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Algebraic integers of pure sextic extensions

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Abstract

Let $\mathbb{K} = \mathbb{Q}(\theta)$, where $\theta = \sqrt[6]{d}$, be a pure sextic field with $d \neq 1$ a square free integer. In this paper, we characterize completely whether $\{1, \theta, \ldots, \theta^5\}$ is an integral basis of $\mathbb{K}$ or do not. When $d \not\equiv \pm 1, \pm 17, \pm 10, -15, -11, -7, -3, 5, 13 (mod\ 36)$ we prove that $\mathbb{K}$ has a power integral basis. Furthermore, for the other cases we present an integral basis.

Keywords: Algebraic number field, algebraic number integer, pure sextic extension.
2010 MSC: 11R04, 11R16, 11R21.

1. Introduction

An algebraic number field $\mathbb{K}$ is a finite degree extension of $\mathbb{Q}$. In this case, $\mathbb{K} = \mathbb{Q}(\theta)$, where $\theta \in \mathbb{C}$ is a root of a monic irreducible polynomial $p(x) \in \mathbb{Q}[x]$. When $p(x) = x^n - d \in \mathbb{Z}[x]$ is irreducible, where $d \neq 1$ is a square free integer, the field $\mathbb{K}$ is called a pure number field. The n distinct roots of $p(x)$, namely, $\theta_1, \theta_2, \ldots, \theta_n$, are the conjugates of θ. If $\sigma : \mathbb{K} \to \mathbb{C}$ is a $\mathbb{Q}$-embedding, then $\sigma(\theta) = \theta_i$ for some $i = 1, 2, \ldots, n$. Furthermore, there are exactly n $\mathbb{Q}$-embeddings σ_i, for $i = 1, 2, \ldots, n$, of $\mathbb{K}$ in $\mathbb{C}$.

An element $\alpha \in \mathbb{K}$ is called an algebraic integer if there is a monic polynomial $f(x)$ with integer coefficients such that $f(\alpha) = 0$. The set $\mathcal{O}_{\mathbb{K}} = \{\alpha \in \mathbb{K} : \alpha \text{ is an algebraic integer}\}$ is a ring, called the ring of algebraic integers of $\mathbb{K}$. It can be shown that $\mathcal{O}_{\mathbb{K}}$, as a $\mathbb{Z}$-module, has a basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ over $\mathbb{Z}$, called integral basis, where n is the degree of $\mathbb{K}$.

The trace and the norm of an element $\alpha \in \mathbb{K}$ over $\mathbb{Q}$ are defined as the rational numbers $Tr_{\mathbb{K}}(\alpha) = \sum_{i=1}^{n} \sigma_i(\alpha)$ and $N_{\mathbb{K}}(\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha)$. If $\alpha \in \mathcal{O}_{\mathbb{K}}$, then $Tr_{\mathbb{K}}(\alpha)$ and $N_{\mathbb{K}}(\alpha)$ are integers. The characteristic polynomial of α is defined as $f_\alpha(x) = (x - \sigma_1(\alpha))(x - \sigma_2(\alpha_2)) \ldots (x - \sigma_n(\alpha))$ and $\alpha \in \mathcal{O}_{\mathbb{K}}$ if and only if the coefficients of the characteristic polynomial are integers. The discriminant of $\mathbb{K}$ over $\mathbb{Q}$ is defined by

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$\mathcal{D}_{\mathbb{K}} = \mathcal{D}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \det_{1 \leq i,j \leq n}(Tr_{\mathbb{K}}(\alpha_i \alpha_j)) = \det_{1 \leq i,j \leq n}(\sigma_i(\alpha_j))^2$, where $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is an integral basis of $\mathbb{K}$ [1].

An element $\theta \in \mathcal{O}_{\mathbb{K}}$ generates a power integral basis $\{1, \theta, \theta^2, \ldots, \theta^{n-1}\}$ for $\mathbb{K}$ if $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\theta]$. When $\mathbb{K}$ has a power integral basis, the field $\mathbb{K}$ is said to be monogenic. The existence of power integral bases in algebraic number fields is a classical problem in algebraic number theory [3, 4, 5]. In this work, we show that the field $\mathbb{Q}(\sqrt[6]{d})$, where $d \neq 1$ is a square free integer and

$$d \not\equiv \pm 1, \pm 17, \pm 10, -15, -11, -7, -3, 5, 13 (mod\ 36),$$

the field $\mathbb{K}$ is monogenic.

2. Pure sextic extensions

Let $\mathbb{K} = \mathbb{Q}(\theta)$, where $\theta = \sqrt[6]{d})$ is root of an irreducible polynomial $p(x) = x^6 - d$, with $d \in \mathbb{Z}$ and $d \neq 1$ is square free. In this case, θ is an algebraic integer and $[\mathbb{Q}(\theta) : \mathbb{Q}] = 6$. The roots of $p(x)$ are θ, $\theta\zeta_6$, $\theta\zeta_6^2$, $\theta\zeta_6^3$, $\theta\zeta_6^4$ and $\theta\zeta_6^5$, where ζ_6 is a primitive 6-th root of unity, and $\zeta_6^1 + \zeta_6^2 + \zeta_6^3 + \zeta_6^4 + \zeta_6^5 = -1$. Let σ_k, $k = 1, 2, \ldots, 6$, be the $\mathbb{Q}$-embeddings of $\mathbb{Q}(\theta)$ in $\mathbb{C}$, that is, $\sigma_k(\theta) = \theta\zeta_6^{k-1}$. Since $\{1, \theta, \theta^2, \theta^3, \theta^4, \theta^5\}$ is a basis of $\mathbb{K}$ over $\mathbb{Q}$, it follows that if $\alpha \in \mathbb{K}$, then $\alpha = a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5$, where $a_i \in \mathbb{Q}$, for $i = 0, 1, 2, 3, 4, 5$.

Proposition 2.1. *If $\alpha = a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5 \in \mathbb{K}$, with $a_0, a_1, a_2, a_3, a_4, a_5 \in \mathbb{Q}$, then the characteristic polynomial of α is given by $f_\alpha(x) = x^6 + f_5x^5 + f_4x^4 + f_3x^3 + f_2x^2 + f_1x + f_0$, where*

$$\left\{\begin{array}{lcl}
f_5 & = & -6a_0 \\
f_4 & = & 3(5a_0^2 - (a_3^2 + 2a_2a_4 + 2a_1a_5)d) \\
f_3 & = & -[2(10a_0^3 + (a_2^3 + 6a_1a_2a_3 - 6a_0a_3^2 + 3a_1^2a_4 - 12a_0a_2a_4 - 12a_0a_1a_5)d \\
 & & +(a_4^3 + 6a_3a_4a_5 + 3a_2a_5^2)d^2)] \\
f_2 & = & [3(5a_0^4 + (-3a_1^2a_2^2 + 2a_0a_2^3 - 2a_1^3a_3 + 12a_0a_1a_2a_3 - 6a_0^2a_3^2 + 6a_0a_1^2a_4 - 12a_0^2a_2a_4 \\
 & & -12a_0^2a_1a_5)d + (a_3^4 + 3a_2^2a_4^2 - 6a_1a_3a_4^2 + 2a_0a_4^3 - 6a_2^2a_3a_5 + 12a_0a_3a_4a_5 + 3a_1^2a_5^2 \\
 & & +6a_0a_2a_5^2)d^2 + (-3a_4^2a_5^2 - 2a_3a_5^3)d^3)] \\
f_1 & = & -[6(a_0^5 + (a_1^4a_2 - 3a_0a_1^2a_2^2 + a_0^2a_2^3 - 2a_0a_1^3a_3 + 6a_0^2a_1a_2a_3 - 2a_0^3a_3^2 + 3a_0^2a_1^2a_4 - 4a_0^3a_2a_4 \\
 & & -4a_0^3a_1a_5)d + (a_2^3a_3^2 - 2a_1a_2a_3^3 + a_0a_3^4 - a_2^4a_4 + 3a_1^2a_3^2a_4 + 3a_0a_2^2a_4^2 - 6a_0a_1a_3a_4^2 + a_0^2a_4^3 \\
 & & +2a_1a_2^3a_5 - 6a_0a_2^2a_3a_5 - -2a_1^3a_4a_5 + 6a_0^2a_3a_4a_5 + 3a_0a_1^2a_5^2 + 3a_0^2a_2a_5^2)d^2 + (a_3^2a_4^3 - a_2a_4^4 \\
 & & -2a_3^3a_4a_5 + 2a_1a_4^3a_5 + 3a_2a_3^2a_5^2 - 3a_0a_4^2a_5^2 - 2a_1a_2a_5^3 - 2a_0a_3a_5^3)d^3 + (a_4a_5^4)d^4)] \\
f_0 & = & [a_0^6 + (-a_1^6 + 6a_0a_1^4a_2 - 9a_0^2a_1^2a_2^2 + 2a_0^3a_2^3 - 6a_0^2a_1^3a_3 + 12a_0^3a_1a_2a_3 - 3a_0^4a_3^2 + 6a_0^3a_1^2a_4 \\
 & & -6a_0^4a_2a_4 - 6a_0^4a_1a_5)d + (a_2^6 - 6a_1a_2^4a_3 + 9a_1^2a_2^2a_3^2 + 6a_0a_2^3a_3^2 - 2a_1^3a_3^3 - 12a_0a_1a_2a_3^3 \\
 & & +3a_0^2a_3^4 + 6a_1^2a_2^3a_4 - 6a_0a_2^4a_4 - 12a_1^3a_2a_3a_4 + 18a_0a_1^2a_3^2a_4 + 3a_1^4a_4^2 + 9a_0^2a_2^2a_4^2 \\
 & & -18a_0^2a_1a_3a_4^2 + 2a_0^3a_4^3 - 6a_1^3a_2^2a_5 + 12a_0a_1a_2^3a_5 + 6a_1^4a_3a_5 - 18a_0^2a_2^2a_3a_5 - 12a_0a_1^3a_4a_5 \\
 & & +12a_0^3a_3a_4a_5 + 9a_0^2a_1^2a_5^2 + 6a_0^3a_2a_5^2)d^2 + (-a_3^6 + 6a_2a_3^4a_4 - 9a_2^2a_3^2a_4^2 - 6a_1a_3^3a_4^2 + 2a_2^3a_4^3 \\
 & & +12a_1a_2a_3a_4^3 + 6a_0a_3^2a_4^3 - 3a_1^2a_4^4 - 6a_0a_2a_4^4 - 6a_2^2a_3^3a_5 + 6a_1a_3^4a_5 + 12a_2^3a_3a_4a_5 \\
 & & -12a_0a_3^3a_4a_5 - 18a_1a_2^2a_4^2a_5 + 12a_0a_1a_4^3a_5 - 3a_2^4a_5^2 - 9a_1^2a_3^2a_5^2 + 18a_0a_2a_3^2a_5^2 + 18a_1^2a_2a_4a_5^2 \\
 & & -9a_0^2a_4^2a_5^2 - 2a_1^3a_5^3 - 12a_0a_1a_2a_5^3 - 6a_0^2a_3a_5^3)d^3 + (a_4^6 - 6a_3a_4^4a_5 + 9a_3^2a_4^2a_5^2 + 6a_2a_4^3a_5^2 \\
 & & -2a_3^3a_5^3 - 12a_2a_3a_4a_5^3 - 6a_1a_4^2a_5^3 + 3a_2^2a_5^4 + 6a_1a_3a_5^4 + 6a_0a_4a_5^4)d^4 + (-a_5^6)d^5].
\end{array}\right.$$

Proof. Let $\alpha_i = \sigma_i(\alpha)$, with $i = 1, 2, 3, 4, 5, 6$. Thus,

$$\left\{\begin{array}{l}
\alpha_1 = a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5, \\
\alpha_2 = a_0 + a_1\theta\zeta_6 + a_2\theta^2\zeta_6^2 + a_3\theta^3\zeta_6^3 + a_4\theta^4\zeta_6^4 + a_5\theta^5\zeta_6^5, \\
\alpha_3 = a_0 + a_1\theta\zeta_6^2 + a_2\theta^2\zeta_6^4 + a_3\theta^3 + a_4\theta^4\zeta_6^2 + a_5\theta^5\zeta_6^4, \\
\alpha_4 = a_0 + a_1\theta\zeta_6^3 + a_2\theta^2 + a_3\theta^3\zeta_6^3 + a_4\theta^4 + a_5\theta^5\zeta_6^3, \\
\alpha_5 = a_0 + a_1\theta\zeta_6^4 + a_2\theta^2\zeta_6^2 + a_3\theta^3 + a_4\theta^4\zeta_6^4 + a_5\theta^5\zeta_6^2 \\
\alpha_6 = a_0 + a_1\theta\zeta_6^5 + a_2\theta^2\zeta_6^4 + a_3\theta^3\zeta_6^3 + a_4\theta^4\zeta_6^2 + a_5\theta^5\zeta_6.
\end{array}\right.$$

The characteristic polynomial of α is given by

$$
\begin{aligned}
f_\alpha(x) &= (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)(x-\alpha_5)(x-\alpha_6)\\
&= x^6 - x^5(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6) + x^4(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_4\\
&\quad +\alpha_3\alpha_4+\alpha_1\alpha_5+\alpha_2\alpha_5+\alpha_3\alpha_5+\alpha_4\alpha_5+\alpha_1\alpha_6+\alpha_2\alpha_6+\alpha_3\alpha_6+\alpha_4\alpha_6+\alpha_5\alpha_6)\\
&\quad -x^3(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4+\alpha_1\alpha_2\alpha_5+\alpha_1\alpha_3\alpha_5+\alpha_2\alpha_3\alpha_5+\alpha_1\alpha_4\alpha_5\\
&\quad +\alpha_2\alpha_4\alpha_5+\alpha_3\alpha_4\alpha_5+\alpha_1\alpha_2\alpha_6+\alpha_1\alpha_3\alpha_6+\alpha_2\alpha_3\alpha_6+\alpha_1\alpha_4\alpha_6+\alpha_2\alpha_4\alpha_6+\alpha_3\alpha_4\alpha_6\\
&\quad +\alpha_1\alpha_5\alpha_6+\alpha_2\alpha_5\alpha_6+\alpha_3\alpha_5\alpha_6+\alpha_4\alpha_5\alpha_6)+x^2(\alpha_1\alpha_2\alpha_3\alpha_4+\alpha_1\alpha_2\alpha_3\alpha_5+\alpha_1\alpha_2\alpha_4\alpha_5\\
&\quad +\alpha_1\alpha_3\alpha_4\alpha_5+\alpha_2\alpha_3\alpha_4\alpha_5+\alpha_1\alpha_2\alpha_3\alpha_6+\alpha_1\alpha_2\alpha_4\alpha_6+\alpha_1\alpha_3\alpha_4\alpha_6+\alpha_2\alpha_3\alpha_4\alpha_6\\
&\quad +\alpha_1\alpha_2\alpha_5\alpha_6+\alpha_1\alpha_3\alpha_5\alpha_6+\alpha_2\alpha_3\alpha_5\alpha_6+\alpha_1\alpha_4\alpha_5\alpha_6+\alpha_2\alpha_4\alpha_5\alpha_6+\alpha_3\alpha_4\alpha_5\alpha_6)\\
&\quad -x(\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5+\alpha_1\alpha_2\alpha_3\alpha_4\alpha_6+\alpha_1\alpha_2\alpha_3\alpha_5\alpha_6+\alpha_1\alpha_2\alpha_4\alpha_5\alpha_6+\alpha_1\alpha_3\alpha_4\alpha_5\alpha_6\\
&\quad +\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6)+(\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6).
\end{aligned}
$$

Since $\zeta_6+\zeta_6^2+\zeta_6^3+\zeta_6^4+\zeta_6^5=-1$, it follows that

- $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6=6a_0$.

- $\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_4+\alpha_3\alpha_4+\alpha_1\alpha_5+\alpha_2\alpha_5+\alpha_3\alpha_5+\alpha_4\alpha_5+\alpha_1\alpha_6$ $+\alpha_2\alpha_6+\alpha_3\alpha_6+\alpha_4\alpha_6+\alpha_5\alpha_6=3(5a_0^2-(a_3^2+2a_2a_4+2a_1a_5)d)$.

- $\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4+\alpha_1\alpha_2\alpha_5+\alpha_1\alpha_3\alpha_5+\alpha_2\alpha_3\alpha_5+\alpha_1\alpha_4\alpha_5$ $+\alpha_2\alpha_4\alpha_5+\alpha_3\alpha_4\alpha_5+\alpha_1\alpha_2\alpha_6+\alpha_1\alpha_3\alpha_6+\alpha_2\alpha_3\alpha_6+\alpha_1\alpha_4\alpha_6+\alpha_2\alpha_4\alpha_6+\alpha_3\alpha_4\alpha_6$ $+\alpha_1\alpha_5\alpha_6+\alpha_2\alpha_5\alpha_6+\alpha_3\alpha_5\alpha_6+\alpha_4\alpha_5\alpha_6=2(10a_0^3+(a_2^3+6a_1a_2a_3-6a_0a_3^2+3a_1^2a_4$ $-12a_0a_2a_4-12a_0a_1a_5)d+(a_4^3+6a_3a_4a_5+3a_2a_5^2)d^2)$.

- $\alpha_1\alpha_2\alpha_3\alpha_4+\alpha_1\alpha_2\alpha_3\alpha_5+\alpha_1\alpha_2\alpha_4\alpha_5+\alpha_1\alpha_3\alpha_4\alpha_5+\alpha_2\alpha_3\alpha_4\alpha_5+\alpha_1\alpha_2\alpha_3\alpha_6+\alpha_1\alpha_2\alpha_4\alpha_6$ $+\alpha_1\alpha_3\alpha_4\alpha_6+\alpha_2\alpha_3\alpha_4\alpha_6+\alpha_1\alpha_2\alpha_5\alpha_6+\alpha_1\alpha_3\alpha_5\alpha_6+\alpha_2\alpha_3\alpha_5\alpha_6+\alpha_1\alpha_4\alpha_5\alpha_6+\alpha_2\alpha_4\alpha_5\alpha_6$ $+\alpha_3\alpha_4\alpha_5\alpha_6=3(5a_0^4+(-3a_1^2a_2^2+2a_0a_2^3-2a_1^3a_3+12a_0a_1a_2a_3-6a_0^2a_3^2+6a_0a_1^2a_4$ $-12a_0^2a_2a_4-12a_0^2a_1a_5)d+(a_3^4+3a_2^2a_4^2-6a_1a_3a_4^2+2a_0a_4^3-6a_2^2a_3a_5+12a_0a_3a_4a_5$ $+3a_1^2a_5^2+6a_0a_2a_5^2)d^2+(-3a_4^2a_5^2-2a_3a_5^3)d^3)$.

- $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5+\alpha_1\alpha_2\alpha_3\alpha_4\alpha_6+\alpha_1\alpha_2\alpha_3\alpha_5\alpha_6+\alpha_1\alpha_2\alpha_4\alpha_5\alpha_6+\alpha_1\alpha_3\alpha_4\alpha_5\alpha_6+\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$ $=6(a_0^5+(a_1^4a_2-3a_0a_1^2a_2^2+a_0^2a_2^3-2a_0a_1^3a_3+6a_0^2a_1a_2a_3-2a_0^3a_3^2+3a_0^2a_1^2a_4-4a_0^3a_2a_4$ $-4a_0^3a_1a_5)d+(a_2^3a_3^2-2a_1a_2a_3^3+a_0a_3^4-a_2^4a_4+3a_1^2a_3^2a_4+3a_0a_2^2a_4^2-6a_0a_1a_3a_4^2+a_0^2a_4^3$ $+2a_1a_2^3a_5-6a_0a_2^2a_3a_5-2a_1^3a_4a_5+6a_0^2a_3a_4a_5+3a_0a_1^2a_5^2+3a_0^2a_2a_5^2)d^2+(a_3^2a_4^3-a_2a_4^4$ $-2a_3^3a_4a_5+2a_1a_4^3a_5+3a_2a_3^2a_5^2-3a_0a_4^2a_5^2-2a_1a_2a_5^3-2a_0a_3a_5^3)d^3+(a_4a_5^4)d^4)$.

- $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6=a_0^6+(-a_1^6+6a_0a_1^4a_2-9a_0^2a_1^2a_2^2+2a_0^3a_2^3-6a_0^2a_1^3a_3+12a_0^3a_1a_2a_3$ $-3a_0^4a_3^2+6a_0^3a_1^2a_4-6a_0^4a_2a_4-6a_0^4a_1a_5)d+(a_2^6-6a_1a_2^4a_3+9a_1^2a_2^2a_3^2+6a_0a_2^3a_3^2-2a_1^3a_3^3$ $-12a_0a_1a_2a_3^3+3a_0^2a_3^4+6a_1^2a_2^3a_4-6a_0a_2^4a_4-12a_1^3a_2a_3a_4+18a_0a_1^2a_3^2a_4+3a_1^4a_4^2+9a_0^2a_2^2a_4^2$ $-18a_0^2a_1a_3a_4^2+2a_0^3a_4^3-6a_1^3a_2^2a_5+12a_0a_1a_2^3a_5+6a_1^4a_3a_5-18a_0^2a_2^2a_3a_5-12a_0a_1^3a_4a_5$ $+12a_0^3a_3a_4a_5+9a_0^2a_1^2a_5^2+6a_0^3a_2a_5^2)d^2+(-a_3^6+6a_2a_3^4a_4-9a_2^2a_3^2a_4^2-6a_1a_3^3a_4^2+2a_2^3a_4^3$ $+12a_1a_2a_3a_4^3+6a_0a_3^2a_4^3-3a_1^2a_4^4-6a_0a_2a_4^4-6a_2^2a_3^3a_5+6a_1a_3^4a_5+12a_2^3a_3a_4a_5-12a_0a_3^3a_4a_5$ $-18a_1a_2^2a_4^2a_5+12a_0a_1a_4^3a_5-3a_2^4a_5^2-9a_1^2a_3^2a_5^2+18a_0a_2a_3^2a_5^2+18a_1^2a_2a_4a_5^2-9a_0^2a_4^2a_5^2$ $-2a_1^3a_5^3-12a_0a_1a_2a_5^3-6a_0^2a_3a_5^3)d^3+(a_4^6-6a_3a_4^4a_5+9a_3^2a_4^2a_5^2+6a_2a_4^3a_5^2-2a_3^3a_5^3$ $-12a_2a_3a_4a_5^3-6a_1a_4^2a_5^3+3a_2^2a_5^4+6a_1a_3a_5^4+6a_0a_4a_5^4)d^4+(-a_5^6)d^5$.

Thus,

$$\begin{array}{rcl} f_\alpha(x) & = & x^6 - x^5[6a_0] + x^4[3(5a_0^2 - (a_3^2 + 2a_2a_4 + 2a_1a_5)d)] - x^3[2(10a_0^3 + (a_2^3 + 6a_1a_2a_3 \\ & & -6a_0a_3^2 + 3a_1^2a_4 - 12a_0a_2a_4 - 12a_0a_1a_5)d + (a_4^3 + 6a_3a_4a_5 + \\ & & +3a_2a_5^2)d^2)] + x^2[3(5a_0^4 + (-3a_1^2a_2^2 + 2a_0a_2^3 - 2a_1^3a_3 + 12a_0a_1a_2a_3 - \\ & & -6a_0^2a_3^2 + 6a_0a_1^2a_4 - 12a_0^2a_2a_4 - 12a_0^2a_1a_5)d + (a_3^4 + 3a_2^2a_4^2 - 6a_1a_3a_4^2 + \\ & & +2a_0a_4^3 - 6a_2^2a_3a_5 + 12a_0a_3a_4a_5 + 3a_1^2a_5^2 + 6a_0a_2a_5^2)d^2 + (-3a_4^2a_5^2 - \\ & & -2a_3a_5^3)d^3)] - x[6(a_0^5 + (a_1^4a_2 - 3a_0a_1^2a_2^2 + a_0^2a_2^3 - 2a_0a_1^3a_3 + 6a_0^2a_1a_2a_3 - \\ & & -2a_0^3a_3^2 + 3a_0^2a_1^2a_4 - 4a_0^3a_2a_4 - 4a_0^3a_1a_5)d + (a_2^3a_3^2 - 2a_1a_2a_3^3 + a_0a_3^4 - \\ & & -a_2^4a_4 + 3a_1^2a_3^2a_4 + 3a_0a_2^2a_4^2 - 6a_0a_1a_3a_4^2 + a_0^2a_4^3 + 2a_1a_2^3a_5 - 6a_0a_2^2a_3a_5 - \\ & & -2a_1^3a_4a_5 + 6a_0^2a_3a_4a_5 + 3a_0a_1^2a_5^2 + 3a_0^2a_2a_5^2)d^2 + (a_3^2a_4^3 - a_2a_4^4 - 2a_3^3a_4a_5 + \\ & & +2a_1a_4^3a_5 + 3a_2a_3^2a_5^2 - 3a_0a_4^2a_5^2 - 2a_1a_2a_5^3 - 2a_0a_3a_5^3)d^3 + (a_4a_5^4)d^4)] + \\ & & +[a_0^6 + (-a_1^6 + 6a_0a_1^4a_2 - 9a_0^2a_1^2a_2^2 + 2a_0^3a_2^3 - 6a_0^2a_1^3a_3 + 12a_0^3a_1a_2a_3 - \\ & & -3a_0^4a_3^2 + 6a_0^3a_1^2a_4 - 6a_0^4a_2a_4 - 6a_0^4a_1a_5)d + (a_2^6 - 6a_1a_2^4a_3 + 9a_1^2a_2^2a_3^2 + \\ & & +6a_0a_2^3a_3^2 - 2a_1^3a_3^3 - 12a_0a_1a_2a_3^3 + 3a_0^2a_3^4 + 6a_1^2a_2^3a_4 - 6a_0a_2^4a_4 - \\ & & -12a_1^3a_2a_3a_4 + 18a_0a_1^2a_3^2a_4 + 3a_1^4a_4^2 + 9a_0^2a_2^2a_4^2 - 18a_0^2a_1a_3a_4^2 + 2a_0^3a_4^3 - \\ & & -6a_1^3a_2^2a_5 + 12a_0a_1a_2^3a_5 + 6a_1^4a_3a_5 - 18a_0^2a_2^2a_3a_5 - 12a_0a_1^3a_4a_5 + \\ & & +12a_0^3a_3a_4a_5 + 9a_0^2a_1^2a_5^2 + 6a_0^3a_2a_5^2)d^2 + (-a_3^6 + 6a_2a_3^4a_4 - 9a_2^2a_3^2a_4^2 - \\ & & -6a_1a_3^3a_4^2 + 2a_2^3a_4^3 + 12a_1a_2a_3a_4^3 + 6a_0a_3^2a_4^3 - 3a_1^2a_4^4 - 6a_0a_2a_4^4 - 6a_2^2a_3^3a_5 + \\ & & +6a_1a_3^4a_5 + 12a_2^3a_3a_4a_5 - 12a_0a_3^3a_4a_5 - 18a_1a_2^2a_4^2a_5 + 12a_0a_1a_4^3a_5 - \\ & & -3a_2^4a_5^2 - 9a_1^2a_3^2a_5^2 + 18a_0a_2a_3^2a_5^2 + 18a_1^2a_2a_4a_5^2 - 9a_0^2a_4^2a_5^2 - 2a_1^3a_5^3 - \\ & & -12a_0a_1a_2a_5^3 - 6a_0^2a_3a_5^3)d^3 + (a_4^6 - 6a_3a_4^4a_5 + 9a_3^2a_4^2a_5^2 + 6a_2a_4^3a_5^2 - 2a_3^3a_5^3 - \\ & & -12a_2a_3a_4a_5^3 - 6a_1a_4^2a_5^3 + 3a_2^2a_5^4 + 6a_1a_3a_5^4 + 6a_0a_4a_5^4)d^4 + (-a_5^6)d^5], \end{array}$$

which proves the result. □

Theorem 2.2. *The ring of algebraic integers of* $\mathbb{K}$ *is given by*

$$\begin{cases} \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2+\mathbb{Z}\theta^3+\mathbb{Z}\theta^4+\mathbb{Z}\theta^5, \ \textit{if } d \not\equiv \pm 1, \pm 17, \pm 10, -15, -11, -7, -3, 5, 13 (mod\ 36) \\ \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2+\mathbb{Z}\left(\frac{1+\theta^3}{2}\right)+\mathbb{Z}\left(\frac{4+3\theta+4\theta^2+\theta^4}{6}\right)+\mathbb{Z}\left(\frac{3+4\theta+3\theta^2+\theta^3+\theta^5}{6}\right), \ \textit{if } d \equiv 1 (mod\ 36) \\ \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2+\mathbb{Z}\theta^3+\mathbb{Z}\left(\frac{1+2\theta^2+\theta^4}{3}\right)+\mathbb{Z}\left(\frac{\theta+2\theta^3+\theta^5}{3}\right), \ \textit{if } d \equiv -10, -1 (mod\ 36) \\ \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2+\mathbb{Z}\left(\frac{1+\theta^3}{2}\right)+\mathbb{Z}\left(\frac{4+3\theta+2\theta^2+\theta^4}{6}\right)+\mathbb{Z}\left(\frac{4\theta+3\theta^2+2\theta^3+\theta^5}{6}\right), \ \textit{if } d \equiv 17 (mod\ 36) \\ \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2+\mathbb{Z}\theta^3+\mathbb{Z}\left(\frac{1+\theta^2+\theta^4}{3}\right)+\mathbb{Z}\left(\frac{\theta+\theta^3+\theta^5}{3}\right), \ \textit{if } d \equiv -17, 10 (mod\ 36) \\ \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2+\mathbb{Z}\left(\frac{1+\theta^3}{2}\right)+\mathbb{Z}\left(\frac{\theta+\theta^4}{2}\right)+\mathbb{Z}\left(\frac{\theta^2+\theta^5}{2}\right), \ \textit{if } d \equiv -15, -11, -7, -3, 5, 13 (mod\ 36). \end{cases}$$

Proof. If $\alpha \in \mathcal{O}_{\mathbb{K}} \subseteq \mathbb{K}$, then $\alpha = a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5$, with $a_0, a_1, a_2, a_3, a_4, a_5 \in \mathbb{Q}$. From [2, Proposition 4], it follows that $6a_0, 6a_1, 6a_2, 6a_3, 6a_4, 6a_5 \in \mathbb{Z}$. Thus, $6a_i = p_i$, with $p_i \in \mathbb{Z}$, for all $i = 0, 1, 2, 3, 4, 5$, that is,

$$a_i = \frac{p_i}{6}, \text{for all } i = 0, 1, 2, 3, 4, 5.$$

Also, p_i can be rewritten as

$$p_i = 6q_i + r_i,$$

where $q_i, r_i \in \mathbb{Z}$ and $r_i \in \{0, 1, 2, 3, 4, 5\}$, for $i = 0, 1, 2, 3, 4, 5$. Therefore,

$$\alpha = a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5 = \frac{p_0}{6} + \frac{p_1}{6}\theta + \frac{p_2}{6}\theta^2 + \frac{p_3}{6}\theta^3 + \frac{p_4}{6}\theta^4 + \frac{p_5}{6}\theta^5 =$$
$$= \frac{6q_0 + r_0}{6} + \frac{6q_1 + r_1}{6}\theta + \frac{6q_2 + r_2}{6}\theta^2 + \frac{6q_3 + r_3}{6}\theta^3 + \frac{6q_4 + r_4}{6}\theta^4 + \frac{6q_5 + r_5}{6}\theta^5 =$$
$$= q_0 + q_1\theta + q_2\theta^2 + q_3\theta^3 + q_4\theta^4 + q_5\theta^5 + \frac{r_0}{6} + \frac{r_1}{6}\theta + \frac{r_2}{6}\theta^2 + \frac{r_3}{6}\theta^3 + \frac{r_4}{6}\theta^4 + \frac{r_5}{6}\theta^5.$$

So,

$$\alpha = q_0 + q_1\theta + q_2\theta^2 + q_3\theta^3 + q_4\theta^4 + q_5\theta^5 + \frac{r_0}{6} + \frac{r_1}{6}\theta + \frac{r_2}{6}\theta^2 + \frac{r_3}{6}\theta^3 + \frac{r_4}{6}\theta^4 + \frac{r_5}{6}\theta^5. \tag{2.1}$$

Since $q = q_0 + q_1\theta + q_2\theta^2 + q_3\theta^3 + q_4\theta^4 + q_5\theta^5 \in \mathcal{O}_{\mathbb{K}}$, it follows that

$$\alpha \in \mathcal{O}_{\mathbb{K}} \Leftrightarrow r = \frac{r_0}{6} + \frac{r_1}{6}\theta + \frac{r_2}{6}\theta^2 + \frac{r_3}{6}\theta^3 + \frac{r_4}{6}\theta^4 + \frac{r_4}{6}\theta^5 \in \mathcal{O}_{\mathbb{K}}. \tag{2.2}$$

Using the magma soft and from Proposition 2.1, it follows that $r \in \mathcal{O}_{\mathbb{K}}$ if and only if

$$\begin{cases} 6[\frac{r_0}{6}] \in \mathbb{Z} \\ 3[(\frac{5r_0^2}{36} - (\frac{r_3^2}{36} + \frac{2r_2r_4}{36} + \frac{2r_1r_5}{36})d)] \in \mathbb{Z}, \\ 2[\frac{10r_0^3}{216} + (\frac{r_2^3}{216} + \frac{6r_1r_2r_3}{216} - \frac{6r_0r_3^2}{216} + \frac{3r_1^2r_4}{216} - \frac{12r_0r_2r_4}{216} - \frac{12r_0r_1r_5}{216})d+ \\ +(\frac{r_4^3}{216} + \frac{6r_3r_4r_5}{216} + \frac{3r_2r_5^2}{216})d^2] \in \mathbb{Z}, \\ 3[\frac{5r_0^4}{1296} + (-\frac{3r_1^2r_2^2}{1296} + \frac{2r_0r_2^3}{1296} - \frac{2r_1^3r_3}{1296} + \frac{12r_0r_1r_2r_3}{1296} - \frac{6r_0^2r_3^2}{1296} + \frac{6r_0r_1^2r_4}{1296} - \frac{12r_0^2r_2r_4}{1296} - \\ -\frac{12r_0^2r_1r_5}{1296})d + (\frac{r_3^4}{1296} + \frac{3r_2^2r_4^2}{1296} - \frac{6r_1r_3r_4^2}{1296} + \frac{2r_0r_4^3}{1296} - \frac{6r_2^2r_3r_5}{1296} + \frac{12r_0r_3r_4r_5}{1296} + \frac{3r_1^2r_5^2}{1296} + \\ +\frac{6r_0r_2r_5^2}{1296})d^2 + (-\frac{3r_4^2r_5^2}{1296} - \frac{2r_3r_5^3}{1296})d^3] \in \mathbb{Z}, \\ 6[\frac{r_0^5}{7776} + (\frac{r_1^4r_2}{7776} - \frac{3r_0r_1^2r_2^2}{7776} + \frac{r_0^2r_2^3}{7776} - \frac{2r_0r_1^3r_3}{7776} + \frac{6r_0^2r_1r_2r_3}{7776} - \frac{2r_0^3r_3^2}{7776} + \frac{3r_0^2r_1^2r_4}{7776} - \frac{4r_0^3r_2r_4}{7776} - \\ -\frac{4r_0^3r_1r_5}{7776})d + (\frac{r_2^3r_3^2}{7776} - \frac{2r_1r_2r_3^3}{7776} + \frac{r_0r_3^4}{7776} - \frac{r_2^4r_4}{7776} + \frac{3r_1^2r_3^2r_4}{7776} + \frac{3r_0r_2^2r_4^2}{7776} - \frac{6r_0r_1r_3r_4^2}{7776} + \\ +\frac{r_0^2r_4^3}{7776} + \frac{2r_1r_2^3r_5}{7776} - \frac{6r_0r_2^2r_3r_5}{7776} - \frac{2r_1^3r_4r_5}{7776} + \frac{6r_0^2r_3r_4r_5}{7776} + \frac{3r_0r_1^2r_5^2}{7776} + \frac{3r_0^2r_2r_5^2}{7776})d^2+ \\ +(\frac{r_3^2r_4^3}{7776} - \frac{r_2r_4^4}{7776} - \frac{2r_3^3r_4r_5}{7776} + \frac{2r_1r_4^3r_5}{7776} + \frac{3r_2r_3^2r_5^2}{7776} - \frac{3r_0r_4^2r_5^2}{7776} - \frac{2r_1r_2r_5^3}{7776} - \frac{2r_0r_3r_5^3}{7776})d^3+ \\ +(\frac{r_4r_5^4}{7776})d^4] \in \mathbb{Z} \text{ and} \\ \frac{r_0^6}{46656} + (-\frac{r_1^6}{46656} + \frac{6r_0r_1^4r_2}{46656} - \frac{9r_0^2r_1^2r_2^2}{46656} + \frac{2r_0^3r_2^3}{46656} - \frac{6r_0^2r_1^3r_3}{46656} + \frac{12r_0^3r_1r_2r_3}{46656} - \frac{3r_0^4r_3^2}{46656} + \\ +\frac{6r_0^3r_1^2r_4}{46656} - \frac{6r_0^4r_2r_4}{46656} - \frac{6r_0^4r_1r_5}{46656})d + (\frac{r_2^6}{46656} - \frac{6r_1r_2^4r_3}{46656} + \frac{9r_1^2r_2^2r_3^2}{46656} + \frac{6r_0r_2^3r_3^2}{46656} - \frac{2r_1^3r_3^3}{46656} - \\ -\frac{12r_0r_1r_2r_3^3}{46656} + \frac{3r_0^2r_3^4}{46656} + \frac{6r_1^2r_2^3r_4}{46656} - \frac{6r_0r_2^4r_4}{46656} - \frac{12r_1^3r_2r_3r_4}{46656} + \frac{18r_0r_1^2r_3^2r_4}{46656} + \frac{3r_1^4r_4^2}{46656} + \\ +\frac{9r_0^2r_2^2r_4^2}{46656} - \frac{18r_0^2r_1r_3r_4^2}{46656} + \frac{2r_0^3r_4^3}{46656} - \frac{6r_1^3r_2^2r_5}{46656} + \frac{12r_0r_1r_2^3r_5}{46656} + \frac{6r_1^4r_3r_5}{46656} - \frac{18r_0^2r_2^2r_3r_5}{46656} - \\ -\frac{12r_0r_1^3r_4r_5}{46656} + \frac{12r_0^3r_3r_4r_5}{46656} + \frac{9r_0^2r_1^2r_5^2}{46656} + \frac{6r_0^3r_2r_5^2}{46656})d^2 + (-\frac{r_3^6}{46656} + \frac{6r_2r_3^4r_4}{46656} - \frac{9r_2^2r_3^2r_4^2}{46656} - \\ -\frac{6r_1r_3^3r_4^2}{46656} + \frac{2r_2^3r_4^3}{46656} + \frac{12r_1r_2r_3r_4^3}{46656} + \frac{6r_0r_3^2r_4^3}{46656} - \frac{3r_1^2r_4^4}{46656} - \frac{6r_0r_2r_4^4}{46656} - \frac{6r_2^2r_3^3r_5}{46656} + \frac{6r_1r_3^4r_5}{46656} + \\ +\frac{12r_2^3r_3r_4r_5}{46656} - \frac{12r_0r_3^3r_4r_5}{46656} - \frac{18r_1r_2^2r_4^2r_5}{46656} + \frac{12r_0r_1r_4^3r_5}{46656} - \frac{3r_2^4r_5^2}{46656} - \frac{9r_1^2r_3^2r_5^2}{46656} + \frac{18r_0r_2r_3^2r_5^2}{46656} + \\ +\frac{18r_1^2r_2r_4r_5^2}{46656} - \frac{9r_0^2r_4^2r_5^2}{46656} - \frac{2r_1^3r_5^3}{46656} - \frac{12r_0r_1r_2r_5^3}{46656} - \frac{6r_0^2r_3r_5^3}{46656})d^3 + (\frac{r_4^6}{46656} - \frac{6r_3r_4^4r_5}{46656} + \\ +\frac{9r_3^2r_4^2r_5^2}{46656} + \frac{6r_2r_4^3r_5^2}{46656} - \frac{2r_3^3r_5^3}{46656} - \frac{12r_2r_3r_4r_5^3}{46656} - \frac{6r_1r_4^2r_5^3}{46656} + \frac{3r_2^2r_5^4}{46656} + \frac{6r_1r_3r_5^4}{46656} + \frac{6r_0r_4r_5^4}{46656})d^4+ \\ +(-\frac{r_5^6}{46656})d^5 \in \mathbb{Z}. \end{cases} \tag{2.3}$$

Consider

$$\omega_1 = \frac{5r_0^2 - (r_3^2 + 2r_2r_4 + 2r_1r_5)d}{12}. \tag{2.4}$$

$$\begin{aligned} \omega_2 = {} & \frac{10r_0^3 + (r_2^3 + 6r_1r_2r_3 - 6r_0r_3^2 + 3r_1^2r_4 - 12r_0r_2r_4 - 12r_0r_1r_5)d}{108} + \\ & + \frac{(r_4^3 + 6r_3r_4r_5 + 3r_2r_5^2)d^2}{108}. \end{aligned} \tag{2.5}$$

$$\begin{aligned} \omega_3 = {} & \frac{5r_0^4 + (-3r_1^2r_2^2 + 2r_0r_2^3 - 2r_1^3r_3 + 12r_0r_1r_2r_3 - 6r_0^2r_3^2 + 6r_0r_1^2r_4)d}{432} + \\ & + \frac{(-12r_0^2r_2r_4 - 12r_0^2r_1r_5)d + (r_3^4 + 3r_2^2r_4^2 - 6r_1r_3r_4^2 + 2r_0r_4^3 - 6r_2^2r_3r_5)d^2}{432} + \\ & + \frac{(12r_0r_3r_4r_5 + 3r_1^2r_5^2 + 6r_0r_2r_5^2)d^2 + (-3r_4^2r_5^2 - 2r_3r_5^3)d^3}{432} \end{aligned} \tag{2.6}$$

$$\begin{aligned}\omega_4 =& \frac{r_0^5 + (r_1^4 r_2 - 3r_0 r_1^2 r_2^2 + r_0^2 r_2^3 - 2r_0 r_1^3 r_3 + 6r_0^2 r_1 r_2 r_3 - 2r_0^3 r_3^2)d}{1296} + \\ &+ \frac{(3r_0^2 r_1^2 r_4 - 4r_0^3 r_2 r_4 - 4r_0^3 r_1 r_5)d + (r_2^3 r_3^2 - 2r_1 r_2 r_3^3 + r_0 r_3^4 - r_2^4 r_4)d^2}{1296} + \\ &+ \frac{(3r_1^2 r_3^2 r_4 + 3r_0 r_2^2 r_4^2 - 6r_0 r_1 r_3 r_4^2 + r_0^2 r_4^3 + 2r_1 r_2^3 r_5 - 6r_0 r_2^2 r_3 r_5)d^2}{1296} + \\ &+ \frac{(-2r_1^3 r_4 r_5 + 6r_0^2 r_3 r_4 r_5 + 3r_0 r_1^2 r_5^2 + 3r_0^2 r_2 r_5^2)d^2 + (r_3^2 r_4^3 - r_2 r_4^4 - 2r_3^3 r_4 r_5)d^3}{1296} + \\ &+ \frac{(2r_1 r_4^3 r_5 + 3r_2 r_3^2 r_5^2 - 3r_0 r_4^2 r_5^2 - 2r_1 r_2 r_5^3 - 2r_0 r_3 r_5^3)d^3 + (r_4 r_5^4)d^4}{1296}.\end{aligned} \tag{2.7}$$

$$\begin{aligned}\omega_5 =& \frac{r_0^6 + (-r_1^6 + 6r_0 r_1^4 r_2 - 9r_0^2 r_1^2 r_2^2 + 2r_0^3 r_2^3 - 6r_0^2 r_1^3 r_3 + 12r_0^3 r_1 r_2 r_3 - 3r_0^4 r_3^2)d}{46656} + \\ &+ \frac{(6r_0^3 r_1^2 r_4 - 6r_0^4 r_2 r_4 - 6r_0^4 r_1 r_5)d + (r_2^6 - 6r_1 r_2^4 r_3 + 9r_1^2 r_2^2 r_3^2 + 6r_0 r_2^3 r_3^2)d^2}{46656} + \\ &+ \frac{(-2r_1^3 r_3^3 - 12r_0 r_1 r_2 r_3^3 + 3r_0^2 r_3^4 + 6r_1^2 r_2^3 r_4 - 6r_0 r_2^4 r_4 - 12r_1^3 r_2 r_3 r_4)d^2}{46656} + \\ &+ \frac{(18r_0 r_1^2 r_3^2 r_4 + 3r_1^4 r_4^2 + 9r_0^2 r_2^2 r_4^2 - 18r_0^2 r_1 r_3 r_4^2 + 2r_0^3 r_4^3 - 6r_1^3 r_2^2 r_5)d^2}{46656} + \\ &+ \frac{(12r_0 r_1 r_2^3 r_5 + 6r_1^4 r_3 r_5 - 18r_0^2 r_2^2 r_3 r_5 - 12r_0 r_1^3 r_4 r_5 + 12r_0^3 r_3 r_4 r_5)d^2}{46656} + \\ &+ \frac{(9r_0^2 r_1^2 r_5^2 + 6r_0^3 r_2 r_5^2)d^2 + (-r_3^6 + 6r_2 r_3^4 r_4 - 9r_2^2 r_3^2 r_4^2 - 6r_1 r_3^3 r_4^2 + 2r_2^3 r_4^3)d^3}{46656} + \\ &+ \frac{(12r_1 r_2 r_3 r_4^3 + 6r_0 r_3^2 r_4^3 - 3r_1^2 r_4^4 - 6r_0 r_2 r_4^4 - 6r_2^2 r_3^3 r_5 + 6r_1 r_3^4 r_5)d^3}{46656} + \\ &+ \frac{(12r_2^3 r_3 r_4 r_5 - 12r_0 r_3^3 r_4 r_5 - 18r_1 r_2^2 r_4^2 r_5 + 12r_0 r_1 r_4^3 r_5 - 3r_2^4 r_5^2 - 9r_1^2 r_3^2 r_5^2)d^3}{46656} + \\ &+ \frac{(18r_0 r_2 r_3^2 r_5^2 + 18r_1^2 r_2 r_4 r_5^2 - 9r_0^2 r_4^2 r_5^2 - 2r_1^3 r_5^3 - 12r_0 r_1 r_2 r_5^3 - 6r_0^2 r_3 r_5^3)d^3}{46656} + \\ &+ \frac{(r_4^6 - 6r_3 r_4^4 r_5 + 9r_3^2 r_4^2 r_5^2 + 6r_2 r_4^3 r_5^2 - 2r_3^3 r_5^3 - 12r_2 r_3 r_4 r_5^3 - 6r_1 r_4^2 r_5^3)d^4}{46656} + \\ &+ \frac{(3r_2^2 r_5^4 + 6r_1 r_3 r_5^4 + 6r_0 r_4 r_5^4)d^4 + (-r_5^6)d^5}{46656}.\end{aligned} \tag{2.8}$$

Therefore,

$$\alpha \in \mathcal{O}_{\mathbb{K}} \Leftrightarrow \omega_1, \omega_2, \omega_3, \omega_4, \omega_5 \in \mathbb{Z}. \tag{2.9}$$

Since $r_0, r_1, r_2, r_3, r_4, r_5 \in \{0, 1, 2, 3, 4, 5\}$, it follows that there exist 46656 possibilities for s_j, where $s_j = (r_0, r_1, r_2, r_3, r_4, r_5)$. Let $\alpha = \left(q_0 + \frac{r_0}{6}\right) + \left(q_1 + \frac{r_1}{6}\right)\theta + \left(q_2 + \frac{r_2}{6}\right)\theta^2 + \left(q_3 + \frac{r_3}{6}\right)\theta^3 + \left(q_4 + \frac{r_4}{6}\right)\theta^4 + \left(q_5 + \frac{r_5}{6}\right)\theta^5$, where $q_i \in \mathbb{Z}$ and $r_i = 0, 1, 2, \ldots, 5$.

1. If $\alpha = z_0 + z_1\theta + z_2\theta^2 + z_3\theta^3 + z_4\theta^4 + z_5\theta^5$, with $z_0, z_1, \ldots, z_5 \in \mathbb{Q}$, then

$$\begin{cases} z_0 = q_0 + \frac{r_0}{6} & (1), \\ z_1 = q_1 + \frac{r_1}{6} & (2), \\ z_2 = q_2 + \frac{r_2}{6} & (3), \\ z_3 = q_3 + \frac{r_3}{6} & (4), \\ z_4 = q_4 + \frac{r_4}{6} & (5), \\ z_5 = q_5 + \frac{r_5}{6} & (6). \end{cases}$$

Thus, $\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$ is an integral basis if and only if $z_0,z_1,z_2,z_3,z_4,z_5 \in \mathbb{Z}$. Thus,
(a) From (1), it follows that $r_0 = 0$.
(b) From (2), it follows that $r_1 = 0$.
(c) From (3), it follows that $r_2 = 0$.
(d) From (4), it follows that $r_3 = 0$.
(e) From (5), it follows that $r_4 = 0$.
(f) From (6), it follwos that $r_5 = 0$.
Therefore, the only solution is $s_1 = (0,0,0,0,0,0)$.

2. If

$$\alpha = z_0 + z_1\theta + z_2\theta^2 + z_3\left(\frac{1+\theta^3}{2}\right) + z_4\left(\frac{4+3\theta+4\theta^2+\theta^4}{6}\right) + z_5\left(\frac{3+4\theta+3\theta^2+\theta^3+\theta^5}{6}\right),$$

with $z_0, z_1, \ldots, z_5 \in \mathbb{Q}$, then

$$\begin{cases} z_0 + \frac{z_3}{2} + \frac{4z_4}{6} + \frac{3z_5}{6} = q_0 + \frac{r_0}{6} \quad (1),\\ z_1 + \frac{3z_4}{6} + \frac{4z_5}{6} = q_1 + \frac{r_1}{6} \quad (2),\\ z_2 + \frac{4z_4}{6} + \frac{3z_5}{6} = q_2 + \frac{r_2}{6} \quad (3),\\ \frac{z_3}{2} + \frac{z_5}{6} = q_3 + \frac{r_3}{6} \quad (4),\\ \frac{z_4}{6} = q_4 + \frac{r_4}{6} \quad (5),\\ \frac{z_5}{6} = q_5 + \frac{r_5}{6} \quad (6). \end{cases}$$

Now, $\left\{1,\theta,\theta^2,\left(\frac{1+\theta^3}{2}\right),\left(\frac{4+3\theta+4\theta^2+\theta^4}{6}\right),\left(\frac{3+4\theta+3\theta^2+\theta^3+\theta^5}{6}\right)\right\}$ is an integral basis if and only if $z_0,z_1,z_2,z_3,z_4,z_5 \in \mathbb{Z}$.
(a) From (6), it follows that $z_5 = 6q_5 + r_5$. Thus, $r_5 = 0,1,2,3,4$ or 5.
(b) From (5), it follows that $z_4 = 6q_4 + r_4$. Thus, $r_4 = 0,1,2,3,4$ or 5.
(c) From (4) and (6), it follows that $z_3 = 2q_3 - 2q_5 + \frac{r_3 - r_5}{3}$. Thus, $r_3 \equiv r_5 (mod\ 3)$.
(d) From (3), (5) and (6), it follows that $z_2 = q_2 - 4q_4 - 3q_5 + \frac{r_2 - 4r_4 - 3r_5}{6}$. Thus, $r_2 \equiv 4r_4 + 3r_5 (mod\ 6)$.
(e) From (2), (5) and (6), it follows that $z_1 = q_1 - 3q_4 - 4q_5 + \frac{r_1 - 3r_4 - 4r_5}{6}$. Thus, $r_1 \equiv 3r_4 + 4r_5 (mod\ 6)$.
(f) From (1), (4), (5) and (6), it follows that $z_0 = q_0 - 2q_3 - 4q_4 - 2q_5 + \frac{r_0 - r_3 - 4r_4 - 2r_5}{6}$. Thus, $r_0 \equiv r_3 + 4r_4 + 2r_5 (mod\ 6)$.

Therefore, the solutions are $s_1 = (0,0,0,0,0,0)$, $s_2 = (0,1,0,4,3,4)$, $s_3 = (0,1,3,4,3,1)$, $s_4 = (0,2,3,2,0,5)$, $s_5 = (0,4,3,4,0,1)$, $s_6 = (0,5,0,2,3,2)$, $s_7 = (0,5,3,2,3,5)$, $s_8 = (1,0,1,3,4,3)$, $s_9 = (1,0,4,3,4,0)$, $s_{10} = (1,1,1,1,1,1)$, $s_{11} = (1,1,4,1,1,4)$, $s_{12} = (1,2,1,5,4,5)$, $s_{13} = (1,2,4,5,4,2)$, $s_{14} = (1,3,1,3,1,3)$, $s_{15} = (1,3,4,3,1,0)$, $s_{16} = (1,4,1,1,4,1)$, $s_{17} = (1,4,4,1,4,4)$, $s_{18} = (1,5,1,5,1,5)$, $s_{19} = (1,5,4,5,1,2)$, $s_{20} = (2,0,5,0,2,3)$, $s_{21} = (2,1,2,4,5,4)$, $s_{22} = (2,1,5,4,5,1)$, $s_{23} = (2,2,5,2,2,5)$, $s_{24} = (2,3,2,0,5,0)$, $s_{25} = (2,3,5,0,5,3)$, $s_{26} = (2,4,5,4,2,1)$, $s_{27} = (2,5,2,2,5,2)$, $s_{28} = (2,5,5,2,5,5)$, $s_{29} = (3,1,0,1,3,4)$, $s_{30} = (3,1,3,1,3,1)$, $s_{31} = (3,2,0,5,0,2)$, $s_{32} = (3,2,3,5,0,5)$, $s_{33} = (3,4,0,1,0,4)$, $s_{34} = (3,4,3,1,0,1)$, $s_{35} = (3,5,0,5,3,2)$, $s_{36} = (3,5,3,5,3,5)$, $s_{37} = (4,0,1,0,4,3)$, $s_{38} = (4,1,1,4,1,1)$, $s_{39} = (4,1,4,4,1,4)$, $s_{40} = (4,2,1,2,4,5)$, $s_{41} = (4,3,1,0,1,3)$, $s_{42} = (4,3,4,0,1,0)$, $s_{43} = (4,4,1,4,4,1)$, $s_{44} = (4,5,1,2,1,5)$, $s_{45} = (4,5,4,2,1,2)$, $s_{46} = (5,0,2,3,2,0)$, $s_{47} = (5,0,5,3,2,3)$, $s_{48} = (5,1,2,1,5,4)$, $s_{49} = (5,1,5,1,5,1)$, $s_{50} = (5,2,2,5,2,2)$, $s_{51} = (5,2,5,5,2,5)$, $s_{52} = (5,3,2,3,5,0)$, $s_{53} = (5,3,5,3,5,3)$, $s_{54} = (5,4,2,1,2,4)$, $s_{55} = (5,4,5,1,2,1)$, $s_{56} = (5,5,2,5,5,2)$, $s_{57} = (5,5,5,5,5,5)$, $s_{58} = (0,2,0,2,0,2)$,

$s_{59} = (0,4,0,4,0,4)$, $s_{60} = (2,0,2,0,2,0)$, $s_{61} = (2,2,2,2,2,2)$, $s_{62} = (2,4,2,4,2,4)$, $s_{63} = (4,0,4,0,4,0)$, $s_{64} = (4,2,4,2,4,2)$, $s_{65} = (4,4,4,4,4,4)$, $s_{66} = (0,0,3,0,0,3)$, $s_{67} = (0,3,0,0,3,0)$, $s_{68} = (0,3,3,0,3,3)$, $s_{69} = (3,0,0,3,0,0)$, $s_{70} = (3,0,3,3,0,3)$, $s_{71} = (3,3,0,3,3,0)$ and $s_{72} = (3,3,3,3,3,3)$.

3. If

$$\alpha = z_0 + z_1\theta + z_2\theta^2 + z_3\theta^3 + z_4\left(\frac{1+2\theta^2+\theta^4}{3}\right) + z_5\left(\frac{\theta+2\theta^3+\theta^5}{3}\right),$$

then

$$\begin{cases} z_0 + \frac{z_4}{3} = q_0 + \frac{r_0}{6} & (1),\\ z_1 + \frac{z_5}{3} = q_1 + \frac{r_1}{6} & (2),\\ z_2 + \frac{2z_4}{3} = q_2 + \frac{r_2}{6} & (3),\\ z_3 + \frac{2z_5}{3} = q_3 + \frac{r_3}{6} & (4),\\ \frac{z_4}{3} = q_4 + \frac{r_4}{6} & (5),\\ \frac{z_5}{3} = q_5 + \frac{r_5}{6} & (6). \end{cases}$$

The set $\left\{1,\theta,\theta^2,\theta^3,\left(\frac{1+2\theta^2+\theta^4}{3}\right),\left(\frac{\theta+2\theta^3+\theta^5}{3}\right)\right\}$ is an integral basis if and only if $z_0,\ldots,z_5 \in \mathbb{Z}$.

(a) From (6), it follows that $z_5 = 3q_5 + \frac{r_5}{2}$. Thus, $r_5 = 0, 2$ or 4.

(b) From (5), it follows that $z_4 = 3q_4 + \frac{r_4}{2}$. Thus, $r_4 = 0, 2$ or 4.

(c) From (4) and (6), it follows that $z_3 = q_3 - 2q_5 + \frac{r_3 - 2r_5}{3}$. Thus, $r_3 \equiv 2r_5 (mod\ 6)$. item From (3) and (5), it follows that $z_2 = q_2 - 2q_4 + \frac{r_2 - 2r_4}{6}$. Thus, $r_2 \equiv 2r_4 (mod\ 6)$.

(d) From (2) and (6), it follows that $z_1 = q_1 - q_5 + \frac{r_1 - r_5}{6}$. Thus, $r_1 = r_5$.

(e) From (1) and (5), it follows that $z_0 = q_0 - q_4 + \frac{r_1 - r_4}{6}$. Thus, $r_0 = r_4$.

Therefore, the solutios are $s_1 = (0,0,0,0,0,0)$, $s_{73} = (0,2,0,4,0,2)$, $s_{74} = (0,4,0,2,0,4)$, $s_{75} = (2,0,4,0,2,0)$, $s_{76} = (2,2,4,4,2,2)$, $s_{77} = (2,4,4,2,2,4)$, $s_{78} = (4,0,2,0,4,0)$, $s_{79} = (4,2,2,4,4,2)$ and $s_{80} = (4,4,2,2,4,4)$.

4. If

$$\alpha = z_0 + z_1\theta + z_2\theta^2 + z_3\left(\frac{1+\theta^3}{2}\right) + z_4\left(\frac{4+3\theta+2\theta^2+\theta^4}{6}\right) + z_5\left(\frac{4\theta+3\theta^2+2\theta^3+\theta^5}{6}\right),$$

then

$$\begin{cases} z_0 + \frac{z_3}{2} + \frac{4z_4}{6} = q_0 + \frac{r_0}{6} & (1),\\ z_1 + \frac{3z_4}{6} + \frac{4z_5}{6} = q_1 + \frac{r_1}{6} & (2),\\ z_2 + \frac{2z_4}{6} + \frac{3z_5}{6} = q_2 + \frac{r_2}{6} & (3),\\ \frac{z_3}{2} + \frac{2z_5}{6} = q_3 + \frac{r_3}{6} & (4),\\ \frac{z_4}{6} = q_4 + \frac{r_4}{6} & (5),\\ \frac{z_5}{6} = q_5 + \frac{r_5}{6} & (6). \end{cases}$$

The set $\left\{1,\theta,\theta^2,\left(\frac{1+\theta^3}{2}\right),\left(\frac{4+3\theta+2\theta^2+\theta^4}{6}\right),\left(\frac{4\theta+3\theta^2+2\theta^3+\theta^5}{6}\right)\right\}$ is an integral basis if and only if $z_0, z_1, \ldots, z_5 \in \mathbb{Z}$.

(a) From (6), it follows that $z_5 = 6q_5 + r_5$. Thus, $r_5 = 0, 1, 2, 3, 4$ or 5.

(b) From (5), it follows that $z_4 = 6q_4 + r_4$. Thus, $r_4 = 0, 1, 2, 3, 4$ or 5.

(c) From (4) and (6), it follows that $z_3 = 2q_3 - 4q_5 + \frac{r_3 - 2r_5}{3}$. Thus, $r_3 \equiv 2r_5 (mod\ 3)$.

(d) From (3), (5) and (6), it follows that $z_2 = q_2 - 2q_4 - 3q_5 + \frac{r_2 - 2r_4 - 3r_5}{6}$. Thus, $r_2 \equiv 2r_4 + 3r_5 (mod\ 6)$.

(e) From (2), (5) and (6), it follows that $z_1 = q_1 - 3q_4 - 4q_5 + \frac{r_1 - 3r_4 - 4r_5}{6}$. Thus, $r_1 \equiv 3r_4 + 4r_5 (mod\ 6)$.

(f) From (1), (4) and (5), it follows that $z_0 = q_0 - q_3 - q_4 + \frac{r_0 - r_3 - 4r_4 + 2r_5}{6}$. Thus, $r_0 \equiv r_3 + 4r_4 - 2r_5 (mod\ 6)$.

Therefore, the solutions are $s_1 = (0,0,0,0,0,0)$, $s_{66} = (0,0,3,0,0,3)$, $s_{67} = (0,3,0,0,3,0)$, $s_{68} = (0,3,3,0,3,3)$, $s_{69} = (3,0,0,3,0,0)$, $s_{70} = (3,0,3,3,0,3)$, $s_{71} = (3,3,0,3,3,0)$, $s_{72} = (3,3,3,3,3,3)$, $s_{73} = (0,2,0,4,0,2)$, $s_{74} = (0,4,0,2,0,4)$, $s_{75} = (2,0,4,0,2,0)$, $s_{76} = (2,2,4,4,2,2)$, $s_{77} = (2,4,4,2,2,4)$, $s_{78} = (4,0,2,0,4,0)$, $s_{79} = (4,2,2,4,4,2)$, $s_{80} = (4,4,2,2,4,4)$, $s_{81} = (0,1,0,2,3,4)$, $s_{82} = (0,1,3,2,3,1)$, $s_{83} = (0,2,3,4,0,5)$, $s_{84} = (0,4,3,2,0,1)$, $s_{85} = (0,5,0,4,3,2)$, $s_{86} = (0,5,3,4,3,5)$, $s_{87} = (1,0,2,3,4,0)$, $s_{88} = (1,0,5,3,4,3)$, $s_{89} = (1,1,2,5,1,4)$, $s_{90} = (1,1,5,5,1,1)$, $s_{91} = (1,2,2,1,4,2)$, $s_{92} = (1,2,5,1,4,5)$, $s_{93} = (1,3,2,3,1,0)$, $s_{94} = (1,3,5,3,1,3)$, $s_{95} = (1,4,2,5,4,4)$, $s_{96} = (1,4,5,5,4,1)$, $s_{97} = (1,5,2,1,1,2)$, $s_{98} = (1,5,5,1,1,5)$, $s_{99} = (2,0,1,0,2,3)$, $s_{100} = (2,1,1,2,5,1)$, $s_{101} = (2,1,4,2,5,4)$, $s_{102} = (2,2,1,4,2,5)$, $s_{103} = (2,3,1,0,5,3)$, $s_{104} = (2,3,4,0,5,0)$, $s_{105} = (2,4,1,2,2,1)$, $s_{106} = (2,5,1,4,5,5)$, $s_{107} = (2,5,4,4,5,2)$, $s_{108} = (3,1,0,5,3,4)$, $s_{109} = (3,1,3,5,3,1)$, $s_{110} = (3,2,0,1,0,2)$, $s_{111} = (3,2,3,1,0,5)$, $s_{112} = (3,4,0,5,0,4)$, $s_{113} = (3,4,3,5,0,1)$, $s_{114} = (3,5,0,1,3,2)$, $s_{115} = (3,5,3,1,3,5)$, $s_{116} = (4,0,5,0,4,3)$, $s_{117} = (4,1,2,2,1,4)$, $s_{118} = (4,1,5,2,1,1)$, $s_{119} = (4,2,5,4,4,5)$, $s_{120} = (4,3,2,0,1,0)$, $s_{121} = (4,3,5,0,1,3)$, $s_{122} = (4,4,5,2,4,1)$, $s_{123} = (4,5,2,4,1,2)$, $s_{124} = (4,5,5,4,1,5)$, $s_{125} = (5,0,1,3,2,3)$, $s_{126} = (5,0,4,3,2,0)$, $s_{127} = (5,1,1,5,5,1)$, $s_{128} = (5,1,4,5,5,4)$, $s_{129} = (5,2,1,1,2,5)$, $s_{130} = (5,2,4,1,2,2)$, $s_{131} = (5,3,1,3,5,3)$, $s_{132} = (5,3,4,3,5,0)$, $s_{133} = (5,4,1,5,2,1)$, $s_{134} = (5,4,4,5,2,4)$, $s_{135} = (5,5,1,1,5,5)$ and $s_{136} = (5,5,4,1,5,2)$.

5. If

$$\alpha = z_0 + z_1\theta + z_2\theta^2 + z_3\theta^3 + z_4\left(\frac{1+\theta^2+\theta^4}{3}\right) + z_5\left(\frac{\theta+\theta^3+\theta^5}{3}\right),$$

with $z_0, z_1, \ldots, z_5 \in \mathbb{Q}$, then

$$\begin{cases} z_0 + \frac{z_4}{3} = q_0 + \frac{r_0}{6} & (1), \\ z_1 + \frac{z_5}{3} = q_1 + \frac{r_1}{6} & (2), \\ z_2 + \frac{z_4}{3} = q_2 + \frac{r_2}{6} & (3), \\ z_3 + \frac{z_5}{3} = q_3 + \frac{r_3}{6} & (4), \\ \frac{z_4}{3} = q_4 + \frac{r_4}{6} & (5), \\ \frac{z_5}{3} = q_5 + \frac{r_5}{6} & (6). \end{cases}$$

The set $\left\{1, \theta, \theta^2, \theta^3, \left(\frac{1+\theta^2+\theta^4}{3}\right), \left(\frac{\theta+\theta^3+\theta^5}{3}\right)\right\}$ is an integral basis if and only if $z_0, \ldots, z_5 \in \mathbb{Z}$.

(a) From (6), it follows that $z_5 = 3q_5 + \frac{r_5}{2}$. Thus, $r_5 = 0, 2$ or 4.

(b) From (5), it follows that $z_4 = 3q_4 + \frac{r_4}{2}$. Thus, $r_4 = 0, 2$ or 4.

(c) From (4) and (6), it follows that $z_3 = q_3 - q_5 + \frac{r_3 - r_5}{6}$. Thus, $r_3 = r_5$.

(d) From (3) and (5), it follows that $z_2 = q_2 - q_4 + \frac{r_2 - r_4}{6}$. Thus, $r_2 = r_4$.

(e) From (2) and (6), it follows that $z_1 = q_1 - q_5 + \frac{r_1 - r_5}{6}$. Thus, $r_1 = r_6$.

(f) From (1) and (5), it follows that $z_0 = q_0 - q_4 + \frac{r_0 - r_4}{6}$. Thus, $r_0 = r_4$.

Therefore, the solutions are $s_1 = (0,0,0,0,0,0)$, $s_{58} = (0,2,0,2,0,2)$, $s_{59} = (0,4,0,4,0,4)$, $s_{60} = (2,0,2,0,2,0)$, $s_{61} = (2,2,2,2,2,2)$, $s_{62} = (2,4,2,4,2,4)$, $s_{63} = (4,0,4,0,4,0)$, $s_{64} = (4,2,4,2,4,2)$ and $s_{65} = (4,4,4,4,4,4)$.

6. If

$$\alpha = z_0 + z_1\theta + z_2\theta^2 + z_3\left(\frac{1+\theta^3}{2}\right) + z_4\left(\frac{\theta+\theta^4}{2}\right) + z_5\left(\frac{\theta^2+\theta^5}{2}\right),$$

wiht $z_0, z-1, \ldots, z_5 \in \mathbb{Q}$, then

$$\begin{cases} z_0 + \frac{z_3}{2} = q_0 + \frac{r_0}{6} & (1), \\ z_1 + \frac{z_4}{2} = q_1 + \frac{r_1}{6} & (2), \\ z_2 + \frac{z_5}{2} = q_2 + \frac{r_2}{6} & (3), \\ \frac{z_3}{2} = q_3 + \frac{r_3}{6} & (4), \\ \frac{z_4}{2} = q_4 + \frac{r_4}{6} & (5) \text{ and} \\ \frac{z_5}{2} = q_5 + \frac{r_5}{6} & (6). \end{cases}$$

The set $\left\{1, \theta, \theta^2, \left(\frac{1+\theta^3}{2}\right), \left(\frac{\theta+\theta^4}{2}\right), \left(\frac{\theta^2+\theta^5}{2}\right)\right\}$ is an integral basis if and only if $z_0, z_1, \ldots, z_5 \in \mathbb{Z}$.

(a) From (6), it follows that $z_5 = 2q_5 + \frac{r_5}{3}$. Thus, $r_5 = 0$ or 3.

(b) From (5), it follows that $z_4 = 2q_4 + \frac{r_4}{3}$. Thus, $r_4 = 0$ or 3.

(c) From (4), it follows that $z_3 = 2q_3 + \frac{r_3}{3}$. Thus, $r_3 = 0$ or 3.

(d) From (3) and (6), it follows that $z_2 = q_2 - q_5 + \frac{r_2 - r_5}{6}$. Thus, $r_2 = r_5$.

(e) From (2) and (5), it follows that $z_1 = q_1 - q_4 + \frac{r_1 - r_4}{6}$. Thus, $r_1 = r_4$.

(f) From (1) and (4), it follows that $z_0 = q_0 - q_3 + \frac{r_0 - r_3}{6}$. Thus, $r_0 = r_3$.

Therefore, the solutions are $s_1 = (0,0,0,0,0,0)$, $s_{66} = (0,0,3,0,0,3)$, $s_{67} = (0,3,0,0,3,0)$, $s_{68} = (0,3,3,0,3,3)$, $s_{69} = (3,0,0,3,0,0)$, $s_{70} = (3,0,3,3,0,3)$, $s_{71} = (3,3,0,3,3,0)$ and $s_{72} = (3,3,3,3,3,3)$.

From items (1), (2), (3), (4), (5) and (6) we found 136 solutions in the form $s_j = (r_0, r_1, r_2, r_3, r_4, r_5)$, with $j = 1, 2, \ldots, 136$. Substituing in Equations (2.4), (2.5), (2.6), (2.7) and (2.8) we found the equivaleces of d modulo 36 such that $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5 \in \mathbb{Z}$. In the Table (1), we present some results Therefore,

1. For $d \not\equiv \pm 1, \pm 17, \pm 10, -15, -11, -7, -3, 5, 13 (mod\ 36)$, the solution is s_1 and an integral basis is given by

$$\left\{1, \theta, \theta^2, \theta^3, \theta^4, \theta^5\right\}.$$

2. For $d \equiv 1 (mod\ 36)$, the solutions are s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}, s_{17}, s_{18}, s_{19}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25}, s_{26}, s_{27}, s_{28}, s_{29}, s_{30}, s_{31}, s_{32}, s_{33}, s_{34}, s_{35}, s_{36}, s_{37}, s_{38}, s_{39}, s_{40}, s_{41}, s_{42}, s_{43}, s_{44}, s_{45}, s_{46}, s_{47}, s_{48}, s_{49}, s_{50}, s_{51}, s_{52}, s_{53}, s_{54}, s_{55}, s_{56} and s_{57}, and thus, an integral basis is given by

$$\left\{1, \theta, \theta^2, \left(\frac{1+\theta^3}{2}\right), \left(\frac{4+3\theta+4\theta^2+\theta^4}{6}\right), \left(\frac{3+4\theta+3\theta^2+\theta^3+\theta^5}{6}\right)\right\}.$$

3. For $d \equiv -10, -1 (mod\ 36)$, the solutions are s_{73}, s_{74}, s_{75}, s_{76}, s_{77}, s_{78}, s_{79} and s_{80}, and thus, an integral basis is given by

$$\left\{1, \theta, \theta^2, \theta^3, \left(\frac{1+2\theta^2+\theta^4}{3}\right), \left(\frac{\theta+2\theta^3+\theta^5}{3}\right)\right\}.$$

Table 1: $\mathbb{K} = \mathbb{Q}(\sqrt[6]{d})$, d livre de quadrados - casos para análise de $\mathcal{O}_{\mathbb{K}}$.

Solutions (s_j)	$d \equiv (?)(mod\ 36)$
s_1	$\forall d$
s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}, s_{17}, s_{18}, s_{19}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25}, s_{26}, s_{27}, s_{28}, s_{29}, s_{30}, s_{31}, s_{32}, s_{33}, s_{34}, s_{35}, s_{36}, s_{37}, s_{38}, s_{39}, s_{40}, s_{41}, s_{42}, s_{43}, s_{44}, s_{45}, s_{46}, s_{47}, s_{48}, s_{49}, s_{50}, s_{51}, s_{52}, s_{53}, s_{54}, s_{55}, s_{56} and s_{57}.	$d \equiv 1$
s_{73}, s_{74}, s_{75}, s_{76}, s_{77}, s_{78}, s_{79} and s_{80}.	$d \equiv -10, -1, 17$
s_{81}, s_{82}, s_{83}, s_{84}, s_{85}, s_{86}, s_{87}, s_{88}, s_{89}, s_{90}, s_{91}, s_{92}, s_{93}, s_{94}, s_{95}, s_{96}, s_{97}, s_{98}, s_{99}, s_{100}, s_{101}, s_{102}, s_{103}, s_{104}, s_{105}, s_{106}, s_{107}, s_{108}, s_{109}, s_{110}, s_{111}, s_{112}, s_{113}, s_{114}, s_{115}, s_{116}, s_{117}, s_{118}, s_{119}, s_{120}, s_{121}, s_{122}, s_{123}, s_{124}, s_{125}, s_{126}, s_{127}, s_{128}, s_{129}, s_{130}, s_{131}, s_{132}, s_{133}, s_{134}, s_{135} and s_{136}.	$d \equiv 17$
s_{58}, s_{59}, s_{60}, s_{61}, s_{62}, s_{63}, s_{64} and s_{65}.	$d \equiv -17, 1, 10$
s_{66}, s_{67}, s_{68}, s_{69}, s_{70}, s_{71} and s_{72}.	$d \equiv -15, -11, -7, -3, 1, 5, 13, 17$

4. For $d \equiv 17(mod\ 36)$, the solutions are s_{81}, s_{82}, s_{83}, s_{84}, s_{85}, s_{86}, s_{87}, s_{88}, s_{89}, s_{90}, s_{91},s_{92}, s_{93}, s_{94}, s_{95}, s_{96}, s_{97}, s_{98}, s_{99}, s_{100}, s_{101}, s_{102}, s_{103}, s_{104}, s_{105}, s_{106}, s_{107}, s_{108}, s_{109}, s_{110}, s_{111}, s_{112}, s_{113}, s_{114}, s_{115}, s_{116}, s_{117}, s_{118}, s_{119}, s_{120}, s_{121}, s_{122}, s_{123}, s_{124}, s_{125}, s_{126}, s_{127}, s_{128}, s_{129}, s_{130}, s_{131}, s_{132}, s_{133}, s_{134}, s_{135} and s_{136}, and thus, an integral basis is given by
$$\left\{1, \theta, \theta^2, \left(\frac{1+\theta^3}{2}\right), \left(\frac{4+3\theta+2\theta^2+\theta^4}{6}\right), \left(\frac{4\theta+3\theta^2+2\theta^3+\theta^5}{6}\right)\right\}.$$
5. For $d \equiv -17, 10(mod\ 36)$, the solutions are s_{58}, s_{59}, s_{60}, s_{61}, s_{62}, s_{63}, s_{64} and s_{65}, and thus, an integral basis is given by
$$\left\{1, \theta, \theta^2, \theta^3, \left(\frac{1+\theta^2+\theta^4}{3}\right), \left(\frac{\theta+\theta^3+\theta^5}{3}\right)\right\}.$$
6. For $d \equiv -15, -11, -7, -3, 5, 13(mod\ 36)$, the solutions are s_{66}, s_{67}, s_{68}, s_{69}, s_{70}, s_{71} and s_{72}, and thus, an integral basis is given by
$$\left\{1, \theta, \theta^2, \left(\frac{1+\theta^3}{2}\right), \left(\frac{\theta+\theta^4}{2}\right), \left(\frac{\theta^2+\theta^5}{2}\right)\right\}.$$

Therefore, the ring of algebraic integers of $\mathbb{K}$ is given by

$$\begin{cases} \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2+\mathbb{Z}\theta^3+\mathbb{Z}\theta^4+\mathbb{Z}\theta^5, \text{ if } d \not\equiv \pm 1, \pm 17, \pm 10, -15, -11, -7, -3, 5, 13(mod\ 36) \\ \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2+\mathbb{Z}\left(\frac{1+\theta^3}{2}\right)+\mathbb{Z}\left(\frac{4+3\theta+4\theta^2+\theta^4}{6}\right)+\mathbb{Z}\left(\frac{3+4\theta+3\theta^2+\theta^3+\theta^5}{6}\right), \text{ if } d \equiv 1(mod\ 36) \\ \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2+\mathbb{Z}\theta^3+\mathbb{Z}\left(\frac{1+2\theta^2+\theta^4}{3}\right)+\mathbb{Z}\left(\frac{\theta+2\theta^3+\theta^5}{3}\right), \text{ if } d \equiv -10, -1(mod\ 36) \\ \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2+\mathbb{Z}\left(\frac{1+\theta^3}{2}\right)+\mathbb{Z}\left(\frac{4+3\theta+2\theta^2+\theta^4}{6}\right)+\mathbb{Z}\left(\frac{4\theta+3\theta^2+2\theta^3+\theta^5}{6}\right), \text{ if } d \equiv 17(mod\ 36) \\ \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2+\mathbb{Z}\theta^3+\mathbb{Z}\left(\frac{1+\theta^2+\theta^4}{3}\right)+\mathbb{Z}\left(\frac{\theta+\theta^3+\theta^5}{3}\right), \text{ if } d \equiv -17, 10(mod\ 36) \\ \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2+\mathbb{Z}\left(\frac{1+\theta^3}{2}\right)+\mathbb{Z}\left(\frac{\theta+\theta^4}{2}\right)+\mathbb{Z}\left(\frac{\theta^2+\theta^5}{2}\right), \text{ if } d \equiv -15, -11, -7, -3, 5, 13(mod\ 36), \end{cases}$$

which proves the result. □

Example 2.3. *Let $\mathbb{K} = Q(\theta)$, with $\theta = \sqrt[6]{7}$. Since $d = 7$ and $7 \equiv 7(mod\ 36)$, it follows that*
$$\mathcal{O}_{\mathbb{K}} = \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2+\mathbb{Z}\theta^3+\mathbb{Z}\theta^4+\mathbb{Z}\theta^5.$$

3. Norm, trace and discriminant

Let $\mathbb{K} = \mathbb{Q}(\theta)$, where $\theta = \sqrt[6]{d}$ with $d \in \mathbb{Z}$, $d \neq 1$ and square free. If $\alpha = a_0+a_1\theta+a_2\theta^2+a_3\theta^3+a_4\theta^4+a_5\theta^5 \in \mathbb{K}$, with $a_0, a_1, a_2, a_3, a_4, a_5 \in \mathbb{Q}$, then

$$Tr_{\mathbb{K}}(\alpha) = 6a_0,$$

$$\begin{aligned}\mathcal{N}(\alpha) = {} & a_0^6 + (-a_1^6 + 6a_0a_1^4a_2 - 9a_0^2a_1^2a_2^2 + 2a_0^3a_2^3 - 6a_0^2a_1^3a_3 + 12a_0^3a_1a_2a_3 - 3a_0^4a_3^2+ \\ & + 6a_0^3a_1^2a_4 - 6a_0^4a_2a_4 - 6a_0^4a_1a_5)d + (a_2^6 - 6a_1a_2^4a_3 + 9a_1^2a_2^2a_3^2 + 6a_0a_2^3a_3^2 - 2a_1^3a_3^3- \\ & - 12a_0a_1a_2a_3^3 + 3a_0^2a_3^4 + 6a_1^2a_2^3a_4 - 6a_0a_2^4a_4 - 12a_1^3a_2a_3a_4 + 18a_0a_1^2a_3^2a_4 + 3a_1^4a_4^2+ \\ & + 9a_0^2a_2^2a_4^2 - 18a_0^2a_1a_3a_4^2 + 2a_0^3a_4^3 - 6a_1^3a_2^2a_5 + 12a_0a_1a_2^3a_5 + 6a_1^4a_3a_5 - 18a_0^2a_2^2a_3a_5- \\ & - 12a_0a_1^3a_4a_5 + 12a_0^3a_3a_4a_5 + 9a_0^2a_1^2a_5^2 + 6a_0^3a_2a_5^2)d^2 + (-a_3^6 + 6a_2a_3^4a_4 - 9a_2^2a_3^2a_4^2- \\ & - 6a_1a_3^3a_4^2 + 2a_2^3a_4^3 + 12a_1a_2a_3a_4^3 + 6a_0a_3^2a_4^3 - 3a_1^2a_4^4 - 6a_0a_2a_4^4 - 6a_2^2a_3^3a_5 + 6a_1a_3^4a_5+ \\ & + 12a_2^3a_3a_4a_5 - 12a_0a_3^3a_4a_5 - 18a_1a_2^2a_4^2a_5 + 12a_0a_1a_4^3a_5 - 3a_2^4a_5^2 - 9a_1^2a_3^2a_5^2+ \\ & + 18a_0a_2a_3^2a_5^2 + 18a_1^2a_2a_4a_5^2 - 9a_0^2a_4^2a_5^2 - 2a_1^3a_5^3 - 12a_0a_1a_2a_5^3 - 6a_0^2a_3a_5^3)d^3+ \\ & + (a_4^6 - 6a_3a_4^4a_5 + 9a_3^2a_4^2a_5^2 + 6a_2a_4^3a_5^2 - 2a_3^3a_5^3 - 12a_2a_3a_4a_5^3 - 6a_1a_4^2a_5^3 + 3a_2^2a_5^4+ \\ & + 6a_1a_3a_5^4 + 6a_0a_4a_5^4)d^4 + (-a_5^6)d^5.\end{aligned}$$

Example 3.1. *Let $\mathbb{K} = Q(\theta)$, with $\theta = \sqrt[6]{7}$ and $\alpha = 1+2(\sqrt[6]{7})^5 \in \mathbb{K}$. The trace of α is given by $Tr_{\mathbb{K}}(\alpha) = 6$ and the norm of α is given by $\mathcal{N}(\alpha) = -1075647$.*

Proposition 3.2. *The discriminant of $\mathbb{K}$ is given by*

$$\mathcal{D}(\mathbb{K}) = \begin{cases} 46656d^5, \ \text{if } d \not\equiv \pm1, \pm17, \pm10, -15, -11, -7, -3, 5, 13 (mod\ 36) \\ 9d^5, \ \text{if } d \equiv 1, 17 (mod\ 36) \\ 576d^5, \ \text{if } d \equiv -17, -10, -1, 10 (mod\ 36) \\ 729d^5, \ \text{if } d \equiv -15, -11, -7, -3, 5, 13 (mod\ 36). \end{cases}$$

Proof. From [2, Proposition 3], it follows that

$$Tr_{\mathbb{K}}(\theta^k) = \begin{cases} 0, \ \text{if } k = 1,2,3,4,5 \\ 6d^s, \ \text{if } k = 6s, \text{ with } s \in \mathbb{N}, \\ 0, \ \text{if } k > 6 \text{ and } k \not\equiv 0 (mod\ 6) \end{cases} \tag{3.1}$$

and $Tr_{\mathbb{K}}(1) = 6$, $Tr_{\mathbb{K}}(\theta) = 0$, $Tr_{\mathbb{K}}(\theta^2) = 0$, $Tr_{\mathbb{K}}(\theta^3) = 0$, $Tr_{\mathbb{K}}(\theta^4) = 0$, $Tr_{\mathbb{K}}(\theta^5) = 0$, $Tr_{\mathbb{K}}(d) = 6d$, $Tr_{\mathbb{K}}(\theta^7) = 0$, $Tr_{\mathbb{K}}(\theta^8) = 0$, $Tr_{\mathbb{K}}(\theta^9) = 0$ and $Tr_{\mathbb{K}}(\theta^{10}) = 0$. If $d \not\equiv \pm1, \pm17, \pm10, -15, -11, -7, -3, 5, 13 (mod\ 36)$, then $\{1, \theta, \theta^2, \theta^3, \theta^4, \theta^5\}$ is an integral basis of $\mathbb{K}$. Thus,

$$\mathcal{D}(1,\theta,\theta^2,\theta^3,\theta^4,\theta^5) = \det \begin{pmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6d \\ 0 & 0 & 0 & 0 & 6d & 0 \\ 0 & 0 & 0 & 6d & 0 & 0 \\ 0 & 0 & 6d & 0 & 0 & 0 \\ 0 & 6d & 0 & 0 & 0 & 0 \end{pmatrix} = 46656d^5.$$

If $d \equiv 1 (mod\ 36)$, then an integral basis is given by $\left\{1, \theta, \theta^2, \frac{1+\theta^3}{2}, \frac{4+3\theta+4\theta^2+\theta^4}{6}, \frac{3+4\theta+3\theta^2+\theta^3+\theta^5}{6}\right\}$. Thus,

$$\mathcal{D}\left(1,\theta,\theta^2,\frac{1+\theta^3}{2},\frac{4+3\theta+4\theta^2+\theta^4}{6},\frac{3+4\theta+3\theta^2+\theta^3+\theta^5}{6}\right) = \det \begin{pmatrix} 6 & 0 & 0 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & d & 0 \\ 3 & 0 & 0 & \frac{3+3d}{2} & 2 & \frac{3+d}{2} \\ 4 & 0 & d & 2 & \frac{8+4d}{3} & 2+d \\ 3 & d & 0 & \frac{3+d}{2} & 2+d & \frac{3+3d}{2} \end{pmatrix} = 9d^5.$$

If $d \equiv -10, -1 (mod\ 36)$, then an integral basis is $\left\{1, \theta, \theta^2, \theta^3, \frac{1+2\theta^2+\theta^4}{3}, \frac{\theta+2\theta^3+\theta^5}{3}\right\}$. Thus,

$$\mathcal{D}\left(1, \theta, \theta^2, \theta^3, \frac{1+2\theta^2+\theta^4}{3}, \frac{\theta+2\theta^3+\theta^5}{3}\right) = \det \begin{pmatrix} 6 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2d \\ 0 & 0 & 0 & 0 & 2d & 0 \\ 0 & 0 & 0 & 6d & 0 & 4d \\ 2 & 0 & 2d & 0 & \frac{2}{3} & \frac{4d}{3} \\ 0 & 2d & 0 & 0 & \frac{4d}{3} & 4d \end{pmatrix} = 576d^5.$$

If $d \equiv 17 (mod\ 36)$, then an integral basis is given by $\left\{1, \theta, \theta^2, \frac{1+\theta^3}{2}, \frac{4+3\theta+2\theta^2+\theta^4}{6}, \frac{4\theta+3\theta^2+2\theta^3+\theta^5}{6}\right\}$. Thus,

$$\mathcal{D}\left(1, \theta, \theta^2, \frac{1+\theta^3}{2}, \frac{4+3\theta+2\theta^2+\theta^4}{6}, \frac{4\theta+3\theta^2+2\theta^3+\theta^5}{6}\right) = \det \begin{pmatrix} 6 & 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & d & 0 \\ 3 & 0 & 0 & \frac{3+3d}{2} & 2 & d \\ 4 & 0 & d & 2 & \frac{8+2d}{3} & d \\ 0 & d & 0 & d & d & 2d \end{pmatrix} = 9d^5.$$

If $d \equiv -17, 10 (mod\ 36)$, then an integral basis is given by $\left\{1, \theta, \theta^2, \theta^3, \frac{1+\theta^2+\theta^4}{3}, \frac{\theta++\theta^3+\theta^5}{3}\right\}$. Thus,

$$\mathcal{D}\left(1, \theta, \theta^2, \theta^3, \frac{1+\theta^2+\theta^4}{3}, \frac{\theta++\theta^3+\theta^5}{3}\right) = \det \begin{pmatrix} 6 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2d \\ 0 & 0 & 0 & 0 & 2d & 0 \\ 0 & 0 & 0 & 6d & 0 & 2d \\ 2 & 0 & 2d & 0 & \frac{2+4d}{3} & 0 \\ 0 & 2d & 0 & 2d & 0 & 2d \end{pmatrix} = 576d^5.$$

If $d \equiv -15, -11, -7, -3, 5, 13 (mod\ 36)$, then an integral basis is given by $\left\{1, \theta, \theta^2, \frac{1+\theta^3}{2}, \frac{\theta+\theta^4}{2}, \frac{\theta^2+\theta^5}{2}\right\}$. Thus,

$$\mathcal{D}\left(1, \theta, \theta^2, \frac{1+\theta^3}{2}, \frac{\theta+\theta^4}{2}, \frac{\theta^2+\theta^5}{2}\right) = \det \begin{pmatrix} 6 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3d \\ 0 & 0 & 0 & 0 & 3d & 0 \\ 3 & 0 & 0 & \frac{3+3d}{2} & 0 & 0 \\ 0 & 0 & 3d & 0 & 0 & 3d \\ 0 & 3d & 0 & 0 & 3d & 0 \end{pmatrix} = 729d^5,$$

which proves the result. □

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