zeta + 4$	$3 - 2\zeta$	$2\xi - \zeta + 3$	0	$0 \leq \zeta \leq 1$
$b_{\eta+2\xi+\beta+\zeta+3}$	$2\xi - \zeta + 2$	$2\zeta + 1$	$2\xi - \zeta + 1$	0	$0 \leq \zeta \leq 1$
$b_{\eta+2\beta-\zeta-2}$	$\zeta + 4$	$2\xi - \zeta + 1$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 4$

Table 16: Representations of inner vertices

3. Harary Graph $H_{\psi,\lambda}$

Harary $H_{\psi,\lambda}$, is an ψ -regular graph with order λ the vertex set $V(H_{\psi,\lambda}) = \{\beta_\epsilon : \epsilon = 1, 2, \dots, \lambda\}$, if ψ is even then $\psi = 2\xi \leq \lambda - 1$ for some integer $\xi \leq \frac{\lambda-1}{2}$. For each ϵ ($1 \leq \epsilon \leq \lambda$), we join β_ϵ to $\beta_{\epsilon+1}, \beta_{\epsilon+2}, \dots, \beta_{\epsilon+\xi}$ and to $\beta_{\epsilon-1}, \beta_{\epsilon-2}, \dots, \beta_{\epsilon-\xi}$. If we think of arranging of the vertices $\beta_1, \beta_2, \dots, \beta_\lambda$ cyclically then each vertex β_ϵ is adjacent to the ξ vertices that immediately follow β_ϵ and the ξ vertices that immediately proceed β_ϵ . Following are some bounds on the partition dimension of Harary graph $H_{\psi,\lambda}$.

Theorem 3.1. *Let $H_{\psi,\lambda}$ is a Harary graph with $\lambda \geq 5$, $\lambda \equiv 0, 2, 3 \pmod{\psi}$ and $\psi = 4$. Then the partition dimension of $H_{\psi,\lambda}$ is ≤ 4 .*

Proof. To prove $pd(H_{\psi,\lambda}) \leq 4$ we split the proof into following cases;

Case 1: $\lambda \equiv 0 \pmod{4}$, $\lambda = 4\xi$, $\xi (\geq 2) \in \mathbb{Z}^+$.

Assume the partition resolving set $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{\beta_1\}$, $\chi_2 = \{\beta_3\}$, $\chi_3 = \{\beta_{\lambda-2}\}$, $\chi_4 = V(H_{\psi,\lambda}) \setminus \{\beta_1, \beta_3, \beta_{\lambda-2}\}$ following are the representations of the entire vertex set of $H_{\psi,\lambda}$ with respect to χ .

$$r(\beta_{2\epsilon}|\chi) = \begin{cases} (\epsilon, \epsilon - 1, \epsilon - 1, 0) & \text{if } \epsilon = 2, 3, \dots, \xi; \\ (\xi, \xi, \xi, 0) & \text{if } \epsilon = \xi + 1; \\ (2\xi - \epsilon + 1, 2\xi - \epsilon + 1, 2\xi - \epsilon + 1, 0) & \text{if } \epsilon = \xi + 2, \dots, 2\xi. \end{cases}$$

$$r(\beta_{2\epsilon+1}|\chi) = \begin{cases} (\epsilon, \epsilon, \epsilon - 1, 0) & \text{if } \epsilon = 2, 3, \dots, \xi; \\ (2\xi - \epsilon, 2\xi - \epsilon + 1, 2\xi - \epsilon + 1, 0) & \text{if } \epsilon = \xi + 1, \dots, 2\xi - 1. \end{cases}$$

Case 2: $\lambda \equiv 2 \pmod{4}$, $\lambda = 4\xi + 2$, $\xi \in \mathbb{Z}^+$.

Assume the partition resolving set $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{\beta_1\}$, $\chi_2 = \{\beta_2\}$, $\chi_3 = \{\beta_3\}$, $\chi_4 = V(H_{\psi,\lambda}) \setminus \{\beta_1, \beta_2, \beta_3\}$ following are the representations of the entire vertex set of $H_{\psi,\lambda}$ with respect to χ .

$$r(\beta_{2\epsilon}|\chi) = \begin{cases} (\epsilon, \epsilon - 1, \epsilon - 1, 0) & \text{if } \epsilon = 2, 3, \dots, \xi + 1; \\ (2\xi - \epsilon + 2, 2\xi - \epsilon + 2, 2\xi - \epsilon + 3, 0) & \text{if } \epsilon = \xi + 2, \dots, 2\xi + 1. \end{cases}$$

$$r(\beta_{2\epsilon+1}|\chi) = \begin{cases} (\epsilon, \epsilon, \epsilon - 1, 0) & \text{if } \epsilon = 2, 3, \dots, \xi; \\ (\xi, \xi + 1, \xi, 0) & \text{if } \epsilon = \xi + 1; \\ (2\xi - \epsilon + 1, 2\xi - \epsilon + 2, 2\xi - \epsilon + 2, 0) & \text{if } \epsilon = \xi + 2, \dots, 2\xi. \end{cases}$$

Case 3: $\lambda \equiv 3(\text{mod } 4)$, $\lambda = 4\xi + 3$, $\xi \in \mathbb{Z}^+$.

Assume the partition resolving set $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{\beta_1\}$, $\chi_2 = \{\beta_2\}$, $\chi_3 = \{\beta_3\}$, $\chi_4 = V(H_{\psi,\lambda}) \setminus \{\beta_1, \beta_2, \beta_3\}$ following are the representations of the entire vertex set of $H_{\psi,\lambda}$ with respect to χ .

$$r(\beta_{2\epsilon}|\chi) = \begin{cases} (\epsilon, \epsilon - 1, \epsilon - 1, 0) & \text{if } \epsilon = 2, 3, \dots, \xi + 1; \\ (2\xi - \epsilon + 2, 2\xi - \epsilon + 3, 2\xi - \epsilon + 3, 0) & \text{if } \epsilon = \xi + 2, \dots, 2\xi + 1. \end{cases}$$

$$r(\beta_{2\epsilon+1}|\chi) = \begin{cases} (\epsilon, \epsilon, \epsilon - 1, 0) & \text{if } \epsilon = 2, 3, \dots, \xi + 1; \\ (2\xi - \epsilon + 2, 2\xi - \epsilon + 2, 2\xi - \epsilon + 3, 0) & \text{if } \epsilon = \xi + 2, \dots, 2\xi + 1. \end{cases}$$

The entire vertex set of $H_{\psi,\lambda}$ w.r.t. to the partition resolving set χ have distinct representations hence, $pd(H_{\psi,\lambda}) \leq 4$. $\square$

4. Conclusion

In this paper provide the sharp bounds of the partition dimension for the generalized Peterson graph $P_{\alpha,\beta}$, and we also studied the partition dimension of Harary graph $H_{\psi,\lambda}$, and we concluded that

$$pd(P_{\alpha,\beta}) = pd(H_{\psi,\lambda}) \leq 4.$$

Theorem 2.1 to 2.4 discussed the partition dimension of the generalized Peterson graph when $\beta \equiv 0, 1, 2, 3(\text{mod } 4)$, respectively and the partitioning of Harary graph studied in the Theorem 3.1.

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