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Chen and Casorati curvature inequalities for the submanifolds of quaternionic Kaehler manifolds endowed with Ricci quarter-symmetric metric connection

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Abstract

In this paper, authors have established Chen's inequalities for the submanifolds of quaternionic Kaehler manifolds characterized by Ricci quarter-symmetric metric connection. Other than these inequalities, generalized normalized Casorati curvature inequalities have been derived.

Keywords: Chen Inequalities, Ricci quarter-symmetric metric connection, Casorati curvature, quaternionic Kaehler manifolds.
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1. Introduction

To provide the answers to Chern's open problem concerning the existence of minimal immersions into Euclidean space of arbitrary dimension, Chen established the famous invariant, $'\delta'$ [2], [26]. The equation involving this invariant, $'\delta'$ for the Riemannian manifold, M is given as:

$$\delta_M = \tau(p) - inf(K)(p), \tag{1.1}$$

where, $'\tau(p)'$ is the scalar curvature of M and $'K(p)'$ is the sectional curvature.
The inequality for the same $'\delta'$ for any Riemannian manifold, M is given as [**?**]:

$$\delta_M \leq \frac{n^2(n-1)}{2(n-1)}\|H\|^2 + \frac{1}{2}(n+1)(n-2)c, \tag{1.2}$$

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where $'c'$ is the constant sectional curvature and $'H'$ is the mean curvature vector for this n-dimensional manifold.
Chen also derived an inequality which establishes a relationship between the extrinsic (K and τ) and intrinsic invariants (squared mean curvature, H) of the submanifold. The same inequality is given as: [3]

$$H^2 \geq \frac{4}{n^2}\{Ric(X) - (n-1)c\}, \tag{1.3}$$

where $Ric(X)$ is Ricci curvature of M at X.

With the basic insight provided by Chen, these inequalities were further extended to different manifolds under various metric and non-metric connections. Chen and Dillen studied the inequalities for Lagrangian submanifolds in complex space forms [4]. For Sasakian manifolds, the inequalities were studied by Mihai [11]. Arslan et al. studied the inequalities in contact manifolds [13]. The study on Lorentzian manifolds was done by Gulbahar et al. 15. Mihai and Ozgur [1] studied Chen inequalities on Riemannian manifolds with semi-symmetric metric connection while Zhang et al. [18], [19] studied geometric inequalities for the submanifolds of Riemannian manifolds of quasi-constant curvature with semi-symmetric metric connection. Inequalities for the submanifolds of real space forms with Ricci quarter-symmetric metric connection were studied by Nergis and Halil [17]. The inequalities of submanifolds in quaternionic space forms were studied by Yoon [8].
Another solution to the problem of establishing relationship between the main extrinsic and intrinsic invariants is given by inequalities involving Casorati Curvatures. Casorati Curvature was originally established for the surfaces in Euclidean 3-space [9] as a normalized sum of squared principal curvatures. It was extended to general case of a submanifold of a Riemannian manifold as the normalized square of the length of second fundamental form of the submanifold by Decu et al.[21], [22]. The inequalities involving this curvature have been studied for various submanifolds. Vilcu [10] obtained an optimal inequality for Casorati Curvature in Lagrangian submanifolds in complex space forms. Brubaker and Suceava [16] obatained the geometrical interpretation of Cauchy-Schwarz inequality in terms of Casorati Curvature. Lee et al. [5], [6] established Casorati curvature for the submanifolds in generalized space forms endowed with semi-symmetric metric connection and Kenmotso space forms. The study of Casorati Curvature on holomorphic statistical manifolds of constant holomorphic curvature was done by Decu et al. [23]. Similarly, the inequalities on slant submanifolds in metallic Riemannian space forms were established by Chaudhary and Blaga [14].
The concept of Ricci Quarter-symmetric metric connection was established by Golab [24] and further developed by Mishra and Pandey [20] and Kamilya and De [7].
In this paper, authors have derived the optimal inequalities for the submanifolds of quaternionic Kaehler manifolds endowed with Ricci quarter-symmetric metric connection.

2. Preliminaries

Let $M^{'}$ be any $4m$-dimensional Riemannian manifold. This $M^{'}$ is called Quaternionic Kaehler manifold if there exists 3-dimensional vector space of tensors of type $(1,1)$ with the local basis of almost Hermitian structure, ψ_1,ψ_2 and ψ_3 such that:

$$\psi_i^2 = -I, \tag{2.1}$$

$$\psi_i\psi_{i+1} = \psi_{i+2} = -\psi_i\psi_{i+1}, \tag{2.2}$$

where I is the identity transformation of tangent space T_pX at point p for the vector X of $M^{'}$.
And

$$\nabla^{`}_X\psi_i = q_{i+2}(X)\psi_{i+1} - q_{i+1}(X)\psi_{i+2}, \tag{2.3}$$

with q_i as the local one-form.
If X is any unit vector on $M^{'}$, then X, $\psi_1(X)$, $\psi_2(X)$ and $\psi_3(X)$ form the orthonormal frame on $M^{`}$. Let the plane spanned by these vectors be $Q(X)$. If X, Y are two unit vectors in $M^{'}$ and let the plane spanned

by these vectors be denoted by $\pi(X,Y)$ then this plane is called Quaternionic plane in $Q(X)$. The sectional curvature of this plane is called as quaternionic sectional curvature. If the sectional curvature is constant say, $4c$, then the quaternionic Kaehler manifold is called as quaternionic space form. Quaternionic Kaehler manifold is called quaternionic space form if and only if its curvature tensor follows the equation as: [25]

$$R^*(X,Y;Z) = c\{g(Y,Z)X - g(X,Z)Y + \sum_{i=1}^{3}(g(\psi_i Y,Z)\psi_i X - g(\psi_i X,Z)\psi_i Y - 2g(\psi_i X,Y)\psi_i Z)\}, \quad (2.4)$$

where X,Y and Z are tangent vectors and g is the Riemannian metric of $M^{'}$.

For this manifold $M^{'}$, if $\nabla^{"}$ is its linear connection then this connection is known as Ricci quarter symmetric metric connection if the torsion tensor, $T^{'}$ satisfies the following equation:

$$T^{'} = \eta(Y)LX - \eta(X)LY, \quad (2.5)$$

where η is a 1-form and L is a $(1,1)$ Ricci tensor defined by:

$$g(LX,Y) = S(X,Y), \quad (2.6)$$

with S as Ricci tensor of $M^{'}$.
With ∇^* as Levi-Civita connection with respect to metric g, the Ricci quarter symmetric metric connection is given by [**?**]:

$$\nabla^{"}_X Y = \nabla^*_X Y + \eta(Y)LX - S(X,Y)P, \quad (2.7)$$

where X and Y are vectors in $M^{'}$ with the unit vector field, P defined as:

$$g(P,X) = \eta(X). \quad (2.8)$$

Let $N^{'}$ be any n-dimensional submanifold immersed in Riemannian manifold, $M^{'}$, then there exist Ricci quarter-symmetric metric connection, ∇ and Levi-Civita metric connection, $\nabla^{'}$ which are induced on this submanifold, $N^{'}$. Let $R^{"}$ and R^* be curvature tensors associated with $\nabla^{"}$ and ∇^*, respectively. Similarly, let R and $R^{'}$ be curvature tensors associated with ∇ and $\nabla^{'}$, respectively.

For this manifold, $M^{'}$, the curvature tensor with respect to Ricci quarter-symmetric metric $\nabla^{"}$ connection is given as:

$$\begin{aligned} R^{"}(X,Y;Z) = R^*(X,Y;Z) - M(Y,Z)LX + M(X,Z)LY - S(Y,Z)QX + S(X,Z)QY \\ + \eta(z)[(\nabla^*_X L)Y - (\nabla^*_Y L)X] - [(\nabla^*_X S)(Y,Z) - (\nabla^*_Y S)(X,Z)]P, \end{aligned} \quad (2.9)$$

where X,Y and Z are tangent vectors in $M^{'}$ and M is a tensor field of type $(0,2)$ defined by;

$$M(X,Y) = g(QX,Y) = (\nabla^*_X \eta)Y - \eta(Y)\eta(LX) + \frac{1}{2}\eta(P)S(X,Y). \quad (2.10)$$

In the above equation, Q is a tensor field of type $(2,1)$ defined by,

$$QX = \nabla^*_X P - (\eta)(LX)P + \frac{1}{2}\eta(P)LX. \quad (2.11)$$

If $N^{'}$ is assumed to be Einstein manifold, then the Ricci curvature S for the linear connection ∇^* is given as: [17]

$$S(X,Y) = \frac{\tau^{'}}{n}g(X,Y), \quad (2.12)$$

with $\tau^{'}$ as the scalar curvature.
Let π be any two dimensional plane of $T_pN^{'}$ generated by $\{e_i, e_j\}$ at point p, then the sectional curvature of the section is given by:

$$K_{ij} = \frac{g(R(e_i, e_j; e_j, e_i)}{g(e_i, e_i)g(e_j, e_j) - g^2(e_i, e_j)}, \tag{2.13}$$

where e_i, e_j are orthonormal vectors in $T_pN^{'}$.
If $(e_1, ..., e_n)$ is the orthonormal basis of $T_pN^{'}$, then at any point p, scalar curvature, τ is given as:

$$\tau(p) = \sum_{1\leq i<j\leq n} K_{ij}. \tag{2.14}$$

For $N^{'}$, the normalized scalar curvature, ρ is given as:

$$\rho = \frac{2\tau}{n(n-1)}. \tag{2.15}$$

Using the equation (2.12) and the fact that connection is metric connection in equation (2.9), we get:

$$R^{"}(X, Y; Z) = R^{*}(X, Y; Z) - \frac{\tau^{'}}{n}\{M(Y, Z)X - M(X, Z)Y + g(Y, Z)QX - g(X, Z)QY\}. \tag{2.16}$$

Contracting the above equation with respect to X, we get:

$$S^{"}(Y, Z) = \frac{\tau^{'}}{n}[g(Y, Z) - \{(n-2)M(Y, Z) + mg(Y, Z)\}], \tag{2.17}$$

where $S^{"}$ is the Ricci tensor of $\nabla^{"}$ and m is the trace of M(Y,Z).
Taking $Y = Z = e_i$ in the above equation, we get,

$$\tau^{"} = \frac{\tau^{'}}{n}[n - 2m(n-1)]. \tag{2.18}$$

Now from the equation (2.4), value of $\tau^{'}$ is given as :

$$\tau^{'} = c\{\sum_{1\leq i<j\leq n} g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i) + \sum_{k=1}^{3}\sum_{1\leq i<j\leq n}(g(\psi_k e_j, e_j)g(\psi_k e_i, e_i) - g(\psi_k e_i, e_j)g(\psi_k e_j, e_i) - 2g(\psi_k e_i, e_j)g(\psi_k e_j, e_i))\}. \tag{2.19}$$

For any p$\in N^{'}$ and X $\in T_pN^{'}$, we have ψ_iX = P_i X+ F_i X where $P_i \in T_pN^{'}$ and $F_i \in T_p^{\perp}$.
Using this fact in equation (2.18), we get:

$$\tau^{'} = c\{n(n-1) + 3\sum_{k=1}^{3}\|P_k\|^2\}, \tag{2.20}$$

where $\|P_k\|^2$=$\sum_{i,j=1}^{n} g^2(P_k e_i, e_j)$.
The mean curvature vector H of N' in $M^{'}$ is given as:

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{2.21}$$

where h is the second fundamental form of $N^{'}$ in $M^{'}$.

Also, set;

$$h_{ij}^{\alpha} = g(h(e_i, e_j), e_\alpha), \tag{2.22}$$

where i, j $\in \{1, ..., n\}$ and $\alpha \in \{n+1, ..., 4m\}$.

The Casorati curvature for the submanifold N' is given as:

$$C = \frac{1}{n} \sum_{\alpha=n+1}^{4m} \sum_{i,j=1}^{n} \left(h_{ij}^{\alpha}\right)^2. \tag{2.23}$$

If V is an r-dimensional subspace of T_pN', and let $\{e_1, ..., e_r\}$ be an orthonormal basis of V, then we have;

$$C(V) = \frac{1}{r} \sum_{\alpha=n+1}^{4m} \sum_{i,j=1}^{r} \left(h_{ij}^{\alpha}\right)^2. \tag{2.24}$$

The generalized normalized δ-Casorati curvatures $\delta_C(r; n-1)$ and $\delta_C\widehat{(r; n-1)}$ for any positive real number, r are given as: [21]

$[\delta_C(r; n-1)]_p = rC_p + \frac{(n-1)(n+r)(n^2-n-r)}{r(n)}$ inf{C(V)| V is a hyperplane of T_pN'},

if $0 < r < n^2 - n$.

$[\delta_C\widehat{(r; n-1)}]_p = rC_p - \frac{(n-1)(n+r)(r-n^2+n)}{r(n)}$ sup{C(V)| V is a hyperplane of T_pN'},

if $r > n^2 - n$.

The normalized δ-Casorati curvatures $\delta_C(n-1)$ and $\delta_C\widehat{(n-1)}$ are given as:

$[\delta_C(n-1)]_p = \frac{1}{2}C_p + \frac{n+1}{2(n)}$ inf{C(V)| V is a hyperplane of T_pN'}.

$[\delta_C\widehat{(n-1)}]_p = 2C_p - \frac{2n-1}{2(n)}$ sup{C(V)| V is a hyperplane of T_pN'}.

The above equations imply that normalized δ- Casorati curvature equalities can be obtained from generalized normalized equalities by providing suitable value to the positive number, $'r'$ [19].

Before establishing the main theorems, we will establish the famous lemma of Chen. [2]

Lemma 2.1: If $n > k \geq 2$ and $a_1, a_2, ...a_n, a$ are real numbers such that:

$$\left(\sum_{i}^{n} a_i\right)^2 = (n-k+1)\left(\sum_{i=1}^{n}(a_i)^2 + a\right). \tag{2.25}$$

Then,

$$2 \sum_{1 \leq i \leq j \leq k} a_i a_j \geq a, \tag{2.26}$$

with equality holding if and only if;

$$a_1 + a_2 + ... + a_k = a_{k+1} = ... = a_n. \tag{2.27}$$

3. Results

(A) Chen invariant, Scalar curvature and Ricci curvature inequalities:

Theorem 3.1: Let $N^{'}$ be any n-dimensional submanifold of quaternionic space form, $M^{'}$ endowed with Ricci quarter-symmetric connection ∇, then τ follows the inequality as :

$$\tau(p) \leq \frac{(n-1)}{2}\left(n\|H\|^2 + c\{(n-1) + 3\sum_{k}^{3}\frac{\|P_k\|^2}{n}\}\frac{[n-2m(n-1)]}{(n-1)}\right). \tag{3.1}$$

Proof: For $M^{'}$ and $N^{'}$, the Gauss equation is given as:

$$R^{"}(X,Y;Z,W) = R(X,Y;Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)). \tag{3.2}$$

Here 'h' is the second fundamental form of $N^{'}$.
Using above equation and equation (2.9), we get the scalar curvature as:

$$2\tau = c\{(n-1) + 3\sum_{k}^{3}\frac{\|P_k\|^2}{n}\}[n-2m(n-1)] + n^2\|H\|^2 - \|h\|^2, \tag{3.3}$$

where $H{=}\frac{1}{n}\sum_{i,j=1}^{n} h(e_i,e_j)$ for any orthonormal basis $\{e_1,e_2,...,e_n\}$ of the tangent space $T_pM^{'}$.

And, $\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j),h(e_i,e_j)$.

The equation (3.3) can be simplified as:

$$n^2\|H\|^2 = 2\tau + c\{(n-1) + 3\sum_{k}^{3}\frac{\|P_k\|^2}{n}\}[2m(n-1)-n] + \|h\|^2. \tag{3.4}$$

Let $p \in N^{'}$ and $e_1,...,e_n,...e_{4m}$ be orthonormal basis at p such that e_{n+1} is parallel to mean curvature vector, H, then the shape operators are given by:

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & \cdots & o \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \tag{3.5}$$

$A_{e_r} = h_{ij}^r$, $i,j = 1,....,n$; $r = n+2,....,4m$; $trace(A_r) = 0$.

Considering the equation (3.4) and the above operators, we get:

$$n^2\|H\|^2 = 2\tau(p) + \sum_{i=2}^{n}a_i^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + c\{(n-1) + 3\sum_{k}^{3}\frac{\|P_k\|^2}{n}\}[2m(n-1)-n]. \tag{3.6}$$

From the algebra, we have:

$$\sum_{i<j}(a_i-a_j)^2 = (n-1)\sum_{i}a_i^2 - 2\sum_{i<j}a_ia_j, \tag{3.7}$$

$$\implies n^2\|H\|^2 = \left(\sum_{i}a_i^2\right)^2 = \sum_{i}a_i^2 + 2\sum_{i<j}a_ia_j \leq n\sum_{i=1}^{n}a_i^2. \tag{3.8}$$

Using above inequality in equation (3.6), we get:

$$n(n-1)\|H\|^2 \geq 2\tau(p) + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + c\{(n-1)+3\sum_{k}^{3}\frac{\|P_k\|^2}{n}\}[2m(n-1)-n], \tag{3.9}$$

or,

$$\tau(p) \leq \frac{(n-1)}{2}\left(n\|H\|^2 + c\{(n-1)+3\sum_{k}^{3}\frac{\|P_k\|^2}{n}\}\frac{[n-2m(n-1)]}{(n-1)}\right). \tag{3.10}$$

In the above inequality, the equality is established only when a_1=a_2=...=a_n and A_{e_r}=0.
$\implies$ p is totally umbilical point.

Remark: For the real space forms, the part involving P_k will be equal to zero as such the inequality will be same as established in [17].

Lemma 3.2: Let $\zeta(x_1, x_2,...x_n)$ be a function in R^n defined by

$$\zeta(x_1, x_2, ...x_n) = x_1\sum_{i=2}^{n}x_i. \tag{3.11}$$

If $x_1 + x_2 + ... + x_n = 2\mu$, then $\zeta \leq \mu^2$.
Proof: From the hypothesis, we have,

$$\zeta = x_1(2\mu - x_1), \tag{3.12}$$

$$\implies \zeta = -(x_1-\mu)^2 + \mu^2, \tag{3.13}$$

$$\implies \zeta \leq \mu^2. \tag{3.14}$$

It follows from the equation (3.13) that equality is established only when $x_1 = \mu$.

Theorem 3.3**:** If $N^{'}$ is n-dimensional submanifold of 4m-dimensional quaternionic space form endowed with Ricci quarter-symmetric connection, then for each unit vector $X \in T_pN^{'}$, the inequality for Ricci curvature, Ric(X), is given as:

$$Ric(X) \leq c\{(n-1)+3\sum_{k}^{3}\sum_{j=2}^{n}g^2(\phi_k X, e_j)\} - c\{(n-1)+3\sum_{k}^{3}\frac{\|P_k\|^2}{n}\}[m+(n-2)M(X,X)] + \frac{n^2\|H\|^2}{4}. \tag{3.15}$$

Proof: Let $\{e_1, e_2, ..., e_n\}$ be the orthonormal basis of $T_pN^{'}$ at any point p.
Using our assumption that $N^{'}$ is Einstein manifold and the equations (3.2) (Gauss Equation) and (2.16), we get;

$$R(e_1,e_j;e_j,e_1) = R^*(e_1,e_j;e_j,e_1) - \frac{\tau'}{n}\{M(e_j,e_j)g(e_1,e_1) - M(e_1,e_j)g(e_j,e_1) + g(e_j,e_j)M(e_1,e_1) - g(e_1,e_j)M(e_1,e_j)\} + \sum_{r=n+1}^{4m}[h_{11}^r h_{jj}^r - (h_{1j}^r)^2], \tag{3.16}$$

with $g(QX,W) = M(X,W)$.

If e_1=X, then the Ricci curvature is defined and given as :

$$Ric(X) = \sum_{i=2}^{n} R(e_1, e_j, e_j, e_1) = c\{(n-1)+3\sum_{k}^{3} g^2(\phi_k X, e_j)\} - c\{c(n-1)+3\sum_{k}^{3} \frac{\|P_k\|^2}{n}\}[m+(n-2)M(X,X)] + \sum_{r=n+1}^{4m} \sum_{2}^{n} [h_{11}^r h_{jj}^r]. \quad (3.17)$$

The above equation implies;

$$Ric(X) \leq c\{(n-1)+3\sum_{k}^{3} g^2(\phi_k X, e_j)\} - c\{(n-1)+3\sum_{k}^{3} \frac{\|P_k\|^2}{n}\}[m+(n-2)M(X,X)] + \sum_{r=n+1}^{4m} \sum_{2}^{n} [h_{11}^r h_{jj}^r]. \quad (3.18)$$

Considering the last term of above inequality, we can make a quadratic form given by:

$$\zeta(h_{11}^r, h_{22}^r, ..., h_{nn}^r) = \sum_{i=2}^{n} h_{11}^r h_{ii}^r. \quad (3.19)$$

Using the Lemma 3.2, we have;

$$\zeta(h_{11}^r, h_{22}^r, ..., h_{nn}^r) \leq \frac{U_r^2}{4}, \quad (3.20)$$

where $U_r = h_{11}^r + h_{22}^r + ... + h_{nn}^r$.
Using this value and the definition of H in equation (3.18) we have;

$$Ric(X) \leq c\{(n-1)+3\sum_{k}^{3}\sum_{j=2}^{n} g^2(\phi_k X, e_j)\} - c\{(n-1)+3\sum_{k}^{3} \frac{\|P_k\|^2}{n}\}[m+(n-2)M(X,X)] + \frac{n^2\|H\|^2}{4}. \quad (3.21)$$

For the condition of equality to hold we should have;

$$h_{1i}^r = 0, \quad (3.22)$$

where i $\neq$ 1 and $\forall$ r.
The another condition follows from the Lemma 3.2, which gives,

$$h_{11}^r = h_{22}^r + h_{33}^r + ... + h_{nn}^r. \quad (3.23)$$

If the equality holds for all unit vectors X in T_pN', then above inequalities are given as;

$$h_{ij}^r = 0, i \neq j, \quad (3.24)$$

$$2h_{ii} = h_{11} + h_{22} + ... + h_{nn}, i \in \{1, ..., n\}, \quad (3.25)$$

$$\implies 2h_{11} = 2h_{22} = ... = 2h_{nn} = h_{11} + h_{22} + ... + h_{nn}, i \in \{1, ..., n\}, \quad (3.26)$$

$$\implies (n-2)(h_{11} + h_{22} + ... + h_{nn}) = 0. \quad (3.27)$$

For $n \neq 2$,

$$h_{ij}^r = 0, i, j = 1, 2, ..., n, r = n+1, ..., 4m. \quad (3.28)$$

Also,

$$h_{ii}^{r} = 0, \tag{3.29}$$

which means the point, p is totally geodesic point.
For $n = 2$,

$$h_{11}^{r} = h_{22}^{r}, r = 3, ..., 4m, \tag{3.30}$$

which means that the point, p is totally umbilical.

Remark: If $N(p)$=$\{X \in T_pN^{'} : h(X,Y) = 0 \ \forall \ Y \in T_pN^{'}\}$, then the above inequality is satisfied by a tangent vector X with $H(p) = 0$.

Theorem 3.3**:** Let $N^{'}$ be n-dimensional submanifold of $4m$-dimensional quaternionic space form, $M^{'}$ with constant sectional curvature given by $4c$. For this submanifold, $N^{'}$ which is characterized by Ricci quarter-symmetric metric connection, the inequality between sectional curvature, K and scalar curvature, τ at any point p $\in N^{'}$ and section plane π is given as:

$$\tau - K(\pi) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{c}{2}\{(n-1) + 3\sum_{k}^{3}\frac{\|P_k\|^2}{n}\}[n + 2mn - 2trace(m_{\pi_k^{\perp}})] - c\{3\sum_{k=1}^{3}\beta_k(\pi) + 1). \tag{3.31}$$

Proof: Let $(e_1, e_2, ..., e_n)$ be the orthonormal basis of $T_pN^{'}$ and $(e_{n+1}, ..., e_{4m})$ be that of $T_p^{\perp}N^{'}$ at any point p such that the section plane π is spanned by (e_1, e_2). The mean curvature vector H is taken to be parallel to e_{n+1}. Then by the equation of Gauss (3.2), we have:

$$K(\pi) = K(e_1 \wedge e_2) = c(1 + 3\sum_{k=1}^{3}\beta_k(\pi)) + c\{(n-1) + 3\sum_{k=1}^{3}\frac{\|P_k\|^2}{n})(trace(m_{\pi_k^{\perp}}) - m) + h_{11}^{n+1}h_{22}^{n+1} + \sum_{r\geq n+2} h_{11}^{r}h_{22}^{r} - \left(h_{12}^{n+1}\right)^2 - \sum_{r\geq n+2}\left(h_{12}^{r}\right)^2, \tag{3.32}$$

where m is the trace of M, $trace(m_{\pi_k^{\perp}})$ is equal to $(m - M(e_1, e_1) - M(e_2, e_2))$ and $\beta_k(\pi)$=$\sum_{1\leq i\leq j\leq 2} g^2(P_ke_i, e_j)$ with $k = 1, 2, 3$.

If we assume;

$$\mu = 2\tau - c((n-1) + 3\sum_{k}^{3}\frac{\|P_k\|^2}{n})[n - 2m(n-1)] - \frac{n^2(n-2)}{n-1}\|H\|^2. \tag{3.33}$$

Then equation (3.3) and above equation gives;

$$n^2\|H\|^2 = (n-1)(\mu + \|h\|^2). \tag{3.34}$$

The above implies;

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\left(\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 + \mu\right). \tag{3.35}$$

Using Lemma 2.1,we get;

$$2\sum_{1\leq i\leq j\leq k} h_{ii}^{n+1}hh_{jj}^{n+1} \geq \mu + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2. \tag{3.36}$$

Using these values in the equation involving $K(\pi)$, we get;

$$K(\pi) \geq c(1+3\sum_{k=1}^{3}\beta_k(\pi)) + c\{(n-1)+3\sum_{k=1}^{3}\frac{\|P_k\|^2}{n}\}(trace(m_{\pi_k^\perp})-m) + \frac{\mu}{2} + \sum_{r=n+1}^{4m}\sum_{j>2}((h_{1j}^r)^2+(h_{2j}^r)^2) + \frac{1}{2}\sum_{i\neq j>2}((h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{4m}\sum_{i,j>2}((h_{ij}^r)^2 + \frac{1}{2}\sum_{r=n+2}^{4m}(h_{11}^r+h_{22}^r)^2. \quad (3.37)$$

On solving above inequality we get,

$$\tau - K(\pi) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{c}{2}\{(n-1)+3\sum_{k}^{3}\frac{\|P_k\|^2}{n}\}[n-2m(n-1)] - c\{1+3\sum_{k=1}^{3}\beta_k(\pi)) - c\{3\sum_{k=1}^{3}\frac{\|P_k\|^2}{n}+(n-1)\}(trace(m_{\pi_k^\perp})-m). \quad (3.38)$$

In the above inequality, the equality condition holds if the terms involving h_{ij}^r is equal to zero and as such we have;

$$h_{1j}^{n+1} = h_{2j}^{n+1} = 0,$$
$$h_{ij}^{n+1} = 0; i \neq j > 2,$$
$$h_{1j}^r = h_{2j}^r = ... = h_{ij}^r; r = n+2, ...4m; i,j > 3,$$
$$h_{11}^r + h_{22}^r = 0; r = n+2, ..., 4m.$$

Choose e_1 and e_2 such that $h_{12}^{n+1} = 0$. Also by Chen's Lemma 2.1, we have;

$$h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = ... = h_{nn}^{n+1}.$$

The above equations imply that the shape operator take the form as :

$$A_{e_{n+1}} = \begin{pmatrix} T & 0 & \cdots & 0 & 0 \\ 0 & U & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 0 & I \end{pmatrix} \quad (3.39)$$

$$A_{e_r} = \begin{pmatrix} T_r & U_r & 0 \\ U_r & -U_r & 0 \\ 0 & 0 & 0_{4m-2} \end{pmatrix} \quad (3.40)$$

where $T+U=I$ and T_r, $U_r \in$ R.

Theorem 3.4**:** If N' is n-dimensional submanifold of 4m-dimensional quaternionic projective space M' $(c>0)$ with constant quaternionic sectional curvature given by 4c and endowed with Ricci quarter-symmetric connection, then for any plane $\pi \in T_pN'$, the inequality for Chen's invariant δ is given as:

$$\delta_{N'} \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{c}{2}(n+8)(n+2mn-2trace(m_{\pi_k^\perp})) - c, \quad (3.41)$$

with the equality holding if and only if $N^{'}$ is invariant.

Proof: For the positive sectional curvature($c > 0$), we have to maximize the term $\frac{3c(n-1)}{2}\sum_{k=1}^{3}\frac{\|P_k\|^2}{n}(n + 2mn - 2trace(m_{\pi_k^{\perp}})) - \frac{3c}{2}\sum_{k=1}^{3}\beta_k(\pi)$ of inequality (61).
For constant c, m and $trace(m_{\pi_k^{\perp}})$, the maximum value can be achieved if $\|P_k\|^2 = n$ and $\beta_k(\pi) = 0$.

Using these values, we get the required inequality.

Remark: The condition, $\|P_k\|^2 = n$ and $\beta_k(\pi) = 0$ implies $N^{'}$ is invariant.

Remark: If $M(X, Y) = 0$, we get the inequality as in [8].

Theorem 3.5**:** If $N^{'}$ is n-dimensional submanifold of 4m-dimensional quaternionic hyperbolic space $M^{'}$ ($c < 0$) with constant quaternionic sectional curvature given by 4c and endowed with Ricci quarter-symmetric connection, then for any plane $\pi \in T_pN^{'}$, the inequality for Chen's invariant δ is given as:

$$\delta_{N'} \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{c}{2}(n-1)(n + 2mn - 2trace(m_{\pi_k^{\perp}})) - c, \tag{3.42}$$

With the equality holding if and only if $N^{'}$ is anti quasi-invariant.

Proof: For $c < 0$, the estimate for $\delta_{N'}$ can be obtained by minimizing the term $\frac{3c(n-1)}{2}\sum_{k=1}^{3}\frac{\|P_k\|^2}{n}(n + 2mn - 2trace(m_{\pi_k^{\perp}})) - \frac{3c}{2}\sum_{k=1}^{3}\beta_k(\pi)$ of inequality (3.31).

By simplification it follows that $\|P_k\|^2 = 0$ and $\beta_k(\pi) = 0$.

The above conclusion exhibits that span of $\pi = span(e_1, e_2)$ is orthogonal to span of $\{span(\phi_k e_i \,|i = 3, ..., n\}$.

(B) Casorati curvature Inequalities:

Theorem 3.6: If $N^{'}$ is n-dimensional submanifold of $4m$-dimensional quaternionic Kaehler manifold, $M^{'}$. Then:

(i). The generalized normalized δ-Casorati curvature $\delta_C(r, n-1)$ satisfies;

$$\delta_C(r, n-1) \geq n(n-1)\rho - c\{(n-1) + 3\sum_{k}^{3}\frac{\|P_k\|^2}{n}\}[n - 2m(n-1)]. \tag{3.43}$$

(ii). The generalized normalized δ-Casorati curvature $\delta_C\widehat{(r, n-1)}$ satisfies;

$$\widehat{\delta_C(r, m)} \geq n(n-1)\rho - c\{(n-1) + 3\sum_{k}^{3}\frac{\|P_k\|^2}{n}\}[n - 2m(n-1)]. \tag{3.44}$$

In the inequality (3.43) , the equality holds if and only if the submanifold $N^{'}$ is invariantly quasi-umbilical with trivial normal connection in $M^{'}$, such that the shape operator A_{e_i} with $i \in \{n+1, ..., 4m\}$ with respect to suitable orthonormal tangent frame $\{e_1, ..., e_n\}$ and normal orthonormal frame $\{e_{n+1}, ..., e_{4m}\}$ takes the

form as:

$$A_{e_{n+1}} = \begin{pmatrix} u & 0 & 0 & \cdots & 0 & 0 \\ 0 & u & 0 & \cdots & 0 & 0 \\ 0 & 0 & u & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & & \\ 0 & 0 & 0 & \cdots & u & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{n(n-1)}{r}u \end{pmatrix} \tag{3.45}$$

$$A_{e_{n+2}} = ..., = A_{e_{4m}} = 0. \tag{3.46}$$

Proof: By equation (3.3) we have;

$$2\tau = c\{(n-1) + 3\sum_{k}^{3} \frac{\|P_k\|^2}{n}\}[n - 2m(n-1)] + n^2\|H\|^2 - \|h\|^2, \tag{3.47}$$

where $H=\frac{1}{n}\sum_{i,j=1}^{n} h(e_i, e_j)$ for any orthonormal basis $\{e_1, e_2, ..., e_n\}$ of the tangent space, $T_pM^{'}$ and m is the trace of $M(X, Y)$.

Now, consider the function T which is associated with the following polynomial in the components $(h_{ij}^{\alpha}), i, j = 1, ..., n; \alpha = n+1, ..., 4m$ of second fundamental form h of $N^{'}$ in $M^{'}$:

$$T = rC + \frac{(n-1)(n+r)(n^2-n-r)}{rn}C(L) - 2\tau + c\{(n-1) + 3\sum_{k}^{3} \frac{\|P_k\|^2}{n}\}[n - 2m(n-1)], \tag{3.48}$$

where L is the hyperplane of $T_pN^{'}$.

If L is assumed to be spanned by $e_1, ..., e_{n-1}$, then we have:

$$T = \frac{r}{n}\sum_{\alpha=n+1}^{4m}\sum_{i,j=1}^{n}(h_{ij})^2 + \frac{(n+r)(n^2-n-r)}{rn}\sum_{\alpha=n+1}^{4m}\sum_{i,j=1}^{n}(h_{ij})^2 - 2\tau + c\{(n-1) + 3\sum_{k}^{3} \frac{\|P_k\|^2}{n}\}[n - 2m(n-1)]. \tag{3.49}$$

Using equation (3.47) we get;

$$T = \frac{n+r}{n}\sum_{\alpha=n+1}^{4m}\sum_{i,j=1}^{n}(h_{ij})^2 + \frac{(n+r)(n^2-n-r)}{rn}\sum_{\alpha=n+1}^{4m}\sum_{i,j=1}^{n}(h_{ij})^2 - \sum_{\alpha=n+1}^{4m}\left(\sum_{i+1}^{n} h_{ii}^{\alpha}\right)^2. \tag{3.50}$$

The above equation can be written as:

$$T = \sum_{\alpha=n+1}^{4m}\sum_{i=1}^{n-1}\left[\frac{(n-1)^2 + (n-1)(r+1) - r)}{r}(h_{ii}^{\alpha})^2 + \frac{2(n+r)}{n}(h_{i(n-1)+1}^{\alpha})^2\right] + \sum_{\alpha=n+1}^{4m}\left[\frac{2(n-1)(n+r)}{r}\sum_{1\le i<j\le m-1}(h_{ij}^{\alpha})^2 - 2\sum_{1\le i<j\le n} h_{ii}^{\alpha}h_{jj}^{\alpha}\right] + \frac{r}{n}\sum_{\alpha=n+1}^{4m}(h_{nn}^{\alpha})^2. \tag{3.51}$$

Hence T is a quadratic polynomial in the components of the second fundamental form. From (3.51), we deduce the critical points, $h^c = \left[h^n_{11}, h^{n+1}_{12}, ..., h^{n+1}_{nn}, ..., h^{4m}_{11}, h^{4m}_{12}, ..., h^{4m}_{nn}\right]$ of T are the solutions of the following system of linear homogeneous equations:

$$\frac{\partial T}{\partial h^\alpha_{ii}} = \frac{2(n-1)(n+r)}{r} h^\alpha_{ii} - 2\sum_{k=1}^{n} h^\alpha_{kk} = 0, \tag{3.52}$$

$$\frac{\partial T}{\partial h^\alpha_{nn}} = \frac{2r}{n} h^\alpha_{n} - 2\sum_{k=1}^{n} h^\alpha_{kk} = 0, \tag{3.53}$$

$$\frac{\partial T}{\partial h^\alpha_{ij}} = \frac{4(n-1)(n+r)}{r} h^\alpha_{ij} = 0, \tag{3.54}$$

$$\frac{\partial T}{\partial h^\alpha_{in}} = \frac{4(n+r)}{n} h^\alpha_{in} = 0, \tag{3.55}$$

with $i, j \in \{1, ..., n-1\}$, $i \neq j$ and $\alpha \in \{n+1, ..., 4m\}$.

The above system of equations shows that every solution, h^c has $h^\alpha_{ij} = 0$ for $i \neq j$. The Hessian matrix can be computed as:

$$H(p) = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix} \tag{3.56}$$

Here H_1 is given as:

$$H_1 = \begin{pmatrix} \frac{2(n-1)(n+r)}{r} - 2 & -2 & -2 & \cdots & -2 & -2 \\ -2 & \frac{2(n-1)(n+r)}{r} - 2 & -2 & \cdots & -2 & -2 \\ -2 & -2 & \frac{2(n-1)(n+r)}{r} - 2 & \cdots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & & \\ -2 & -2 & -2 & \cdots & \frac{2(n-1)(n+r)}{r} - 2 & -2 \\ -2 & -2 & -2 & \cdots & -2 & \frac{2r}{n} \end{pmatrix} \tag{3.57}$$

$H_2 = diag\left(\frac{4(n-1)(n+r)}{r}, \frac{4(n-1)(n+r)}{r}, ..., \frac{4(n-1)(n+r)}{r}\right)$,

$H_3 = diag\left(\frac{4(n+r)}{n}, \frac{4(n+r)}{n}, ..., \frac{4(n+r)}{n}\right)$.

It can be found by direct computation that the Hessian matrix $H(T)$ of T has the eigenvalues as follows:

$\lambda_{11} = 0$, $\lambda_{22} = \frac{2(n-1)n^2+r^2)}{rn}$, $\lambda_{33} = ..., = \lambda_{m+1m+1} = \frac{2(n-1)(n+r)}{r}$,

$\lambda_{ij} = \frac{4(n-1)(n+r)}{r}$,$\lambda_{i(n-1)+1} = \frac{4(n+r)}{n}$;, $\forall$ i,j $\in \{1, ..., n-1\}$, $i \neq j$.

It follows that the Hessian matrix is positive semi-definite and admits one eigenvalue equal to zero. Hence, we can see that T is parabolic and reaches a minimum at h^c. In fact, at the critical point h^c, the function T reaches the global minimum. So, we have $T(h^c) = 0$. Therefore, we deduce $T \geq 0$ and this implies;

$$2\tau \leq rC + \frac{(n-1)(n+r)(n^2-n-r)}{rn} C(L) + c\{(n-1) + 3\sum_{k}^{3} \frac{\|P_k\|^2}{n}\}[n - 2m(n-1)]. \tag{3.58}$$

We know that normalized scalar curvature ρ is given as:

$$\rho = \frac{2\tau}{n(n-1)}. \tag{3.59}$$

Using this value in above equation we get;

$$\rho \leq \frac{r}{n(n-1)}C + \frac{(n+r)(n^2-m-r)}{rn^2}C(L) + c\{(n-1) + 3\sum_{k}^{3}\frac{\|P_k\|^2}{n}\}\frac{[n-2m(n-1)]}{n(n-1)}. \tag{3.60}$$

The above statement is equivalent to;

$$rC + \frac{(n-1)(n+r)(n^2-m-r)}{rn}C(L) \geq n(n-1)\rho - c\{(n-1) + 3\sum_{k}^{3}\frac{\|P_k\|^2}{n}\}[n-2m(n-1)]. \tag{3.61}$$

The required inequalities follow by taking the infimum and supremum respectively, over all the tangent hyperplanes $T_pN^{'}$.

On the other hand, it is clear that the equality sign follows if and only if; $h^{\alpha}_{ij} = 0$, $\forall$ i,j $\in \{1, ..., n\}$, $i \neq j$:
and

$$h^{\alpha}_{nn} = \frac{n(n-1)}{r}h^{\alpha}_{11} =, ..., \frac{n(n-1)}{r}h^{\alpha}_{n-1n-1}, \tag{3.62}$$

for all $\alpha \in \{n+1, ..., 4m\}$.

Since, $h(e_i, \zeta_p) \neq 0$, we have that the equality holds only when the submanifold is invariantly quasi-umbilical and shape operator takes the form given in equations (3.45) and (3.46).
Similarly, inequality (3.44) can be proved.

Corollary 3.7**:** If $N^{'}$ is n-dimensional submanifold of $4m$-dimensional quaternionic Kaehler manifold, $M^{'}$. Then:

(i). The normalized δ-Casorati curvature $\delta_C(n-1)$ satisfies;

$$\delta_C(n-1) \geq \rho - c\{(n-1) + 3\sum_{k}^{3}\frac{\|P_k\|^2}{n}\}\frac{[n-2m(n-1)]}{n(n-1)}. \tag{3.63}$$

(ii). The normalized δ-Casorati curvature $\delta_{\widehat{C(n-1)}}$ satisfies;

$$\delta_{\widehat{C(n-1)}} \geq \rho - c\{(n-1) + 3\sum_{k}^{3}\frac{\|P_k\|^2}{n}\}\frac{[n-2m(n-1)]}{n(n-1)}. \tag{3.64}$$

In the inequalities (3.63) and (3.64) , the equality holds if and only if the submanifold, $N^{'}$ is invariantly quasi-umbilical with trivial normal connection in $M^{'}$, such that the shape operator A_{e_i} with $i \in \{n+1, ..., 4m\}$ with respect to suitable orthonormal tangent frame $\{e_1, ..., e_n\}$ and normal orthonormal frame $\{e_{n+1}, ..., e_{4m}\}$ takes the form as:

$$A_{e_{n+1}} = \begin{pmatrix} u & 0 & 0 & \cdots & 0 & 0 \\ 0 & u & 0 & \cdots & 0 & 0 \\ 0 & 0 & u & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & & \\ 0 & 0 & 0 & \cdots & u & 0 \\ 0 & 0 & 0 & \cdots & 0 & 2u \end{pmatrix} \tag{3.65}$$

$$A_{e_{n+2}} = ..., = A_{e_{4m}} = 0. \tag{3.66}$$

and

$$A_{e_{n+1}} = \begin{pmatrix} 2u & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2u & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2u & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & & \\ 0 & 0 & 0 & \cdots & 2u & 0 \\ 0 & 0 & 0 & \cdots & 0 & u \end{pmatrix} \tag{3.67}$$

$$A_{e_{n+2}} = ..., = A_{e_{4m}} = 0 \tag{3.68}$$

Proof: It can be easily seen that:

$$[\delta_C(\frac{n(n-1)}{2}, m)]_p = n(n-1)[\delta_C(n-1)]_p, \tag{3.69}$$

and,

$$[\delta_C(2n\widehat{(n-1)}, m)]_p = n(n-1)[\delta_C\widehat{(n-1)}]_p. \tag{3.70}$$

The proof follows by putting $r = \frac{n(n-1)}{2}$ in (3.43) and $r = 2n(n-1)$ in (3.44) and using above two equations.

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