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New results on periodic solutions for a nonlinear fourth-order iterative differential equation

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Abstract

The key task of this paper is to investigate a nonlinear fourth-order delay differential equation. By virtue of the fixed point theory and the Green's functions method, we establish some new results on the existence, uniqueness and continuous dependence on parameters of periodic solutions. In addition, an example is given to corroborate the validity of our main results. Up to now, no work has been carried out on this topic. So, the findings of this paper are new and complement the available works in the literature to some degree.

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1. Introduction

Our principal purposes in the present work is to establish the existence, uniqueness and continuous dependence on parameters of periodic solutions for the following nonlinear fourth-order iterative differential equation:

$$x^{(4)}(t) + px^{(2)}(t) + qx(t) = x(t) \sum_{k=1}^{n} c_k(t)\, x^{[k]}(t) + H(t), \tag{1.1}$$

where $p = \beta_1^2 + \beta_2^2$, $q = \beta_1^2 \beta_2^2$, β_1, β_2 are non-zero constants and $x^{[m]}(t)$ is the m th iterate of the function $x(t)$, i.e. $x^{[2]}(t) = x(t) \circ x(t), ..., x^{[n]}(t) = x(t) \circ x^{[n-1]}(t)$.

We may encounter fourth-order differential equations in many practical and real situations such as viscoelastic and inelastic flows, deflection patterns in physics, deformation of elastic beams in structures such as aircraft, buildings and ships, vibrational motion in bridges, heartbeats in physiology, soil settlement, the folding of rock layers and electric circuits (we refer the reader to the paper [6] and the book [15] and references therein). However, in practice, many fourth-order differential equations whether with or without

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delays can be deduced from partial differential equations such as in studying the beam equation which serves as a model of an elastic bar with nonconvex stored energy density, the model of water driven capillarity-gravity waves and also in the investigation of stationary solutions for the extended Fisher-Kolmogorov equation and the Swift–Hohenberg equation that describes the formation and evolution of patterns in a wide range of systems. For example (but not limited to), the authors of [15] revealed that the use of an argument based on the center manifold theorem, a system of partial differential equations modeling gravity-capillary water waves can be reduced to the following fourth-order ordinary differential equation:

$$x^{(4)}(t) + px^{(2)}(t) + x(t) = x^2(t).$$

As another important example is the famous Swift–Hohenberg equation

$$x^{(4)}(t) + px^{(2)}(t) + r(t) f(x(t)) = 0,$$

which models the formation and evolution of patterns.

In the last decades, periodic solutions for such equations with delays have received considerable portion of attention of many researchers. In fact, it is very crucial both in theory and practice to prove the existence theorems of periodic solutions of fourth order differential equations describing the formation of periodic patterns (beams), oscillatory phenomena occurring in bridges and periodic waves. Moreover, the reliance on the past occurs usually in the aforementioned applications where the delay plays a prominent role in gleaning a better understanding of the underlying mechanisms that lead to the formation and evolution of patterns, the deformation of beams and the formation of waves. In what follows, we review some motivating works for the current paper.

In [3], Balamuralitharan employed the coincidence degree continuation theorem for studying the existence of positive periodic solutions for the following fourth-order differential equation with time-varying delay:

$$x^{(4)}(t) + ax^{(3)}(t) + \lambda f\left(x''(t-\tau(t))\right) + \lambda g\left(x'(t-\tau(t))\right) + h(x(t-\tau(t))) = \lambda p(t).$$

Tunç [17] used the Liapunov functional approach to study the asymptotic stability of zero solution of the following class of fourth-order non-linear differential equations with constant delay:

$$x^{(4)}(t) + \varphi\left(x''(t)\right) x^{(3)}(t) + h\left(x'(t)\right) x''(t) + \phi\left(x'(t-r)\right) + f(x(t-r)) = 0.$$

By means of Krasnoselskii's fixed point theorem, the authors of [14] established the existence of positive periodic solutions for the following class of fourth-order nonlinear neutral equations:

$$\frac{d^4}{dt^4}(x(t) - c(t)x(t-\tau(t))) = a(t)x(t) - f(t, x(t-\tau(t))).$$

Iterations in equation (1.1) result from time and state dependent delays such as the time taken for the observation to be available for use in control, delayed feedback control of piezoelectric elastic beams and so on and so forth.

To the best of the author's knowledge, iterative problems were initiated by Babbage [2], Schröder [16] and Abel [1] in the early of the last century. They can be regarded as a special type of the so-called differential equations with time and state-dependent delays resulting from many factors such as the competition for food and habitat during larval stage in insect populations and the dependence on the history of the mature cell population in hematopoiesis models and so on. Such equations which appear in the modeling of many real phenomena such as infectious disease transmission models and the two-body problem of classical electrodynamics, were and still are deemed unpopular subject among scholars. Perhaps this is due to the fact that the study of such equations has often been hampered by a lack of theoretical basis and also by the existence of the iterative terms which obstruct the application of several methods. For this, there exist only very few results about this type of equations for instance, first and second order iterative differential equations have been investigated only by a few researchers (see for example [5, 7, 9, 12, 18, 19]) while there are only our three works [8, 10, 13] that investigated third order iterative differential equations. However, as far as we know, no papers have appeared in the literature that focussed on fourth-order iterative differential equations except the very recent work of the first author.

Filling this gap is a top priority for us. For this reason and motivated by the previously mentioned works and applications, it is highly desirable to establish some criteria that guarantee the existence, uniqueness and continuous dependence on parameters of periodic solutions for equation (1.1) which implicitly involves multiple time and state-dependent delays leading to the appearance of the iterates. The technique used in this paper consists in applying the Krasnoselskii's fixed point theorem for a sum of two mappings and the contraction mapping principle together with some constructed properties of an obtained Green's function (for more details about Green's functions for linear fourth order differential equations and their properties, one can refer to our paper [4]).

The main contributions of this article are summarized as follows:

- We presented a model including iterative terms which involve implicitly complex delays depending on time and state.
- We constructed some interesting properties of the Green's function needed to reach our goals and also can be used to investigate many other problems.
- This work is a continuation of our recent papers on iterative problems. Up to now there are no publications on fourth order iterative differential equations except our recent paper [6].
- Our approach will provide a good reference to study other iterative problems. It is based on the construction of an appropriate Banach space and a subset of it that ensure the belongingness of the iterates to them, the conversion of the iterative problem into an equivalent integral equation whose kernel is a Green's function, the establishment of some useful properties of this kernel and also the choice of the suitable fixed point theorem.

The rest of the paper is organized as follows: In the beginning of this paper, we introduce some definitions and preliminary results needed to understand the subsequent sections. Furthermore, we state and prove some of the properties of a Green's kernel for a linear fourth order differential equation. In the third section, by using the Krasnoselskii's fixed point theorem for a sum of two mappings and some constructed properties of the obtained Green's function, we establish some sufficient conditions which allow us to achieve our goals. In the fourth section, we present an illustrative example to demonstrate the feasibility of our obtained findings. Finally, the conclusion is drawn to end this article in the last section.

2. Preliminaries

For $w > 0$ and $L,\, M \geq 0$, let

$$P_w = \left\{ x \in \mathcal{C}\left(\mathbb{R}, \mathbb{R}\right),\ x\left(t\right) = x\left(t + w\right) \right\},$$

equipped with the norm

$$\left\| x \right\| = \sup_{t \in \mathbb{R}} \left| x\left(t\right) \right| = \sup_{t \in [0,w]} \left| x\left(t\right) \right|,$$

and

$$P_w\left(L, M\right) = \left\{ x \in P_w,\ \ \left\| x \right\| \leq L,\ \ \left| x\left(t_2\right) - x\left(t_1\right) \right| \leq M \left| t_2 - t_1 \right|,\ \forall t_1, t_2 \in \mathbb{R} \right\}.$$

Then $(P_w, \|.\|)$ is a Banach space and $P_w\left(L, M\right)$ is a closed convex and bounded subset of P_w.

Furthermore we assume that $\beta_1 \neq \beta_2$, $w\beta_1 \neq 2k\pi$ and $w\beta_2 \neq 2k\pi$, $k \in \mathbb{Z}$.

Lemma 2.1. *For $h \in P_w$ the equation*

$$\left\{ \begin{array}{l} x^{(4)}\left(t\right) + \left(\beta_1^2 + \beta_2^2\right) x^{(2)}\left(t\right) + \beta_1^2 \beta_2^2 x\left(t\right) = h(t), \\ x^{(k)}(0) = x^{(k)}(w),\ \ k \in \left\{0, 1, 2, 3\right\}, \end{array} \right. \tag{2.1}$$

has a unique $w-$periodic solution

$$x(t) = \int_t^{t+w} G(t, s) h\left(s\right) ds,$$

where $s \in [t, t+w]$ *and*

$$G(t,s) = \frac{1}{\beta_1 (2\cos w\beta_1 - 2)} \frac{\sin \beta_1 (s-t) + \sin \beta_1 (t-s+w)}{\beta_1^2 - \beta_2^2} - \frac{1}{\beta_2 (2\cos w\beta_2 - 2)} \frac{\sin \beta_2 (s-t) + \sin \beta_2 (t-s+w)}{\beta_1^2 - \beta_2^2}. \tag{2.2}$$

Proof. The associated homogeneous equation of (2.1) is

$$x^{(4)}(t) + \left(\beta_1^2 + \beta_2^2\right) x^{(2)}(t) + \beta_1^2 \beta_2^2 x(t) = 0, \tag{2.3}$$

where its characteristic equation is

$$\lambda^4 + \left(\beta_1^2 + \beta_2^2\right)\lambda^2 + \beta_1^2\beta_2^2 = 0,$$

and the roots of this last characteristic equation are $\lambda_1 = i\beta_1, \lambda_2 = -i\beta_1, \lambda_3 = i\beta_2$ and $\lambda_4 = -i\beta_2$, so, the solution of the homogeneous equation (2.3) is

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t} + c_4 e^{\lambda_4 t}.$$

The only periodic solution of the associated homogeneous problem (2.1) with the periodic properties is the trivial solution.
For the nonhomogeneous problem (2.1) with the periodic properties, we use the method of variation of parameters, to arrive at

$$c_1'(t) = ih(t)\frac{e^{-t\lambda_1}}{2\beta_1^3 - 2\beta_1\beta_2^2},\ c_2'(t) = -ih(t)\frac{e^{-t\lambda_2}}{2\beta_1^3 - 2\beta_1\beta_2^2},$$
$$c_3'(t) = ih(t)\frac{e^{-t\lambda_3}}{2\beta_2^3 - 2\beta_1^2\beta_2},\ c_4'(t) = -ih(t)\frac{e^{-t\lambda_4}}{2\beta_2^3 - 2\beta_1^2\beta_2},$$

and hence

$$c_1(t+w) = c_1(t) + \int_t^{t+w} i\frac{e^{-s\lambda_1}}{2\beta_1^3 - 2\beta_1\beta_2^2} h(s)\,ds,$$
$$c_2(t+w) = c_2(t) - \int_t^{t+w} i\frac{e^{-s\lambda_2}}{2\beta_1^3 - 2\beta_1\beta_2^2} h(s)\,ds,$$
$$c_3(t+w) = c_3(t) + \int_t^{t+w} i\frac{e^{-s\lambda_3}}{2\beta_2^3 - 2\beta_1^2\beta_2} h(s)\,ds,$$
$$c_4(t+w) = c_4(t) - \int_t^{t+w} i\frac{e^{-s\lambda_4}}{2\beta_2^3 - 2\beta_1^2\beta_2} h(s)\,ds.$$

Since we are looking for w-periodic solutions of (2.1), we get

$$c_1(t) = -\int_t^{t+w} i\frac{e^{-\lambda_1(s-w)}}{2\beta_1\left(e^{w\lambda_1} - 1\right)\left(\beta_1^2 - \beta_2^2\right)} h(s)\,ds,$$
$$c_2(t) = \int_t^{t+w} i\frac{e^{-\lambda_2(s-w)}}{2\beta_1\left(e^{w\lambda_2} - 1\right)\left(\beta_1^2 - \beta_2^2\right)} h(s)\,ds,$$
$$c_3(t) = \int_t^{t+w} i\frac{e^{-\lambda_3(s-w)}}{2\beta_2\left(e^{w\lambda_3} - 1\right)\left(\beta_1^2 - \beta_2^2\right)} h(s)\,ds,$$
$$c_4(t) = -\int_t^{t+w} i\frac{e^{-\lambda_4(s-w)}}{2\beta_2\left(e^{w\lambda_4} - 1\right)\left(\beta_1^2 - \beta_2^2\right)} h(s)\,ds.$$

Therefore,

$$\begin{aligned} u(t) &= c_1(t)e^{\lambda_1 t} + c_2(t)e^{\lambda_2 t} + c_3(t)e^{\lambda_3 t} + c_4(t)e^{\lambda_4 t} \\ &= \int_t^{t+w} G(t,s)\,h(s)\,ds. \end{aligned}$$

This completes the proof of Lemma 2.1. □

Lemma 2.2. *Function $G(t,s)$ satisfies*

$$\int_t^{t+w} G(t,s)\,ds = \frac{1}{\beta_1^2\beta_2^2}, \tag{2.4}$$

and

$$|G(t,s)| \le A,$$

where

$$A = \frac{1}{|\beta_1^2-\beta_2^2|}\left(\frac{1}{|\beta_1|\,|\cos(w\beta_1)-1|} + \frac{1}{|\beta_2|\,|\cos(w\beta_2)-1|}\right). \tag{2.5}$$

Proof. Let

$$g_1(t,s) = \frac{1}{\beta_1(2\cos w\beta_1 - 2)}\frac{\sin\beta_1(s-t)+\sin\beta_1(t-s+w)}{\beta_1^2-\beta_2^2},$$

and

$$g_2(t,s) = -\frac{1}{\beta_2(2\cos w\beta_2 - 2)}\frac{\sin\beta_2(s-t)+\sin\beta_2(t-s+w)}{\beta_1^2-\beta_2^2}.$$

We have

$$\int_t^{t+w} g_1(t,s)\,ds = -\frac{1}{\beta_1^2(\beta_1^2-\beta_2^2)} \text{ and } \int_t^{t+w} g_2(t,s)\,ds = \frac{1}{\beta_2^2(\beta_1^2-\beta_2^2)}.$$

So

$$\int_t^{t+w} G(t,s)\,ds = -\frac{1}{\beta_1^2(\beta_1^2-\beta_2^2)} + \frac{1}{\beta_2^2(\beta_1^2-\beta_2^2)} = \frac{1}{\beta_1^2\beta_2^2}.$$

On the other hand

$$\begin{aligned} |G(t,s)| &\le \frac{1}{|\beta_1(2\cos(w\beta_1)-2)|}\frac{|\sin\beta_1(s-t)+\sin\beta_1(t-s+w)|}{|\beta_1^2-\beta_2^2|} \\ &\quad + \frac{1}{|\beta_2(2\cos w(\beta_2)-2)|}\frac{|\sin\beta_2(s-t)+\sin\beta_2(t-s+w)|}{|\beta_1^2-\beta_2^2|} \\ &\le \frac{1}{|\beta_1^2-\beta_2^2|}\left(\frac{1}{|\beta_1|\,|\cos(w\beta_1)-1|} + \frac{1}{|\beta_2|\,|\cos(w\beta_2)-1|}\right), \end{aligned}$$

which completes the proof. □

Lemma 2.3. *[19] For any $\varphi, \theta \in P_w(L,M)$,*

$$\left\|\varphi^{[m]} - \theta^{[m]}\right\| \le \sum_{j=0}^{m-1} M^j \|\varphi - \theta\|,\ m = 1,2,...$$

Lemma 2.4. *[18] It holds that*

$$P_w(L,M) = \{x \in P_w,\ \ \|x\| \le L,\ \ |x(t_2) - x(t_1)| \le M|t_2 - t_1|,\ \forall t_1,\ t_2 \in [0,w]\}.$$

Lemma 2.5. *[5] For any* $\varphi, \theta \in P_w(L, M)$,

$$\left\|\varphi \times \varphi^{[i]} - \theta \times \theta^{[i]}\right\| \leq L\left(1 + \sum_{j=0}^{i-1} M^j\right)\|\varphi - \theta\|, \; i = 1, 2, ...$$

Theorem 2.6. *(Krasnoselskii's fixed point theorem). Let* $\mathbb{M}$ *be a closed convex nonempty subset of a Banach space* $(\mathbb{B}, \|.\|)$. *Suppose that* $\mathcal{A}$ *and* $\mathcal{B}$ *map* $\mathbb{M}$ *into* $\mathbb{B}$ *such that*
(i) $x, y \in \mathbb{M}$, *implies* $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$,
(ii) $\mathcal{A}$ *is compact and continuous,*
(iii) $\mathcal{B}$ *is a contraction mapping.*
Then there exists $z \in \mathbb{M}$ *with* $z = \mathcal{A}z + \mathcal{B}z$.

3. Main results

Lemma 3.1. $x \in P_w(L, M) \cap \mathcal{C}^4(\mathbb{R}, \mathbb{R})$ *is a solution of (1.1) if and only if* $x \in P_w(L, M)$ *is a solution of the following integral equation:*

$$x(t) = \sum_{k=1}^{n} \int_t^{t+w} G(t,s)\, x(s)\, c_k(s)\, x^{[k]}(s)\, ds + \int_t^{t+w} G(t,s)\, H(s)\, ds. \tag{3.1}$$

Proof. Thanks to Lemma 2.1, we can convert equation (1.1) into the integral equation (3.1). □

3.1. Existence of periodic solutions

In this section, we will apply the Krasnoselskii's fixed point theorem for a sum of contraction and compact mappings to show the existence of at least one periodic solution of (1.1). To this end and from Lemma 3.1, we define two operators $\mathcal{N}_1, \mathcal{N}_2 : P_w(L, M) \to P_w$ as follows:

$$(\mathcal{N}_1\varphi)(t) = \sum_{k=1}^{n} \int_t^{t+w} G(t,s)\, \varphi(s)\, c_k(s)\, \varphi^{[k]}(s)\, ds, \tag{3.2}$$

and

$$(\mathcal{N}_2\varphi)(t) = \int_t^{t+w} G(t,s)\, H(s)\, ds. \tag{3.3}$$

Remark 3.2. The condition $\|x\| \leq L$ in the definition of $P_w(L, M)$ shows that this subset is uniformly bounded and from the condition $|x(t_2) - x(t_1)| \leq M|t_2 - t_1|$, $\forall t_1, t_2 \in \mathbb{R}$, it follows that $P_w(L, M)$ is equicontinuous. Consequently, the Arzelà-Ascoli theorem guarantees that $P_w(L, M)$ is a compact subset of $\mathcal{C}([0, w], \mathbb{R})$.

Lemma 3.3. *If* $c_k \in P_w(L_{c_k}, M_{c_k})$, $k = \overline{1, n}$, *then operator* $\mathcal{N}_1$ *defined by (3.2) is continuous and compact on* $P_w(L, M)$.

Proof. Let $\varphi, \theta \in P_w(L, M)$ and $c_k \in P_w(L_{c_k}, M_{c_k})$, $k = \overline{1, n}$. From Lemma 2.2, we have

$$\begin{aligned} |(\mathcal{N}_1\varphi)(t) - (\mathcal{N}_1\theta)(t)| &\leq \sum_{k=0}^{n} \int_t^{t+w} G(t,s)\, |c_k(s)| \left|\varphi(s)\, \varphi^{[k]}(s) - \theta(s)\, \theta^{[k]}(s)\right| ds \\ &\leq A \sum_{k=1}^{n} L_{c_k} \int_t^{t+w} \left|\varphi(s)\, \varphi^{[k]}(s) - \theta(s)\, \theta^{[k]}(s)\right| ds. \end{aligned}$$

Using Lemma 2.5, we obtain

$$|(\mathcal{N}_1\varphi)(t) - (\mathcal{N}_1\theta)(t)| \leq ALw \sum_{k=0}^{n} L_{c_k} \left(1 + \sum_{j=0}^{k-1} M^j\right) \|\varphi - \theta\|,$$

which proves the continuity of $\mathcal{N}_1$.
According to Remark 3.2, since $\mathcal{N}_1$ is a continuous operator and since any continuous operator maps compact sets into compact sets, the compactness of the operator $\mathcal{N}_1$ follows immediately from its continuity. □

Lemma 3.4. *If $\mathcal{N}_2$ is given by (3.3), then $\mathcal{N}_2$ is a contraction mapping on $P_w(L, M)$.*

Proof. For φ, θ in $P_w(L, M)$, we have

$$|(\mathcal{N}_2\varphi)(t) - (\mathcal{N}_2\theta)(t)| \leq \int_t^{t+w} |G(t,s) H(s) - G(t,s) H(s)| \, ds = 0,$$

so

$$\begin{aligned}\|\mathcal{N}_2\varphi - \mathcal{N}_2\theta\| &= \sup_{t\in[0,w]} |(\mathcal{N}_2\varphi)(t) - (\mathcal{N}_2\theta)(t)| \\ &\leq \Gamma \|\varphi - \theta\|,\end{aligned}$$

for any $\Gamma \in [0, 1[$. Consequently, $\mathcal{N}_2$ is a contraction mapping on $P_w(L, M)$. □

Lemma 3.5. *If t_1, $t_2 \in [0, w]$ with $t_1 \leq t_2$, then*

$$\int_{t_1}^{t_1+w} |G(t_1, s) - G(t_2, s)| \, ds \leq \mu |t_2 - t_1|, \tag{3.4}$$

where

$$\mu = \frac{w}{|\beta_1^2 - \beta_2^2|} \left(\frac{1}{|\cos(w\beta_1) - 1|} + \frac{1}{|\cos(w\beta_2) - 1|} \right).$$

Proof. We have

$$\begin{aligned}&\int_{t_1}^{t_1+w} |G(t_1, s) - G(t_2, s)| \, ds \\ &\leq \frac{1}{2|\beta_1| |\beta_1^2 - \beta_2^2| |\cos(w\beta_1) - 1|} \int_{t_1}^{t_1+w} |\sin\beta_1(s - t_1) - \sin\beta_1(s - t_2)| \, ds \\ &+ \frac{1}{2|\beta_1| |\beta_1^2 - \beta_2^2| |\cos(w\beta_1) - 1|} \int_{t_1}^{t_1+w} |\sin\beta_1(t_1 - s + w) - \sin\beta_1(t_2 - s + w)| \, ds \\ &+ \frac{1}{2|\beta_2| |\beta_1^2 - \beta_2^2| |\cos(w\beta_2) - 1|} \int_{t_1}^{t_1+w} |\sin\beta_2(s - t_1) - \sin\beta_2(s - t_2)| \, ds \\ &+ \frac{1}{2|\beta_2| |\beta_1^2 - \beta_2^2| |\cos(w\beta_2) - 1|} \int_{t_1}^{t_1+w} |\sin\beta_2(t_1 - s + w) - \sin\beta_2(t_2 - s + w)| \, ds \\ &\leq \frac{w}{|\beta_1^2 - \beta_2^2| |\cos(w\beta_1) - 1|} |t_2 - t_1| \\ &+ \frac{w}{|\beta_1^2 - \beta_2^2| |\cos(w\beta_2) - 1|} |t_2 - t_1| \\ &= \frac{w}{|\beta_1^2 - \beta_2^2|} \left(\frac{1}{|\cos(w\beta_1) - 1|} + \frac{1}{|\cos(w\beta_2) - 1|} \right) |t_1 - t_2|.\end{aligned}$$

The proof is completed. □

Lemma 3.6. *Let φ, $\theta \in P_w(L, M)$, $c_k \in P_w(L_{c_k}, M_{a_k})$ with $k = \overline{1, n}$, and $H \in P_w(L_H, M_H)$. If*

$$Aw \left(L^2 \sum_{k=1}^{n} L_{c_k} \left(1 + \sum_{j=0}^{k-1} M^j \right) + L_H \right) \leq L, \tag{3.5}$$

then

$$|(\mathcal{N}_1\varphi)(t) + (\mathcal{N}_2\theta)(t)| \leq L.$$

Proof. We have

$$\begin{aligned}
\left|\left(\mathcal{N}_1\varphi\right)(t)+\left(\mathcal{N}_2\theta\right)(t)\right| &\leq \left|\left(\mathcal{N}_1\varphi\right)(t)\right|+\left|\left(\mathcal{N}_2\theta\right)\right| \\
&\leq \sum_{k=1}^{n}\int_{t}^{t+w} G(t,s)\left|c_k(s)\right|\left|\varphi(s)\times\varphi^{[k]}(s)\right| ds \\
&+\int_{t}^{t+w} G(t,s)\left|H(s)\right| ds.
\end{aligned}$$

From Lemma 2.5, we obtain

$$\left\|\varphi\times\varphi^{[k]}\right\| \leq L\left(1+\sum_{j=0}^{k-1}M^j\right)\|\varphi\| \leq L^2\left(1+\sum_{j=0}^{k-1}M^j\right). \tag{3.6}$$

Using Lemma 2.2 and (3.6), we get

$$\begin{aligned}
\left|\left(\mathcal{N}_1\varphi\right)(t)+\left(\mathcal{N}_2\theta\right)(t)\right| &\leq L^2 Aw\sum_{k=1}^{n}L_{c_k}\left(1+\sum_{j=0}^{k-1}M^j\right)+AwL_H \\
&= Aw\left(L^2\sum_{k=1}^{n}L_{c_k}\left(1+\sum_{j=0}^{k-1}M^j\right)+L_H\right).
\end{aligned}$$

Thanks to condition (3.5), we can obtain the desired estimate. □

Lemma 3.7. *Let $t_1,\ t_2\in[0,w]$ with $t_2>t_1$. If*

$$\left[L^2\sum_{k=1}^{n}L_{c_k}\left(1+\sum_{j=0}^{k-1}M^j\right)+L_H\right](2A+\mu)\leq M, \tag{3.7}$$

then

$$\left|\left(\left(\mathcal{N}_1\varphi\right)(t_2)+\left(\mathcal{N}_2\theta\right)(t_2)\right)-\left(\left(\mathcal{N}_1\varphi\right)(t_1)+\left(\mathcal{N}_2\theta\right)(t_1)\right)\right|\leq M\left|t_2-t_1\right|,\ \forall\varphi,\ \theta\in P_w(L,M).$$

Proof. For all $t_1,\ t_2\in[0,w]$ and $\varphi,\ \theta\in P_w(L,M)$, we get

$$\begin{aligned}
&\left|\left(\left(\mathcal{N}_1\varphi\right)(t_2)+\left(\mathcal{N}_2\theta\right)(t_2)\right)-\left(\left(\mathcal{N}_1\varphi\right)(t_1)+\left(\mathcal{N}_2\theta\right)(t_1)\right)\right| \\
&\leq\left|\left(\mathcal{N}_1\varphi\right)(t_2)-\left(\mathcal{N}_1\varphi\right)(t_1)\right|+\left|\left(\mathcal{N}_2\theta\right)(t_2)-\left(\mathcal{N}_2\theta\right)(t_1)\right|.
\end{aligned}$$

We have

$$\begin{aligned}
&\left|\left(\mathcal{N}_1\varphi\right)(t_2)-\left(\mathcal{N}_1\varphi\right)(t_1)\right| \\
&=\left|\sum_{k=1}^{n}\int_{t_2}^{t_2+w}G(t_2,s)\,c_k(s)\,\varphi(s)\,\varphi^{[k]}(s)\,ds-\sum_{k=1}^{n}\int_{t_1}^{t_1+w}G(t_1,s)\,c_k(s)\,\varphi(s)\,\varphi^{[k]}(s)\,ds\right| \\
&\leq\sum_{k=1}^{n}\int_{t_2}^{t_1}G(t_2,s)\left|c_k(s)\right|\left|\varphi(s)\,\varphi^{[k]}(s)\right|ds+\sum_{k=1}^{n}\int_{t_1+w}^{t_2+w}G(t_2,s)\left|c_k(s)\right|\left|\varphi(s)\,\varphi^{[k]}(s)\right|ds \\
&+\sum_{k=1}^{n}\int_{t_1}^{t_1+w}\left|c_k(s)\right|\left|\varphi(s)\,\varphi^{[k]}(s)\right|\left|G(t_2,s)-G(t_1,s)\right|ds.
\end{aligned}$$

By using (3.6) and Lemmas 2.2 and 3.5, we arrive at

$$\begin{aligned}|(\mathcal{N}_1\varphi)(t_2)-(\mathcal{N}_1\varphi)(t_1)| &\le 2AL^2\sum_{k=1}^{n}L_{c_k}\left(1+\sum_{j=0}^{k-1}M^j\right)|t_2-t_1| \\ &+\mu L^2\sum_{k=1}^{n}L_{c_k}\sum_{k=1}^{n}\left(1+\sum_{j=0}^{k-1}M^j\right)|t_2-t_1| \\ &=(2A+\mu)L^2\sum_{k=1}^{n}L_{c_k}\left(1+\sum_{j=0}^{k-1}M^j\right)|t_2-t_1|.\end{aligned}$$

On the other hand, we get

$$\begin{aligned}|(\mathcal{N}_2\theta)(t_2)-(\mathcal{N}_2\theta)(t_1)| &= \left|\int_{t_2}^{t_2+w}G(t_2,s)H(s)\,ds-\int_{t_1}^{t_1+w}G(t_1,s)H(s)\,dsds\right| \\ &\le \int_{t_2}^{t_1}G(t_2,s)|H(s)|\,ds+\int_{t_1+w}^{t_2+w}G(t_2,s)|H(s)|\,ds \\ &+\int_{t_1}^{t_1+w}|H(s)||G(t_2,s)-G(t_1,s)|\,ds.\end{aligned}$$

It follows from Lemma 2.2 that

$$\begin{aligned}|(\mathcal{N}_2\theta)(t_2)-(\mathcal{N}_2\theta)(t_1)| &\le 2AL_H|t_2-t_1|+\mu L_H|t_2-t_1| \\ &= L_H(2A+\mu)|t_2-t_1|.\end{aligned}$$

Consequently

$$\begin{aligned}&|((\mathcal{N}_1\varphi)(t_2)+(\mathcal{N}_2\theta)(t_2))-((\mathcal{N}_1\varphi)(t_1)+(\mathcal{N}_2\theta)(t_1))| \\ &\le\left[L^2\sum_{k=1}^{n}L_{c_k}\left(1+\sum_{j=0}^{k-1}M^j\right)+L_H\right](2A+\mu)|t_2-t_1|.\end{aligned}$$

From (3.7), we have

$$|((\mathcal{N}_1\varphi)(t_2)+(\mathcal{N}_2\theta)(t_2))-((\mathcal{N}_1\varphi)(t_1)+(\mathcal{N}_2\theta)(t_1))|\le M|t_2-t_1|.$$

The lemma is proved. □

Theorem 3.8. *If $c_k\in P_w(L_{c_k},M_{c_k})$, $k=\overline{1,n}$ and $H\in P_w(L_H,M_H)$, then (1.1) has at least one solution x in $P_w(L,M)$.*

Proof. From Lemma 3.1, fixed points of $\mathcal{N}_1+\mathcal{N}_2$ are solutions of (1.1) and vice versa. By virtue of Lemmas 2.4, 3.3, 3.4, 3.6 and 3.7, all the hypotheses of the Krasnoselskii's fixed point theorem are satisfied. Thus, we can conclude that the operator $\mathcal{N}_1+\mathcal{N}_2$ has at least one fixed point in $P_w(L,M)$ which shows that equation (1.1) has at least one periodic solution in $P_w(L,M)$. □

3.2. Uniqueness of periodic solutions

Theorem 3.9. *Let $c_k\in P_w(L_{c_k},M_{c_k})$, $k=\overline{1,n}$ and $H\in P_w(L_H,M_H)$. If*

$$LAw\sum_{k=0}^{n}L_{c_k}\left(1+\sum_{j=0}^{k-1}M^j\right)<1, \tag{3.8}$$

then (1.1) has a unique solution x in $P_w(L,M)$.

Proof. For $\varphi, \theta \in P_w(L, M)$, $c_k \in P_w(L_{c_k}, M_{c_k})$ with $k = \overline{1,n}$ and $H \in P_w(L_H, M_H)$, we have

$$|(\mathcal{N}_1 + \mathcal{N}_2)(\varphi)(t) - (\mathcal{N}_1 + \mathcal{N}_2)(\theta)(t)| = |(\mathcal{N}_1\varphi)(t) - (\mathcal{N}_1\theta)(t)|.$$

Similarly to the proof of Lemma 3.3, we arrive at

$$|(\mathcal{N}_1 + \mathcal{N}_2)(\varphi)(t) - (\mathcal{N}_1 + \mathcal{N}_2)(\theta)(t)| \leq LAw \sum_{k=0}^{n} L_{c_k} \left(1 + \sum_{j=0}^{k-1} M^j\right) \|\varphi - \theta\|.$$

From (3.8) and by the help of the contraction mapping principle, $\mathcal{N}_1 + \mathcal{N}_2$ has a unique fixed point in $P_w(L, M)$ and from Lemma 3.1, this fixed point is the unique solution of (1.1). □

3.3. Continuous dependence on parameters of the periodic solution

Theorem 3.10. *The solution obtained in Theorem 3.9 depends continuously upon the functions c_k, $k = \overline{1,n}$ and the function H.*

Proof. Under the assumptions of Theorem 3.9, since x is a solution of (1.1), so it satisfies the integral equation (3.1), i.e.

$$x(t) = \sum_{k=1}^{n} \int_t^{t+w} G(t,s)\, x(s)\, a_k(s)\, x^{[k]}(s)\, ds + \int_t^{t+w} G(t,s)\, H_1(s)\, ds.$$

Let y be a solution of the perturbed equation with small perturbations in functions c_k, $k = \overline{1,n}$ and H that satisfy the conditions of Theorem 3.9. Then y satisfies the following integral equation:

$$y(t) = \sum_{k=1}^{n} \int_t^{t+w} G(t,s)\, y(s)\, b_k(s)\, y^{[k]}(s)\, ds + \int_t^{t+w} G_2(s,t)\, H_2(s)\, ds,$$

where a_k, $b_k \in P_w(L_{c_k}, M_{c_k})$, $k = \overline{1,n}$.

Estimating the difference between solutions x and y, we have

$$\begin{aligned}
|x(t) - y(t)| &\leq \sum_{k=1}^{n} \int_t^{t+w} G(t,s) \left| x(s)\, a_k(s)\, x^{[k]}(s) - y(s)\, b_k(s)\, y^{[k]}(s) \right| ds \\
&+ \int_t^{t+w} G(t,s) \left| H_1(s) - H_2(s) \right| ds \\
&\leq \sum_{k=1}^{n} \int_t^{t+w} G(t,s) \left| x(s)\, a_k(s)\, x^{[k]}(s) - y(s)\, a_k(s)\, y^{[k]}(s) \right| ds \\
&+ \sum_{k=1}^{n} \int_t^{t+w} G(t,s) \left| y(s)\, a_k(s)\, y^{[k]}(s) - y(s)\, b_k(s)\, y^{[k]}(s) \right| ds \\
&+ \int_t^{t+w} G(t,s) \left| H_1(s) - H_2(s) \right| ds.
\end{aligned}$$

By using the same technique as that in the proof of Lemma 3.3, and by taking into account (3.6), we get

$$\begin{aligned}
|x(t) - y(t)| &\leq LAw \sum_{k=0}^{n} L_{c_k} \left(1 + \sum_{j=0}^{k-1} M^j\right) \|x - y\| \\
&+ Aw \sum_{k=1}^{n} L^2 \left(1 + \sum_{j=0}^{k-1} M^j\right) \|a_k - b_k\| + Aw \|H_1 - H_2\|.
\end{aligned}$$

It follows from (3.8) that

$$\|x-y\| \leq Aw \frac{\sum_{k=1}^{n} L^2 \left(1+\sum_{j=0}^{k-1} M^j\right) + \|H_1 - H_2\|}{1 - LAw \sum_{k=0}^{n} L_{c_k} \left(1+\sum_{j=0}^{k-1} M^j\right)}.$$

This completes the proof. □

4. Example

We consider the following equation:

$$x^{(4)}(t) + \frac{5}{2}\pi^2 x^{(2)}(t) + \frac{9}{16}\pi^4 x(t) = \frac{1}{5}x^2(t)\sin 2\pi t + \frac{1}{7}x(t)\,x^{[2]}(t)\sin 2\pi t + \frac{1}{3}\sin 2\pi t, \tag{4.1}$$

where

$$\begin{aligned} \beta_1 &= \frac{\pi}{2},\ \beta_2 = \frac{3}{2}\pi, \\ c_1(t) &= \frac{1}{5}\sin 2\pi t \in P_1\left(\frac{1}{5}, \frac{2}{5}\pi\right), \\ c_2(t) &= \frac{1}{7}\sin 2\pi t \in P_1\left(\frac{1}{7}, \frac{2}{7}\pi\right), \\ H(t) &= \frac{1}{3}\sin 2\pi t \in P_1\left(\frac{1}{3}, \frac{2}{3}\pi\right). \end{aligned}$$

Equation (4.1) can be found in studying the extended Fisher–Kolmogorov and Swift–Hohenberg equations with forcing terms and can also generalize many delay models such as those describing water waves driven by gravity and capillarity and those modeling the deflection patterns of elastic struts resting on elastic foundations where x in the last model stands for the deflection, p is the compressive axial load, $H(t)$ is the forcing term, the nonlinearity $f\left(t, x(t), x^2(t)\right) = qx(t) - c_1(t)\,x^2(t) - c_1(t)\,x(t)\,x^{[2]}(t)$ is introduced as the resisting force per unit length of the foundation but in this case t is the spatial coordinate (see [11] and [15]).

By taking $w = 1$, $L = \pi$ and $M = 2\pi$, we find

$$\begin{gathered} A = \frac{4}{3\pi^3},\ \mu = \frac{1}{\pi^2}, \\ Aw\left(L^2 \sum_{k=1}^{n} L_{c_k}\left(1+\sum_{j=0}^{k-1} M^j\right) + L_H\right) \simeq 0.6863 \leq L = \pi, \\ \left[L^2 \sum_{k=1}^{n} L_{c_k}\left(1+\sum_{j=0}^{k-1} M^j\right) + L_H\right](2A+\mu) \simeq 2.9897 \leq M = 2\pi, \\ LAw \sum_{k=0}^{n} L_{c_k}\left(1+\sum_{j=0}^{k-1} M^j\right) \simeq 0.2139 < 1. \end{gathered}$$

All conditions of Theorems 3.8, 3.9 and 3.10 are satisfied and consequently (4.1) has a unique solution in $P_w(\pi, 2\pi)$ depends continuously on the functions $c_1(t)$, $c_2(t)$ and $H(t)$.

5. Conclusion

With the help of the Krasnoselskii's fixed point theorem for a sum of 2 mappings, the contraction mapping principle and the Green's functions method, we established some new sufficient conditions ensuring the existence, uniqueness and continuous dependence on parameters of periodic solutions for a nonlinear fourth-order differential equation with an iterative source term. Furthermore, we supported our findings by an example to illustrate their effectiveness.

We would like to point out that the derived results of this work are completely innovative and extend those of the previous studies in [3], [5]-[14], [17]-[19].

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