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Journal of Prime Research in Mathematics, 18(1) (2022), 96-107

Journal of Prime Research in Mathematics

Quadrics with finite Chen-type Gauss map

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Abstract

In this article, we investigate the family of quadric surfaces in $\mathbb{E}^3$ according to its finite type Gauss map. We prove that spheres, circular cylinders, and planes are the only quadric surfaces with finite Chen type Gauss map corresponding to the first fundamental form I.

Keywords: Surfaces of finite Chen-type, Surfaces in the Euclidean 3-space, Beltrami-Laplace operator, Quadric surfaces.
2010 MSC: 47A75, 53A05.

1. Introduction

In 1983 B. Y. Chen introduced the concept of Euclidean immersions of finite type, [12] and since then the research into this field has been widely spread as one can see in the literature in this area. Let $\boldsymbol{r} : M^m \to \mathbb{E}^n$, be the position vector field of a (connected) submanifold M^m in the Euclidean n-space and $\boldsymbol{H}$ the mean curvature field of M^m. Then, a well-known formula is

$$\Delta^I \boldsymbol{r} = -n\boldsymbol{H}.$$

where Δ^I denotes the second Laplace operator corresponding to the first fundamental form I of M^m. From the above formula, it can be seen that M^m is a minimal submanifold when all coordinate functions of $\boldsymbol{r}$, are eigenfunctions of Δ^I with eigenvalue $\mu = 0$.

In general, a submanifold M^m is called of finite type if all components of the vector $\boldsymbol{r}$ of M^m, can be written as a finite sum of eigenfunctions of the Laplacian Δ^I. Following this, a surface S in $\mathbb{E}^3$ is of finite type if the components of $\boldsymbol{r}$ are of finite sum of eigenfunctions of its Laplacian Δ^I. Geometrically, in the Euclidean 3-space $\mathbb{E}^3$, a surface M is said to have finite type if it is homeomorphic to a closed surface with a finite number of points removed. An important problem in classical differential geometry is the classification of properly embedded finite type surfaces of constant mean curvature in $\mathbb{E}^3$. For more details see [13].

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Received : 23 January 2022; *Accepted:* 09 July 2022; *Published Online:* 22 August 2022

In this context, Chen and Piccini in [15], introduced in the same way the theory of submanifolds of finite type Gauss map. A special case for $\mathbb{E}^3$ is classifying all surfaces in $\mathbb{E}^3$ with finite type Gauss map.

Concerning this, in [10] two kinds of surfaces were studied, namely, tubes and ruled surfaces. In [7], it was proved that the only spiral surfaces in $\mathbb{E}^3$ with finite type Gauss maps are the planes. Further, C. Baikoussis and others proved in [9] that for the compact cyclides of Dupin and the noncompact also, the Gauss map is of infinite type. Many results concerning the theory of Gauss map, one can find in [2, 4, 5, 6, 8, 11, 16, 17, 19].

A new type of research one can follow, is computing the Laplace operator by using the definition of the fractional vector operators [18]. In this present paper, we will study the class of quadric surfaces in $\mathbb{E}^3$ in the sense of its finite type Gauss map. Our main result is

Theorem 1.1. *The only quadric surfaces (even locally) in $\mathbb{E}^3$ whose Gauss map $\boldsymbol{N}$ is of finite type are the planes, the circular cylinders and the spheres.*

In [14], Chen classified the class of quadric surfaces of finite type. A similar study was done in [3] by investigating the same class according to the third fundamental form.

2. Basic concepts

Let $\boldsymbol{r} = \boldsymbol{r}(u, v)$ be an isometric immersion of a surface S in $\mathbb{E}^3$ referred to any coordinate system. The inner product on $\mathbb{E}^3$ is

$$\boldsymbol{X} \cdot \boldsymbol{Y} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

where $\boldsymbol{X} = (x_1, x_2, x_3)$, and $\boldsymbol{Y} = (y_1, y_2, y_3)$. As it's known the first fundamental form of S is given by

$$I = Edu^2 + 2Fdudv + Gdv^2.$$

The Laplacian Δ^I is given as follows

$$\Delta^I = -\frac{1}{\sqrt{EG-F^2}}\left[\frac{\partial}{\partial u}\left(\frac{G\frac{\partial}{\partial u} - F\frac{\partial}{\partial v}}{\sqrt{EG-F^2}}\right) - \frac{\partial}{\partial v}\left(\frac{F\frac{\partial}{\partial u} - E\frac{\partial}{\partial v}}{\sqrt{EG-F^2}}\right)\right]. \tag{2.1}$$

We denote by $\boldsymbol{N}$ the Gauss map of S, and K, H, the Gauss and mean curvature of S respectively. Applying the above equation for the position vectors $\boldsymbol{r}$ and $\boldsymbol{N}$ we obtain

$$\Delta^I \boldsymbol{r} = -2H\boldsymbol{N}. \tag{2.2}$$

$$\Delta^I \boldsymbol{N} = grad^I 2H + (4H^2 - 2K)\boldsymbol{N}, \tag{2.3}$$

where $grad^I 2H = \nabla^I(2H, \boldsymbol{r})$ is the first Laplace operator with regarding the first fundamental form. From (2.2) we know the following two facts [20]

- S is part of a sphere exactly when all coordinate functions of $\boldsymbol{r}$ are eigenfunctions of Δ^I with a nonzero eigenvalue.
- S is minimal exactly when all coordinate functions of $\boldsymbol{r}$ are eigenfunctions of Δ^I with eigenvalue $\mu = 0$.

Now we prove our main result.

3. Quadric surfaces

We consider now a quadric surface S in $\mathbb{E}^3$. Then S is either ruled, or of one of the following two types [1]

$$Z^2 = c - pX^2 - qY^2, \quad p, q, c \in \mathbb{R}, \quad pq \neq 0, \quad c > 0, \tag{3.1}$$

or

$$Z = \frac{p}{2}X^2 + \frac{q}{2}Y^2, \quad p, q \in \mathbb{R}, \quad p, q > 0. \tag{3.2}$$

If S is ruled, then it follows from [10] S is a plane or a circular cylinder. We first prove that, If a quadric of the form (3.1) is of finite type, then $p = q = -1$, that is, S in this case is a part of a sphere. Next we prove that a quadric of the form (3.2) is of infinite type.

3.1. Quadric surfaces of the 1^{st} type

This type of surfaces can be parameterized in the following way

$$\boldsymbol{r}(u,v)=\left(u,v,\sqrt{c+pu^2+qv^2}\right).$$

For simplicity, we denote $c+pu^2+qv^2$ by σ. The partial derivatives $\{\boldsymbol{r}_u,\boldsymbol{r}_v\}$ of S are

$$\boldsymbol{r_u}=\left(1,0,\frac{pu}{\sqrt{\sigma}}\right),\qquad \boldsymbol{r_v}=\left(0,1,\frac{qv}{\sqrt{\sigma}}\right).$$

The components of the first fundamental form are defined as follows

$$E=<\boldsymbol{r_u},\boldsymbol{r_u}>=1+\frac{(pu)^2}{\sigma},$$
$$F=<\boldsymbol{r_u},\boldsymbol{r_v}>=\frac{pquv}{\sigma},$$
$$G=<\boldsymbol{r_v},\boldsymbol{r_v}>=1+\frac{(qv)^2}{\sigma}.$$

Therefore the Laplace operator Δ^I of S is given as follows

$$\begin{aligned}\Delta^I \;=\;& \frac{p^2u^2}{\Phi}\Big(\frac{\partial^2}{\partial u^2}-\frac{p(p+1)u}{\Phi}\frac{\partial}{\partial u}\Big)-\frac{\partial^2}{\partial u^2}+\frac{pu}{\Phi}(p+q)\frac{\partial}{\partial u}\\ &+\frac{q^2v^2}{\Phi}\Big(\frac{\partial^2}{\partial v^2}-\frac{q(q+1)v}{\Phi}\frac{\partial}{\partial v}\Big)-\frac{\partial^2}{\partial v^2}+\frac{qv}{\Phi}(p+q)\frac{\partial}{\partial v}\\ &+\frac{2pquv}{\Phi}\frac{\partial^2}{\partial u\partial v}-\frac{pquv}{\Phi^2}\Big(q(q+1)v\frac{\partial}{\partial u}+p(p+1)u\frac{\partial}{\partial v}\Big),\end{aligned}\tag{3.3}$$

where $\Phi:=EG-F^2=p(p+1)u^2+q(q+1)v^2+c$.

For the normal vector $\boldsymbol{N}$ of S, we have

$$\boldsymbol{N}=\frac{\boldsymbol{r_u}\times\boldsymbol{r_v}}{\sqrt{\Phi}},$$

which becomes after simple calculations

$$\boldsymbol{N}=\Big(-\frac{pu}{\sqrt{\Phi}},-\frac{qv}{\sqrt{\Phi}},\frac{\sqrt{\sigma}}{\sqrt{\Phi}}\Big).$$

We denote by (N_1,N_2,N_3) the components of the vector $\boldsymbol{N}$. For the partial derivatives of N_1, we have

$$\frac{\partial N_1}{\partial u}=-\frac{p}{\sqrt{\Phi}}+\frac{p^2(p+1)u^2}{\Phi\sqrt{\Phi}},$$
$$\frac{\partial N_1}{\partial v}=\frac{pq(q+1)uv}{\Phi\sqrt{\Phi}},$$
$$\frac{\partial^2 N_1}{\partial u^2}=\frac{3p^2(p+1)u}{\Phi\sqrt{\Phi}}-\frac{3p^3(p+1)^2u^3}{\Phi^2\sqrt{\Phi}},$$
$$\frac{\partial^2 N_1}{\partial v^2}=\frac{pq(q+1)u}{\Phi\sqrt{\Phi}}-\frac{3pq^2(q+1)^2uv^2}{\Phi^2\sqrt{\Phi}},$$
$$\frac{\partial^2 N_1}{\partial u\partial v}=\frac{pq(q+1)v}{\Phi\sqrt{\Phi}}-\frac{3p^2q(p+1)(q+1)u^2v}{\Phi^2\sqrt{\Phi}},$$

Similar relations can be drawn for the partial derivatives of N_2. Inserting N_1 and N_2 in (3.3), gives

$$\begin{aligned}\Delta^I N_1 = \Delta^I\Big(-\frac{pu}{\sqrt{\Phi}}\Big) &= -\frac{4pu}{\Phi^{\frac{7}{2}}}\left[p^2(p+1)u^2+q^2(q+1)v^2\right]^2\\ &\quad+\frac{pu}{\Phi^{\frac{5}{2}}}F(u,v),\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}F(u,v) &= p(p+1)[4p^2-q(q+p)]u^2\\ &\quad+q(q+1)(3pq-4q-4p^2-2q^2-3p)v^2\\ &\quad-[p(p+q)+3p(p+1)+q(q+1)]c,\end{aligned}$$

$$\begin{aligned}\Delta^I N_2 = \Delta^I\Big(-\frac{qv}{\sqrt{\Phi}}\Big) &= -\frac{4qv}{\Phi^{\frac{7}{2}}}\left[p^2(p+1)u^2+q^2(q+1)v^2\right]^2\\ &\quad+\frac{qv}{\Phi^{\frac{5}{2}}}G(u,v),\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}G(u,v) &= q(q+1)(4q^2-p(p+q)]v^2\\ &\quad+q(q+1)(3pq-4p-4q^2-2p^2-3q)u^2\\ &\quad-[q(p+q)+p(p+1)+3q(q+1)]c.\end{aligned}$$

We write (3.4) and (3.5) as follows

$$\Delta^I N_1 = -\frac{4p^5(p+1)^2u^5}{\Phi^{\frac{7}{2}}}+\frac{pu}{\Phi^{\frac{7}{2}}}f_{11}(u,v)+\frac{pu}{\Phi^{\frac{5}{2}}}f_{12}(u,v), \tag{3.6}$$

where

$$f_{11}(u,v) = -4q^2(q+1)\Big(2p^2(p+1)u^2+q^2(q+1)v^2\Big)v^2,$$

$$\begin{aligned}f_{12}(u) &= p(p+1)[4p^2-q(q+p)]u^2\\ &\quad+q(q+1)(3pq-4q-4p^2-2q^2-3p)v^2\\ &\quad-[p(p+q)+3p(p+1)+q(q+1)]c.\end{aligned}$$

Here we have $f_{11}(u,0)=0$, and $f_{12}(u,v)$ is a polynomial of the parameters u,v, with $deg(f_{12}(u,0))\leq 2$. Similarly,

$$\Delta^I N_2 = -\frac{4q^5(q+1)^2v^5}{\Phi^{\frac{7}{2}}}+\frac{qv}{\Phi^{\frac{7}{2}}}g_{11}(u,v)+\frac{qv}{\Phi^{\frac{5}{2}}}g_{12}(u,v), \tag{3.7}$$

where

$$g_{11}(u,v) = -4p^2(p+1)\Big(2q^2(q+1)v^2+p^2(p+1)u^2\Big)u^2,$$

$$\begin{aligned}g_{12}(u,v) &= q(q+1)[4q^2-p(p+q)]v^2\\ &\quad+p(p+1)(3pq-4p-4q^2-2p^2-3q)u^2\\ &\quad-[q(p+q)+p(p+1)+3q(q+1)]c.\end{aligned}$$

Here we also have $g_{11}(0,v)=0$, and $g_{12}(u,v)$ is a polynomial of the parameters u,v with $deg(g_{12}(0,v))\leq 2$. Applying (3.3) for the relations (3.6) and (3.7), we find

$$(\Delta^I)^2N_1=-\frac{(4)(70)p^9(p+1)^4u^9}{\Phi^{\frac{13}{2}}}+\frac{pu}{\Phi^{\frac{13}{2}}}f_{21}(u,v)+\frac{pu}{\Phi^{\frac{11}{2}}}f_{22}(u,v), \tag{3.8}$$

$$(\Delta^I)^2N_2=-\frac{(4)(70)q^9(q+1)^4v^9}{\Phi^{\frac{13}{2}}}+\frac{qv}{\Phi^{\frac{13}{2}}}g_{21}(u,v)+\frac{qv}{\Phi^{\frac{11}{2}}}g_{22}(u,v), \tag{3.9}$$

where $f_{21}(u,0)=g_{21}(0,v)=0$, $f_{22}(u,v)$ and $g_{22}(u,v)$ are polynomials of the parameters u,v of degree at most 6.

We need the following two lemmas which can be easily proved by a direct computation

Lemma 3.1. *The relation*

$$\begin{aligned}\left(\Delta^I\right)^k N_1 \;&=\; -\left(\prod_{i=1}^{k}(6i-5)(6i-2)\right)\left(\frac{p^{4k+1}(p+1)^{2k}u^{4k+1}}{\Phi^{\frac{6k+1}{2}}}\right)\\ &\quad+\frac{pu}{\Phi^{\frac{6k+1}{2}}}f_{k1}(u,v)+\frac{pu}{\Phi^{\frac{6k-1}{2}}}f_{k2}(u,v)\end{aligned} \tag{3.10}$$

holds true, where $f_{k1}(u,0)=0$*, and* $f_{k2}(u,v)$ *is a polynomial of the parameters* u,v *with* $\deg(f_{k2}(u,0))=4k-2$*.*

Proof. The proof goes by induction on k. For $k=1$ the formula follows immediately from (3.6). Suppose the Lemma is true for $k-1$. Then

$$\begin{aligned}(\Delta^I)^{k-1}N_1 \;&=\; -\Big(\prod_{i=1}^{k-1}(6i-5)(6i-2)\Big)\Big(\frac{p^{4k-3}(p+1)^{2k-2}u^{4k-3}}{\Phi^{\frac{6k-5}{2}}}\Big)\\ &\quad+\frac{pu}{\Phi^{\frac{6k-5}{2}}}f_{(k-1)1}(u,v)+\frac{pu}{\Phi^{\frac{6k-7}{2}}}f_{(k-1)2}(u,v),\end{aligned}$$

where $f_{(k-1)1}(u,0)=0$, and $f_{(k-1)2}(u,v)$ is a polynomial of the parameters u,v with $\deg(f_{(k-1)2}(u,0))=$

$4k-6$. Taking into account (3.3) and $v=0$ we obtain

$$
\begin{aligned}
&\left(\Delta^I\right)^k N_1 = \Delta^I\Big(\left(\Delta^I\right)^{k-1} N_1\Big) = \\
&-\Big(\prod_{i=1}^{k-1}(6i-5)(6i-2)\Big)\frac{p^2u^2}{\Phi}\left[\frac{\partial^2}{\partial u^2}\Big(\frac{p^{4k-3}(p+1)^{2k-2}u^{4k-3}}{\Phi^{\frac{6k-5}{2}}}\Big)\right.\\
&\left.-\frac{p(p+1)u}{\Phi}\frac{\partial}{\partial u}\Big(\frac{p^{4k-3}(p+1)^{2k-2}u^{4k-3}}{\Phi^{\frac{6k-5}{2}}}\Big)\right]\\
&+\Big(\prod_{i=1}^{k-1}(6i-5)(6i-2)\Big)\frac{\partial^2}{\partial u^2}\Big(\frac{p^{4k-3}(p+1)^{2k-2}u^{4k-3}}{\Phi^{\frac{6k-5}{2}}}\Big)\\
&-\Big(\prod_{i=1}^{k-1}(6i-5)(6i-2)\Big)\frac{pu}{\Phi}(p+q)\frac{\partial}{\partial u}\Big(\frac{p^{4k-3}(p+1)^{2k-2}u^{4k-3}}{\Phi^{\frac{6k-5}{2}}}\Big)\\
&-\Big(\prod_{i=1}^{k-1}(6i-5)(6i-2)\Big)\frac{p^2u^2}{\Phi}\left[\frac{\partial^2}{\partial u^2}\Big(\frac{pu}{\Phi^{\frac{6k-5}{2}}}f_{(k-1)1}(u,v)+\frac{pu}{\Phi^{\frac{6k-7}{2}}}f_{(k-1)2}(u,v)\Big)\right.\\
&\left.-\frac{p(p+1)u}{\Phi}\frac{\partial}{\partial u}\Big(\frac{pu}{\Phi^{\frac{6k-5}{2}}}f_{(k-1)1}(u,v)+\frac{pu}{\Phi^{\frac{6k-7}{2}}}f_{(k-1)2}(u,v)\Big)\right]\\
&+\Big(\prod_{i=1}^{k-1}(6i-5)(6i-2)\Big)\frac{\partial^2}{\partial u^2}\Big(\frac{pu}{\Phi^{\frac{6k-5}{2}}}f_{(k-1)1}(u,v)+\frac{pu}{\Phi^{\frac{6k-7}{2}}}f_{(k-1)2}(u,v)\Big)\\
&-\Big(\prod_{i=1}^{k-1}(6i-5)(6i-2)\Big)\frac{pu}{\Phi}(p+q)\frac{\partial}{\partial u}\Big(\frac{pu}{\Phi^{\frac{6k-5}{2}}}f_{(k-1)1}(u,v)+\frac{pu}{\Phi^{\frac{6k-7}{2}}}f_{(k-1)2}(u,v)\Big)
\end{aligned}
$$

where after a lot of calculations and taking into account

$$
\begin{aligned}
&\Delta^I\Big(\frac{p^m(p+1)^n u^m}{\Phi^{\frac{r}{2}}}\Big) = (6m-5)(6m-2)\frac{p^{4m+1}(p+1)^{n+2}u^{4m+1}}{\Phi^{\frac{6r+1}{2}}}\\
&+\frac{pu}{\Phi^{\frac{6r+1}{2}}}f_{r1}(u,v)+\frac{pu}{\Phi^{\frac{6k-1}{2}}}f_{r2}(u,v),
\end{aligned}
$$

we finally get (3.10). □

Similarly, we can prove

Lemma 3.2. *The relation*

$$
\begin{aligned}
\left(\Delta^I\right)^k N_2 \;&=\; -\left(\prod_{i=1}^{k}(6i-5)\,(6i-2)\right)\left(\frac{q^{4k+1}\,(q+1)^{2k}\,v^{4k+1}}{\Phi^{\frac{6k+1}{2}}}\right)\\
&\quad+\frac{qv}{\Phi^{\frac{6k+1}{2}}}g_{k1}(u,v)+\frac{qv}{\Phi^{\frac{6k-1}{2}}}g_{k2}(u,v) \qquad (3.11)
\end{aligned}
$$

holds true, where $g_{k1}(0,v)=0$, and $g_{k2}(u,v)$ is a polynomial of the parameters u,v with $\deg(g_{k2}(u,v)) = 4k-2$.

We suppose now that the Gauss map of S is of finite type k. Then there exist real numbers $c_1,\ldots,c_k$ such that [12]

$$\left(\Delta^I\right)^k \boldsymbol{N}+c_1\left(\Delta^I\right)^{k-1}\boldsymbol{N}+\ldots+c_{k-1}\Delta^I\boldsymbol{N}+c_k\boldsymbol{N}=\boldsymbol{0}. \qquad (3.12)$$

Inserting the coordinate functions N_1 and N_2 of $\boldsymbol{N}$ of S in (3.12), we get

$$\left(\Delta^I\right)^k N_1 + c_1\left(\Delta^I\right)^{k-1} N_1 + \cdots + c_{k-1}\Delta^I N_1 + c_k N_1 = 0, \tag{3.13}$$

$$\left(\Delta^I\right)^k N_2 + c_1\left(\Delta^I\right)^{k-1} N_2 + \cdots + c_{k-1}\Delta^I N_2 + c_k N_2 = 0, \tag{3.14}$$

Using Lemma 3.1 and relations (3.6), (3.8), relation (3.13) becomes

$$\begin{aligned}
&-\left(\prod_{i=1}^{k}(6i-5)(6i-2)\right)\left(\frac{p^{4k+1}(p+1)^{2k}u^{4k+1}}{\Phi^{\frac{6k+1}{2}}}\right) + \frac{pu}{\Phi^{\frac{6k+1}{2}}} f_{k1}(u,v)\\
&+\frac{pu}{\Phi^{\frac{6k-1}{2}}} f_{k2}(u,v) - c_1\left(\prod_{i=1}^{k-1}(6i-5)(6i-2)\right)\left(\frac{p^{4k-3}(p+1)^{2k-2}u^{4k-3}}{\Phi^{\frac{6k-5}{2}}}\right)\\
&+\frac{c_1 pu}{\Phi^{\frac{6k-5}{2}}} f_{k-1\,1}(u,v) + \frac{c_1 pu}{\Phi^{\frac{6k-7}{2}}} f_{k-1\,2}(u,v) + \cdots - \frac{70c_{k-2}p^9(p+1)^4u^9}{\Phi^{\frac{13}{2}}}\\
&+\frac{c_{k-2}pu}{\Phi^{\frac{13}{2}}} f_{21}(u,v) + \frac{c_{k-2}pu}{\Phi^{\frac{11}{2}}} f_{22}(u,v) - \frac{4c_{k-1}p^5(p+1)^2u^5}{\Phi^{\frac{7}{2}}}\\
&+\frac{c_{k-1}pu}{\Phi^{\frac{7}{2}}} f_{11}(u,v) + \frac{c_{k-1}pu}{\Phi^{\frac{5}{2}}} f_{12}(u,v) - \frac{c_k pu}{\Phi^{\frac{1}{2}}} = 0.
\end{aligned} \tag{3.15}$$

If we put $v=0$ in (3.15), multiplying (3.15) by $\Phi^{\frac{6k+1}{2}}$ and replacing Φ by $[p(p+1)u^2+c]$, then (3.15) becomes

$$\begin{aligned}
&-\left(\prod_{i=1}^{k}(6i-5)(6i-2)\right)\left(p^{4k}(p+1)^{2k}u^{4k}\right) + [p(p+1)u^2+c]^2 f_{k2}(u,0)\\
&-c_1\left(\prod_{i=1}^{k-1}(6i-5)(6i-2)\right)\left(p^{4k-4}(p+1)^{2k-2}u^{4k-4}\right)\\
&+c_1[p(p+1)u^2+c]^2 f_{k2}(u,0) + \cdots - 70c_{k-2}p^8(p+1)^4u^8[p(p+1)u^2+c]^{3k-6}\\
&+c_{k-2}[p(p+1)u^2+c]^{3k-5} f_{22}(u,0) - 4c_{k-1}p^4(p+1)^2u^4[p(p+1)u^2+c]^{3k-3}\\
&+c_{k-1}[p(p+1)u^2+c]^{3k-2} f_{12}(u,0) - c_k[p(p+1)u^2+c]^{3k} = 0.
\end{aligned} \tag{3.16}$$

Relation (3.16) is a nontrivial polynomial in u of degree $6k$ with constant coefficients. Since this equation holds true for all values of u so we must have all coefficients of the polynomial equal zeros, taking into account $p \neq 0$, a case that cannot be satisfied unless only $p+1=0$.

Following the same procedure for relation (3.14) we must have also $q+1=0$. Hence equation (3.16) becomes $u^2+v^2=c$ which is a part of a sphere.

When $p=-1$ and $q=-1$, then $\Phi=c$, and $\boldsymbol{N}=\left(\frac{u}{\sqrt{c}}, \frac{v}{\sqrt{c}}, \frac{\sqrt{\sigma}}{\sqrt{c}}\right)$.

The Gauss and mean curvature of S become

$$K=\frac{1}{c}, \quad 2H=\frac{1}{\sqrt{c}}.$$

Thus from (2.3) we have

$$\Delta^I \boldsymbol{N} = \frac{1}{c}\boldsymbol{N}.$$

That is, the Gauss map of S is of finite type 1 with corresponding eigenvalue $\mu=\frac{1}{c}$.

3.2. Quadric surfaces of the 2^{nd} type

This type of surfaces can be parameterized in the following way

$$\boldsymbol{r}(u,v)=\Big(u,v,\frac{p}{2}u^2+\frac{q}{2}v^2\Big). \tag{3.17}$$

We have

$$\boldsymbol{r_u}=(1,0,pu)\,,\qquad \boldsymbol{r_v}=(0,1,qv)\,.$$

The components of the first fundamental form are defined as follows

$$E=(pu)^2+1,\quad F=pquv,\quad G=(qv)^2+1.$$

Therefore the Laplace operator Δ^I of S is given by

$$\begin{aligned}\Delta^I \;=\;& \frac{p^2u^2}{g}\Big(\frac{\partial^2}{\partial u^2}-\frac{p^2u}{g}\frac{\partial}{\partial u}\Big)-\frac{\partial^2}{\partial u^2}+\frac{pu}{g}(p+q)\frac{\partial}{\partial u}\\ &+\frac{q^2v^2}{g}\Big(\frac{\partial^2}{\partial v^2}-\frac{q^2v}{g}\frac{\partial}{\partial v}\Big)-\frac{\partial^2}{\partial v^2}+\frac{qv}{g}(p+q)\frac{\partial}{\partial v}\\ &+\frac{2pquv}{g}\frac{\partial^2}{\partial u\partial v}-\frac{pquv}{g^2}\Big(q^2v\frac{\partial}{\partial u}+p^2u\frac{\partial}{\partial v}\Big),\end{aligned} \tag{3.18}$$

where

$$g:\;=EG-F^2=p^2\,u^2+q^2\,v^2+1$$

The normal vector field of S is

$$\boldsymbol{N}=\Big(-\frac{pu}{\sqrt{g}},-\frac{qv}{\sqrt{g}},\frac{1}{\sqrt{g}}\Big).$$

We denote by (N_1,N_2,N_3) the components of $\boldsymbol{N}$, For the partial derivatives of N_1, we have

$$\frac{\partial N_1}{\partial u}=-\frac{p}{\sqrt{g}}+\frac{p^3u^2}{g\sqrt{g}},\qquad \frac{\partial N_1}{\partial v}=\frac{pq^2uv}{g\sqrt{g}},$$

$$\frac{\partial^2 N_1}{\partial u^2}=\frac{3p^3u}{g\sqrt{g}}-\frac{3p^5u^3}{g^2\sqrt{g}},\qquad \frac{\partial^2 N_1}{\partial v^2}=\frac{pq^2u}{g\sqrt{g}}-\frac{3pq^4uv^2}{g^2\sqrt{g}},$$

$$\frac{\partial^2 N_1}{\partial u\partial v}=\frac{pq^2v}{g\sqrt{g}}-\frac{3p^3q^2u^2v}{g^2\sqrt{g}},$$

Similar relations can be drawn for the partial derivatives of N_2. Applying (3.18) for the functions N_1 and N_2, we find

$$\begin{aligned}&\Delta^IN_1=\Delta^I\Big(-\frac{pu}{\sqrt{g}}\Big)=-\frac{4pu}{g^{\frac{7}{2}}}\left[p^3u^2+q^3v^2\right]^2\\ &+\frac{pu}{g^{\frac{5}{2}}}\Big((4p^2-q^2)p^2u^2+(4q^2-p^2+3pq)q^2v^2-(4p^2+q^2+pq)\Big).\end{aligned} \tag{3.19}$$

$$\begin{aligned}&\Delta^IN_2=\Delta^I\Big(-\frac{qv}{\sqrt{g}}\Big)=-\frac{4qv}{g^{\frac{7}{2}}}\left[p^3u^2+q^3v^2\right]^2\\ &+\frac{qv}{g^{\frac{5}{2}}}\Big((4q^2-p^2)q^2v^2+(4p^2-q^2+3pq)p^2u^2-(p^2+4q^2+pq)\Big).\end{aligned} \tag{3.20}$$

We write (3.19) and (3.20) as follows

$$\Delta^I N_1 = -\frac{4p^7u^5}{g^{\frac{7}{2}}} + \frac{pu}{g^{\frac{7}{2}}} f_{11}(u,v) + \frac{pu}{g^{\frac{5}{2}}} f_{12}(u,v), \tag{3.21}$$

where

$$f_{11}(u,v) = -4q^3\Big(2p^3u^2 + q^3v^2\Big)v^2,$$

$$f_{12}(u) = (4p^2 - q^2)p^2u^2 + (4q^2 - p^2 + 3pq)q^2v^2 - (4p^2 + q^2 + pq).$$

Here we have $f_{11}(u,0) = 0$, and $f_{12}(u,v)$ is a polynomial of the parameters u, v, with $deg(f_{12}(u,0)) \leq 2$. Similarly,

$$\Delta^I N_2 = -\frac{4q^7v^5}{g^{\frac{7}{2}}} + \frac{qv}{g^{\frac{7}{2}}} g_{11}(u,v) + \frac{qv}{g^{\frac{5}{2}}} g_{12}(u,v), \tag{3.22}$$

where

$$g_{11}(u,v) = -4p^3\Big(p^3u^2 + 2q^3v^2\Big)u^2,$$

$$g_{12}(u,v) = (4q^2 - p^2)q^2v^2 + (4p^2 - q^2 + 3pq)p^2u^2 - (p^2 + 4q^2 + pq).$$

Here we also have $g_{11}(0,v) = 0$, and $g_{12}(u,v)$ is a polynomial of the parameters u, v with $deg(g_{12}(0,v)) \leq 2$. Applying (3.18) for the relations (3.19) and (3.20), we find

$$(\Delta^I)^2 N_1 = -\frac{(4)(70)p^{13}u^9}{g^{\frac{13}{2}}} + \frac{pu}{g^{\frac{13}{2}}} f_{21}(u,v) + \frac{pu}{g^{\frac{11}{2}}} f_{22}(u,v), \tag{3.23}$$

where $f_{21}(u,0) = 0$, $f_{22}(u,v)$ is a polynomial of the parameters u, v with $deg(f_{22}(u,0)) \leq 6$. Similarly,

$$(\Delta^I)^2 N_2 = -\frac{(4)(70)q^{13}v^9}{g^{\frac{13}{2}}} + \frac{qv}{g^{\frac{13}{2}}} g_{21}(u,v) + \frac{qv}{g^{\frac{11}{2}}} g_{22}(u,v), \tag{3.24}$$

where similarly, we find that $g_{21}(0,v) = 0$, and $g_{22}(u,v)$ is a polynomial of the parameters u, v with $deg(g_{22}(0,v)) \leq 6$. In general it can be easily proved

Lemma 3.3. *The relation*

$$\begin{aligned} \left(\Delta^I\right)^k N_1 = & -\left(\prod_{i=1}^{k} (6i-5)(6i-2)\right)\left(\frac{p^{6k+1}u^{4k+1}}{g^{\frac{6k+1}{2}}}\right) \\ & + \frac{pu}{g^{\frac{6k+1}{2}}} f_{k1}(u,v) + \frac{pu}{g^{\frac{6k-1}{2}}} f_{k2}(u,v) \end{aligned} \tag{3.25}$$

holds true, where $f_{k1}(u,0) = 0$, and $f_{k2}(u,v)$ are polynomials of the parameters u, v with $\deg(f_{k2}(u,0)) = 4k - 2$.

Proof. The proof goes by induction on k. For $k=1$ the formula follows immediately from (3.7). Suppose the Lemma is true for $k-1$. Then

$$\begin{aligned}(\Delta^I)^{k-1}N_1 &= -\Big(\prod_{i=1}^{k-1}(6i-5)(6i-2)\Big)\Big(\frac{p^{6k-5}(p+1)^{2k-4}u^{4k-3}}{g^{\frac{6k-5}{2}}}\Big)\\ &+\frac{pu}{g^{\frac{6k-5}{2}}}f_{(k-1)1}(u,v)+\frac{pu}{g^{\frac{6k-7}{2}}}f_{(k-1)2}(u,v),\end{aligned}$$

where $f_{(k-1)1}(u,0)=0$, and $f_{(k-1)2}(u,v)$ are polynomials of the parameters u,v with $\deg(f_{(k-1)2}(u,0)) = 4k-6$. Taking into account (3.3) and $v=0$ we obtain

$$\begin{aligned}&\left(\Delta^I\right)^k N_1=\Delta^I\Big(\left(\Delta^I\right)^{k-1}N_1\Big)=\\ &-\Big(\prod_{i=1}^{k-1}(6i-5)(6i-2)\Big)\frac{p^2u^2}{g}\left[\frac{\partial^2}{\partial u^2}\Big(\frac{p^{6k-5}u^{4k-3}}{g^{\frac{6k-5}{2}}}\Big)-\frac{p^2u}{g}\frac{\partial}{\partial u}\Big(\frac{p^{6k-5}u^{4k-3}}{g^{\frac{6k-5}{2}}}\Big)\right]\\ &+\Big(\prod_{i=1}^{k-1}(6i-5)(6i-2)\Big)\frac{\partial^2}{\partial u^2}\Big(\frac{p^{6k-5}u^{4k-3}}{g^{\frac{6k-5}{2}}}\Big)\\ &-\Big(\prod_{i=1}^{k-1}(6i-5)(6i-2)\Big)\frac{pu}{g}(p+q)\frac{\partial}{\partial u}\Big(\frac{p^{6k-5}u^{4k-3}}{g^{\frac{6k-5}{2}}}\Big)\\ &-\Big(\prod_{i=1}^{k-1}(6i-5)(6i-2)\Big)\frac{p^2u^2}{g}\left[\frac{\partial^2}{\partial u^2}\Big(\frac{pu}{g^{\frac{6k-5}{2}}}f_{(k-1)1}(u,v)+\frac{pu}{g^{\frac{6k-7}{2}}}f_{(k-1)2}(u,v)\Big)\right.\\ &\left.-\frac{p^2u}{g}\frac{\partial}{\partial u}\Big(\frac{pu}{\Phi^{\frac{6k-5}{2}}}f_{(k-1)1}(u,v)+\frac{pu}{g^{\frac{6k-7}{2}}}f_{(k-1)2}(u,v)\Big)\right]\\ &+\Big(\prod_{i=1}^{k-1}(6i-5)(6i-2)\Big)\frac{\partial^2}{\partial u^2}\Big(\frac{pu}{g^{\frac{6k-5}{2}}}f_{(k-1)1}(u,v)+\frac{pu}{g^{\frac{6k-7}{2}}}f_{(k-1)2}(u,v)\Big)\\ &-\Big(\prod_{i=1}^{k-1}(6i-5)(6i-2)\Big)\frac{pu}{g}(p+q)\frac{\partial}{\partial u}\Big(\frac{pu}{g^{\frac{6k-5}{2}}}f_{(k-1)1}(u,v)+\frac{pu}{g^{\frac{6k-7}{2}}}f_{(k-1)2}(u,v)\Big)\end{aligned}$$

where after a lot of calculations and taking into account

$$\begin{aligned}&\Delta^I\Big(\frac{p^{m+1}u^m}{g^{\frac{r}{2}}}\Big)=(6m-5)(6m-2)\frac{p^{4m+4}u^{4m+1}}{g^{\frac{6r+1}{2}}}\\ &+\frac{pu}{g^{\frac{6r+1}{2}}}f_{r1}(u,v)+\frac{pu}{g^{\frac{6k-1}{2}}}f_{r2}(u,v),\end{aligned}$$

we finally get (3.25). □

Similarly, we have

Lemma 3.4. *The relation*

$$\begin{aligned}\left(\Delta^I\right)^k N_2 &= -\left(\prod_{i=1}^{k}(6i-5)(6i-2)\right)\left(\frac{q^{6k+1}v^{4k+1}}{g^{\frac{6k+1}{2}}}\right)\\ &+\frac{qv}{g^{\frac{6k+1}{2}}}g_{k1}(u,v)+\frac{qv}{\Phi^{\frac{6k-1}{2}}}g_{k2}(u,v)\end{aligned}\tag{3.26}$$

holds true, where $g_{k1}(0,v)=0$, *and* $g_{k2}(u,v)$ *is a polynomial of the parameters* u,v *with* $\deg(g_{k2}(u,v)) = 4k-2$.

From relation (3.13), using Lemma 3.3 and relations (3.19), (3.23), we get

$$\begin{aligned}
&-\left(\prod_{i=1}^{k}(6i-5)(6i-2)\right)\left(\frac{p^{6k+1}u^{4k+1}}{g^{\frac{6k+1}{2}}}\right)+\frac{pu}{g^{\frac{6k+1}{2}}}f_{k1}(u,v)\\
&+\frac{pu}{g^{\frac{6k-1}{2}}}f_{k2}(u,v)-c_1\left(\prod_{i=1}^{k-1}(6i-5)(6i-2)\right)\left(\frac{p^{6k-5}u^{4k-3}}{g^{\frac{6k-5}{2}}}\right)\\
&+\frac{c_1pu}{g^{\frac{6k-5}{2}}}f_{k-1\,1}(u,v)+\frac{c_1pu}{g^{\frac{6k-7}{2}}}f_{k-1\,2}(u,v)+\cdots-\frac{70c_{k-2}p^{13}u^9}{g^{\frac{13}{2}}}\\
&+\frac{c_{k-2}pu}{g^{\frac{13}{2}}}f_{21}(u,v)+\frac{c_{k-2}pu}{g^{\frac{11}{2}}}f_{22}(u,v)-\frac{4c_{k-1}p^7u^5}{g^{\frac{7}{2}}}\\
&+\frac{c_{k-1}pu}{g^{\frac{7}{2}}}f_{11}(u,v)+\frac{c_{k-1}pu}{g^{\frac{5}{2}}}f_{12}(u,v)-\frac{c_kpu}{\Phi^{\frac{1}{2}}}=0. \qquad (3.27)
\end{aligned}$$

If we put $v=0$ in (3.27), multiplying (3.27) by $g^{\frac{6k+1}{2}}$ and replacing g by $[p^2u^2+c]$, then (3.27) becomes

$$\begin{aligned}
&-\left(\prod_{i=1}^{k}(6i-5)(6i-2)\right)\left(p^{6k}u^{4k}\right)+[p^2u^2+c]^2f_{k2}(u,0)\\
&-c_1\left(\prod_{i=1}^{k-1}(6i-5)(6i-2)\right)\left(p^{6k-6}u^{4k-4}\right)\\
&+c_1[p^2u^2+c]^2f_{k2}(u,0)+\cdots-70c_{k-2}p^{12}u^8[p^2u^2+c]^{3k-6}\\
&+c_{k-2}[p^2u^2+c]^{3k-5}f_{22}(u,0)-4c_{k-1}p^6u^4[p^2u^2+c]^{3k-3}\\
&+c_{k-1}[p^2u^2+c]^{3k-2}f_{12}(u,0)-c_k[p^2u^2+c]^{3k}=0. \qquad (3.28)
\end{aligned}$$

Relation (3.21) is a nontrivial polynomial in u of degree $6k$ with constant coefficients. Since this equation holds true for all values of u so we must have all coefficients of the polynomial equal zeros, a case that cannot be satisfied unless only $p=0$. A contradiction since we have $p>0$.

Following the same procedure for relation (3.14) we obtain also a nontrivial polynomial in v of degree $6k$ with constant coefficients. Since this equation holds true for all values of v so we must have all coefficients of the polynomial equal zeros, a case that cannot be satisfied unless only $q=0$, which is clearly impossible since $q>0$.

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