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Optimal System, Group Invariant Solutions and Conservation Laws of the Non-linear Elastic Wave Equation and Damped Elastic Wave Equation

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Abstract

Non-linear wave equations are created by the elastic wave propagation through inelastic material. We obtain the Lie point symmetries for the non-linear elastic wave equation and the optimal system of the symmetry algebra using Lie symmetry approach. Numerous solutions that are group invariant are obtained under the optimal system of subalgebras of Lie algebra. Additionally, the variational symmetries are obtained via Noether approach and the corresponsing conservation laws are presented. The non-linear elastic wave equation with a damping term is also studied. The local conservation laws using the direct approach are also discussed in this study.

Keywords: Optimal system, Group invariant solutions, Conservation laws, Variational Symmetries, Non-linear elastic waves
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1. Introduction

The non-linear theory of elastic waves has come under the spotlight in recent years because of its numerous applications. In a recent work, J.J. Rushchitsky [1] has presented the developments in the non-linear elastic waves describing many real problems arising in applications. The linear theory of elasticity is based upon the assumption that the deformations in the elastic body are small. In this case, the strain tensor ϵ_{ij} is given by

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{1.1}$$

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Many real life situations can be studied using the non-linear model in a number of materials such as rubber exhibits large deformations under stress. J.J. Rushchitsky [1] has presented the developments of such a phenomenon in recent work [1]. Thus the strain has to be defined as a non-linear function ϵ_{ij} given by

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}). \tag{1.2}$$

In the event of the strain tensor being non linear, the Hooke's law is no longer applicable. There are quite a few models to take into account the non linearity of the strain. We use a unidirectional motion in which $u_1 = u$ and (1.2) has only one component given by

$$\epsilon_{11} = u_x + \frac{1}{2}{u_x}^2. \tag{1.3}$$

The Murnaghan equation of state describes the connection between the volume of a body and the deformation to which it is subjected. For one-dimensional case, Murnaghan potential upto third order takes the form, presented in [1] given by

$$W = (\frac{1}{2}\lambda + \mu){u_x}^2 + (\frac{1}{2}\lambda + \mu + \frac{A}{3} + B + \frac{C}{3}){u_x}^3. \tag{1.4}$$

If the continuum body is in static equilibrium, it can be shown that the components of the Cauchy stress tensor in every material point in the body meet the equilibrium equations. This is in accordance with the concept of conservation of linear momentum. The Cauchy equation of motion [2] in absence of body force is given by

$$t_{ij,j} = \rho \ddot{u}_i, \tag{1.5}$$

where t_{ij} is the stress tensor and is given by $t_{ij} = \frac{\partial W}{\partial u_{j,i}}$.$\rho$ is the density of a body. The equation (1.5) with the values of $\alpha = \frac{\lambda+2\mu}{\rho}$ and $\beta = \frac{3(\lambda+2\mu)+2(A+3B+C)}{\rho}$ can then be written as

$$u_{tt} - (\alpha + \beta u_x)u_{xx} = 0. \tag{1.6}$$

In reference [3], Apostol has studied equation (1.6) using asymptotic series method. In another reference, Bokhari et. al. [4] have performed Lie symmetry analysis of equation (1.6) and presented some invariant solutions. In a study, Mustafa and Masood [5] have studied equation (1.6) with third order anharmonic corrections. In another study [21], (2 + 1) non-linear elastic wave equation is studied via Lie symmetry approach.

In non-linear analysis, finding exact solutions of non-linear DEs is a challenging task. Lie theory [7, 8] provides a technique to obtain group-invariant solutions of non-linear PDEs that are the special type of solutions admitted by Lie algebra of PDEs. The major goal of this study is to classify the invariant solutions by identifying the optimal system of Lie sub algebras for the symmetry algebra admitted by equation (1.6) and constructing the conservation laws by different approach. In this study, we use the approach proposed by P.J. Olver [8] to construct the optimal system of subalgebras of their Lie algebras. There is a detailed discussion of how to create the family of invariant solutions in [9, 20].

In earlier studies [4, 5], the Noether symmetries, conservation laws and multiplier approach were not discussed. In this study, we perform Lie symmetry analysis, obtained invariant solutions, presented Noether symmetries and study conservation laws by two methods. The multiplier method has led to some new conservation laws. The exact invariant solutions and conservation laws provides information about physical phenomenon and helps in understanding the system.

The study of conservation laws is important because they as a vector remains invariant on the surface generated by solutions of the differential equation. We discussed the conservation laws of equation (1.6) in this work. Due to the fact that they provide conserved quantities for every solution and can be used to assess the accuracy of numerical solution techniques, conservation laws are of vital importance in the study of PDEs [14, 15, 16]. Conservation laws do double reduction to obtain the solutions of given PDEs.

The outline of the paper is following. In section (2), the Lie symmetry analysis of equation (1.6) is performed in order to find Lie symmetries of equation (1.6). The discussion of the one-dimensional optimal system is used to categorise and divide the Lie symmetries generators into disjoint, non-equivalent classes. Complete set of reductions by similarity variables, forms of group invariant solutions under the optimal systems are presented in section (3). In section (4), variational symmetries of equation (1.6) and conservation laws are obtained using approach. Furthermore, the multiplier approach is used to study the elastic wave equation in section (5). In section (6), we reviewed the classification of Lie symmetries of non-linear damped elastic wave equation which was already performed by Bokhari et. al. in [4]. In this work, we extended the study by presenting optimal system of sub algebras, reductions under optimal systems, Noether symmetries and conservation laws by two methods, Noether approach and by multipliers method.

2. Lie Symmetries and One-Dimensional Optimal System

The procedure for identifying classical symmetries of a PDE is well-known and is covered in several literature, e.g. [10, 13]. For the symmetry algebra of equation (1.6), we consider the infinitesimal generator given by

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{2.1}$$

Using the method discussed in [6, 7], we obtain the Lie algebra of infinitesimal symmetries for equation (1.6) spanned by six dimensional algebra given by

$$\begin{gathered} X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial u}, \quad X_4 = t\frac{\partial}{\partial u}, \\ X_5 = -5x\frac{\partial}{\partial x} - 6t\frac{\partial}{\partial t} + (-3u + \frac{2\alpha}{\beta}x)\frac{\partial}{\partial u}, \\ X_6 = 4x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} + (2u - \frac{2\alpha}{\beta}x)\frac{\partial}{\partial u}. \end{gathered} \tag{2.2}$$

Now by using the technique outlined in [11], we will obtain the optimal system of sub algebras of Lie algebra of equation (2.2) by dividing the previously mentioned Lie symmetry generators into disjoint non-equivalent classes. The commutator relation for the Lie algebra (2.2) is given by

$$[X_i, X_j] = X_jX_i - X_iX_j, \quad (i,j = 1,2,...,6),$$

Commutator table for the Lie algebra (2.2) is presented in Table-(2) given by

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	0	$-5X_1 + \frac{2\alpha}{\beta}X_3$	$4X_1 - \frac{2\alpha}{\beta}X_3$
X_2	0	0	0	X_3	$-6X_2$	$5X_2$
X_3	0	0	0	0	$-3X_3$	$2X_3$
X_4	0	$-X_3$	0	0	$3X_4$	$-3X_4$
X_5	$5X_1 - \frac{2\alpha}{\beta}X_3$	$6X_2$	$3X_3$	$-3X_4$	0	0
X_6	$-4X_1 + \frac{2\alpha}{\beta}X_3$	$-5X_2$	$-2X_3$	$3X_4$	0	0

Table 1: Commutator Table

The adjoint representation is given by

$$Ad(\exp(\epsilon X_i).X_j) = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2!}[X_i,[X_i,X_j]] - \cdots . \tag{2.3}$$

The adjoint representation table can be constructed using the adjoint expression (2.2), and presented in Table-(2) given by

$Ad(e^{\epsilon})$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	X_1	X_2	X_3	X_4	$5\epsilon X_1 - \frac{2\alpha\epsilon}{\beta}X_3 + X_5$	$-4\epsilon X_1 + \frac{2\alpha\epsilon}{\beta}X_3 + X_6$
X_2	X_1	X_2	X_3	$X_4 - \epsilon X_3$	$6\epsilon X_2 + X_5$	$-5\epsilon X_2 + X_6$
X_3	X_1	X_2	X_3	X_4	$3\epsilon X_3 + X_5$	$-2\epsilon X_3 + X_6$
X_4	X_1	$X_2 + \epsilon X_3$	X_3	X_4	$-3\epsilon X_4 + X_5$	$3\epsilon X_4 + X_6$
X_5	$e^{-5\epsilon}X_1 - \frac{\alpha}{\beta}e^{-3\epsilon}(e^{-2\epsilon}-1)X_3$	$e^{-6\epsilon}X_2$	$e^{-3\epsilon}X_3$	$e^{3\epsilon}X_4$	X_5	X_6
X_6	$e^{4\epsilon}X_1 - \frac{\alpha}{\beta}e^{2\epsilon}(e^{2\epsilon}-1)X_3$	$e^{5\epsilon}X_2$	$e^{2\epsilon}X_3$	$e^{-3\epsilon}X_4$	X_5	X_6

Table 2: Adjoint Table

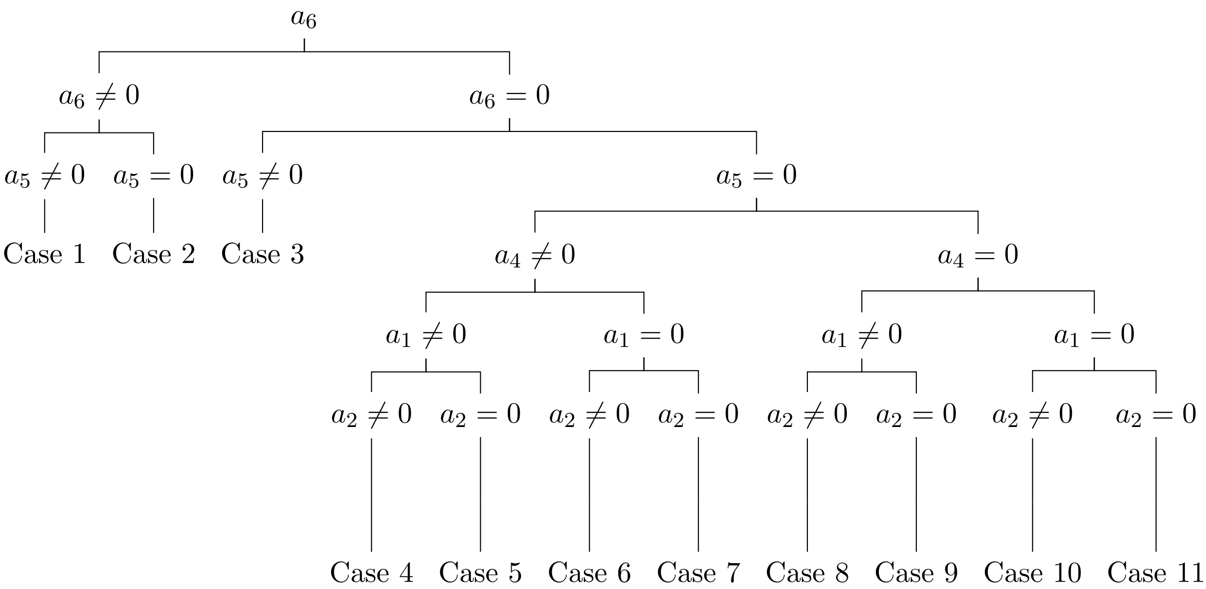

Figure 1: Tree leaf diagram of the case by case study of optimal system

Theorem 2.1. *Let $\mathcal{L}^6$ be the Lie algebra of Lie symmetries with basis (8). The optimal system of one-dimensional subalgebras is then provided by the following generators given by*

$$\begin{aligned}
\mathcal{S}^1 &= cX_5 + X_6, \quad c \neq 0\\
\mathcal{S}^2 &= X_6,\\
\mathcal{S}^3 &= X_5,\\
\mathcal{S}^4 &= \pm X_1 + X_2 \pm X_4,\\
\mathcal{S}^5 &= X_1 \pm X_4,\\
\mathcal{S}^6 &= X_2 \pm X_4,\\
\mathcal{S}^7 &= X_4,\\
\mathcal{S}^8 &= X_1 \pm X_2,\\
\mathcal{S}^9 &= X_1,\\
\mathcal{S}^{10} &= X_2,\\
\mathcal{S}^{11} &= X_3.
\end{aligned} \tag{2.4}$$

Proof: Consider a general element $X \in \mathcal{L}^6$. We have,

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 + a_6X_6. \tag{2.5}$$

Case 1: $a_6 \neq 0$, $a_5 \neq 0$. Under the adjoint action on X, we have

$$X' = Ad(e^{\epsilon}X_1)X = a_2X_2 + a_4X_4 + a_5X_5 + a_6X_6, \tag{2.6}$$

$$X'' = Ad(e^{\epsilon}X_2)X' = a_4X_4 + a_5X_5 + a_6X_6, \tag{2.7}$$

$$X''' = Ad(e^{\epsilon}X_4)X'' = a_5X_5 + a_6X_6. \tag{2.8}$$

We can scale $a_6 = 1$.

$$\mathcal{S}^1 = cX_5 + X_6, \quad c \neq 0, \tag{2.9}$$

Case 2: $a_6 \neq 0, a_5 = 0$.

$$X''' = a_6X_6, \tag{2.10}$$

$$\mathcal{S}^2 = X_6. \tag{2.11}$$

Case 3: $a_6 = 0, a_5 \neq 0$.

$$X''' = a_5X_5, \tag{2.12}$$

$$\mathcal{S}^3 = X_5. \tag{2.13}$$

Case 4: $a_6 = 0, a_5 = 0, a_1 \neq 0, a_2 \neq 0, a_4 \neq 0$.

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4, \tag{2.14}$$

$$X' = Ad(e^{\epsilon}X_5)X = a_1e^{-5\epsilon}X_1 + e^{-6\epsilon}a_2X_2 + e^{3\epsilon}a_4X_4, \tag{2.15}$$

$$X' = a_1e^{\epsilon}X_1 + a_2X_2 + a_4e^{9\epsilon}X_4. \tag{2.16}$$

We can scale $a_2 = 1$.

$$\mathcal{S}^4 = \pm X_1 + X_2 \pm X_4, \tag{2.17}$$

Case 5: $a_6 = 0, a_5 = 0, a_2 = 0, a_1 \neq 0, a_4 \neq 0$.

$$X' = e^{\epsilon}a_1X_1 + a_4e^{9\epsilon}X_4, \tag{2.18}$$

$$X' = a_1X_1 + a_4e^{8\epsilon}X_4. \tag{2.19}$$

We can take $a_1 = 1$.

$$\mathcal{S}^5 = X_1 \pm X_4. \tag{2.20}$$

Case 6: $a_6 = 0, a_1 = 0, a_5 = 0, a_2 \neq 0, a_4 \neq 0$.

$$X' = Ad(e^{\epsilon}X_4)X = a_2X_2 + a_4X_4, \tag{2.21}$$

$$X'' = Ad(e^{\epsilon}X_5)X' = e^{-6\epsilon}a_2X_2 + a_4e^{3\epsilon}X_4, \tag{2.22}$$

$$X'' = a_2X_2 + a_4e^{9\epsilon}X_4. \tag{2.23}$$

We can scale $a_2 = 1$.

$$\mathcal{S}^6 = X_2 \pm X_4. \tag{2.24}$$

Case 7: $a_6 = 0, a_5 = 0, a_2 = 0, a_1 = 0, a_4 \neq 0$.

$$X' = Ad(e^{\epsilon}X_2)X = a_4X_4, \tag{2.25}$$

$$\mathcal{S}^7 = X_4. \tag{2.26}$$

Case 8: $a_6 = 0, a_4 = 0, a_5 = 0, a_2 \neq 0, a_1 \neq 0$.

$$X' = Ad(e^{\epsilon}X_6)X = e^{4\epsilon}a_1X_1 + e^{5\epsilon}a_2X_2, \tag{2.27}$$

$$X' = a_1X_1 + e^{\epsilon}a_2X_2. \tag{2.28}$$

Take $a_1 = 1$.

$$\mathcal{S}^8 = X_1 \pm X_2. \tag{2.29}$$

Case 9: $a_6 = 0, a_4 = 0, a_5 = 0, a_2 = 0, a_1 \neq 0$.

$$X' = Ad(e^{\epsilon} X_6)X = e^{4\epsilon} a_1 X_1, \tag{2.30}$$

$$\mathcal{S}^9 = X_1. \tag{2.31}$$

Case 10: $a_5 = 0, a_6 = 0, a_4 = 0, a_1 = 0, a_2 \neq 0$.

$$X' = Ad(e^{\epsilon} X_4)X = a_2 X_2, \tag{2.32}$$

$$\mathcal{S}^{10} = X_2. \tag{2.33}$$

Case 11: $a_6 = 0, a_4 = 0, a_5 = 0, a_1 = 0, a_2 = 0$.

$$X = a_3 X_3, \tag{2.34}$$

$$\mathcal{S}^{11} = X_3. \tag{2.35}$$

By considering the general element from $\mathcal{L}^6$ and by applying the suitable adjoint action from the Table-(2), we have constructed the optimal system of sub algebras of their Lie algebras in this section. The case by case study of optimal system is presented with the help of tree-leaf diagram.

3. Similarity Reductions and Group Invariant Solutions

In this section the reduction under the optimal system (2.4) has been performed and the exact invariant solutions are presented. The given $(1+1)$-dimensional non-linear elastic wave equation can be reduced to ODEs using similarity reduction. They are frequently referred to as similarity reduction equations and can produce group invariant solutions. For the symmetry generator X_1, the associated characteristic equation is given by

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}.$$

The similarity variables are $s(x,t) = t$ and $u = f(s)$. Similarity variables $s(x,t) = t$ and $u = f(s)$, further reduce the equation (1.6) into the reduced ODE given by

$$f'' = 0. \tag{3.1}$$

The reduced ODE $f'' = 0$ yields the form of an exact solution $f(s) = c_1 s + c_2$. Consequently, the given PDE (1.6) has an exact solution that is invariant under X_1 presented by

$$u(x,t) = c_1 t + c_2.$$

Corresponding to the symmetry generator X_2, the characteristic equation is given by

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}.$$

The invariant variables $s(x,t) = x$ and $u = f(s)$ reduced the (1.6) to a reduced ODE given by

$$-f''(\alpha + \beta f') = 0. \tag{3.2}$$

Thus, the equation $\alpha + \beta f' = 0$ yields the function $f(s) = c_1 - \frac{\alpha}{\beta} s$ which is an exact solution of the PDE (1.6) that is invariant under the symmetry generator X_2 given by

$$u(x,t) = c_1 - \frac{\alpha}{\beta} x.$$

For the symmetry generator $X_1 + X_2$, we have

$$\frac{dx}{1} = \frac{dt}{1} = \frac{du}{0}.$$

The similarity variables are $s(x,t) = t - x$ and $u = f(s)$. Using these transformations PDE (1.6) reduces to an ODE given by

$$f''(1 - \alpha + \beta f') = 0. \tag{3.3}$$

The equation $1 - \alpha + \beta f' = 0$ yields $f(s) = \frac{\alpha}{\beta}s - \frac{s}{\beta} + c_1$. Consequently, the PDE (1.6) has an exact solution that is invariant under $X_1 + X_2$ given by

$$u(x,t) = c_1 + \frac{\alpha}{\beta}(t - x) + \frac{x - t}{\beta}.$$

For generator $X_1 + X_4$, the associated characteristic equation is given by

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{t}.$$

The invariant variables are $s(x,t) = t$ and $u = tx + f(s)$. Using these transformations equation (1.6) reduces to a ODE given by

$$f'' = 0. \tag{3.4}$$

The equation $f'' = 0$ yields $f(s) = c_1 s + c_2$. Consequently, the PDE (1.6) has an exact solution that is invariant under $X_1 + X_4$ given by

$$u(x,t) = (x + c_1 t) + c_2.$$

For symmetry generator $X_2 + X_4$, the invariant variables $s(x,t) = x$ and $u = \frac{t^2}{2} + f(s)$ reduce the PDE (1.6) to a ODE given by

$$1 + (-\alpha - \beta f')f'' = 0. \tag{3.5}$$

The equation $1 + (-\alpha - \beta f')f'' = 0$ thus yields an exact solution of the given PDE (1.6) that is invariant under $X_2 + X_4$ given by:

$$u(x,t) = \frac{((4x + 4c_1)\beta + 2\alpha^2)}{6\beta^2}\sqrt{(2c_1 + 2x)\beta + \alpha^2} - \frac{\alpha}{\beta}x + \frac{t^2}{2} + c_2.$$

Similarly, corresponding to the symmetry $X_1 + X_2 + X_4$, the PDE (1.6) reduces to a ODE given by

$$(1 - \alpha - \beta s + \beta f')f'' - \beta f' + \beta s + \alpha = 0, \tag{3.6}$$

where the similarity variables are $s(x,t) = -x + t$ and $u = \frac{-x^2}{2} + tx + f(s)$. This implies

$$f(s) = -\frac{(2\beta c_1 + \alpha^2 - 2\beta s - 2\alpha + 1)^{\frac{3}{2}}}{3\beta^2} + \frac{\alpha}{\beta}s + \frac{s^2}{2} - \frac{s}{\beta} + c_2. \tag{3.7}$$

Consequently, the PDE (1.6) has an exact solution that is invariant under $X_1 + X_2 + X_4$ given by

$$\begin{aligned} u(x,t) =& \frac{((-4x - 4c_1 + 4t)\beta - 2(\alpha - 1)^2)}{6\beta^2}\sqrt{(2c_1 + 2x - 2t)\beta + (\alpha - 1)^2} \\ &+ \frac{(\alpha - 1)}{\beta}(t - x) + \frac{t^2}{2} + c_2. \end{aligned}$$

The complete symmetry reduction is performed in this section under the optimal system of sub algebras. The characteristic method is used to performed the reduction. The reduced ODEs are obtained and the exact invariant solutions are presented corresponding to each symmetry generators under the optimal system of sub algebras of equation (1.6).

4. Conservation Laws via Noether Approach

A non-trivial conservation law of (1.6) exists if there exist a vector (T^t, T^x) whose divergence $D_tT^t + D_xT^x$ vanishes on the solutions of (6). To obtain the conserved vectors, we will apply Noether theorem [17, 7]. The Noether approach is linked with the variational problem and the Lagrangian. Following Noether approach first we will find the variational symmetries which are also called Noether symmetries. Then by using Noether theorem formulas we can obtain the Noether conservation laws. The Lagrangian of Euler-Lagrange equation (1.6) is given by

$$\mathcal{L} = \frac{\alpha u_x^2}{2} + \frac{\beta u_x^3}{6} - \frac{u_t^2}{2}. \tag{4.1}$$

The variational symmetry has the form given by

$$\mathbf{X} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}, \tag{4.2}$$

which is also known as Noether symmetry of equation (1.6) associated with the Lagrangian (4.1) that satisfies the identity given by

$$\mathbf{X}^{[1]}\mathcal{L} + \mathcal{L}(D_x\xi + D_t\tau) = D_tG^1 + D_xG^2, \tag{4.3}$$

where $G^i(x,t,u), \quad i = 1,2$ are gauge terms. We now obtain a set of determining equations from the expression (4.3) given by

$$\begin{aligned}
\alpha\eta_x - G_u^2 &= 0\\
\alpha\eta_u - \alpha\xi_x - \frac{\beta}{2}\eta_x + \frac{\alpha}{2}\tau_t + \frac{\alpha}{2}\xi_x &= 0\\
-\alpha\xi_u - \frac{\beta}{2}\eta_u + \frac{\beta}{2}\xi_x + \frac{\alpha}{2}\xi_u + \frac{\beta}{6}\tau_t + \frac{\beta}{6}\xi_x &= 0,\\
\frac{\beta}{2}\xi_u + \frac{\beta}{6}\xi_u &= 0\\
-\alpha\tau_x + \xi_t &= 0,\\
-\alpha\tau_u + \frac{\beta}{2}\tau_x + \frac{\alpha}{2}\tau_u &= 0\\
\beta\tau_u &= 0\\
\xi_u &= 0\\
\frac{\beta}{6}\tau_u &= 0\\
-\eta_t - G_u^1 &= 0\\
-\eta_u + \tau_t - \frac{\tau_t}{2} - \frac{\xi_x}{2} &= 0\\
\tau_u - \frac{\tau_u}{2} &= 0\\
G_t^1 + G_x^2 &= 0
\end{aligned}$$

By solving the above set of determining equations, we obtain the following infinitesimals

$$\xi = \frac{c_1}{11}x + c_3, \quad \tau = c_1t + c_2, \quad \eta = \frac{5c_1}{11}u + \frac{20\alpha c_1}{11\beta}x + c_4t + c_5,$$

and the gauge terms G^1 and G^2 given by

$$G^1 = -c_4u + f_x, \quad G^2 = -f_t + \frac{20\alpha^2c_1}{11\beta}u + g(t).$$

We can set $f = g = 0$, then we have $G^1 = -c_4 u$, $\quad G^2 = \frac{20\alpha^2 c_1}{11\beta} u$. Noether symmetries are given by

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial x}; \quad G^1 = 0, G^2 = 0,\\
X_2 &= \frac{\partial}{\partial t}; \quad G^1 = 0, G^2 = 0,\\
X_3 &= \frac{\partial}{\partial u}; \quad G^1 = 0, G^2 = 0,\\
X_4 &= t\frac{\partial}{\partial u}; \quad G^1 = -u, G^2 = 0,\\
X_5 &= \frac{x}{11}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + (\frac{5}{11}u + \frac{20\alpha x}{11\beta})\frac{\partial}{\partial u}; \quad G^1 = 0, G^2 = \frac{20\alpha^2 u}{11\beta}.
\end{aligned}
$$

The Noether symmetry generators generate indeed a subalgebra of the algebra generated by Lie symmetry generators. We will now find the conserved vectors corresponding to the previously discussed Noether symmetries by using the formula [18] given by

$$T^i = G^i - \xi^i \mathcal{L} - (\eta^\alpha - \xi^j u_j^\alpha)\frac{\delta \mathcal{L}}{\delta u_i^\alpha} - \sum_{s\geq 1} D_{i_1\ldots i_s}(\eta^\alpha - \xi^i u_j^\alpha)\frac{\delta \mathcal{L}}{\delta u^\alpha_{i_1\ldots i_s}}, \tag{4.4}$$

where $\frac{\delta}{\delta u_i^\alpha}$ is the Euler operator defined as

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1}(-1)^s D_{j_1\ldots j_s}\frac{\partial}{\partial u^\alpha_{ij_1\ldots j_s}}. \tag{4.5}$$

Here, D_i represents the total derivative operator given by,

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots . \tag{4.6}$$

By the expression (4.4), we obtain the following set of conservation laws which are presented in the form of conserved vectors given by

1. For $X_1 = \frac{\partial}{\partial x}; \quad G^1 = 0, G^2 = 0$
   $T^t = -u_x u_t,$
   $T^x = \frac{\alpha}{2}u_x^2 + \frac{\beta}{3}u_x^3 + \frac{u_t^2}{2}.$
2. For $X_2 = \frac{\partial}{\partial t}; \quad G^1 = 0, G^2 = 0$
   $T^t = \frac{-\alpha}{2}u_x^2 - \frac{\beta}{6}u_x^3 - \frac{u_t^2}{2},$
   $T^x = \alpha u_x u_t + \frac{\beta}{2}u_x^2 u_t,$
3. For $X_3 = \frac{\partial}{\partial u}; \quad G^1 = 0, G^2 = 0$
   $T^t = u_t,$
   $T^x = -\alpha u_x - \frac{\beta}{2}u_x^2.$
4. For $X_4 = t\frac{\partial}{\partial u}; \quad G^1 = -u, G^2 = 0$
   $T^t = -u + tu_t,$
   $T^x = -t(\alpha u_x + \frac{\beta}{2}u_x^2).$
5. For $X_5 = \frac{x}{11}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + (\frac{5}{11}u + \frac{20\alpha x}{11\beta})\frac{\partial}{\partial u}; \quad G^1 = 0, G^2 = \frac{20\alpha^2 u}{11\beta}$
   $T^t = -t(\frac{\alpha u_x^2}{2} + \frac{\beta u_x^3}{6} - \frac{u_t^2}{2}) + (\frac{5u}{11} + \frac{20\alpha x}{11\beta} - \frac{x}{11}u_x - tu_t)u_t,$
   $T^x = \frac{20\alpha^2 u}{11\beta} - \frac{x}{11}(\frac{\alpha u_x^2}{2} + \frac{\beta u_x^3}{6} - \frac{u_t^2}{2}) - (\alpha u_x + \frac{\beta u_x^2}{2})(\frac{5u}{11} + \frac{20\alpha x}{11\beta} - \frac{x}{11}u_x - tu_t).$

The Noether conservation laws are presented in the form of conserved vector $T = (T^t, T^x)$ corresponding to Noether symmetries such that it holds for all solutions of equation (1.6) and satisfy the divergence expression $D_tT^t+D_xT^x = 0$. Some of the conservation laws describe physical quantities and explain natural phenomena like as momentum, energy, and object motion, the rest of the conservation laws describe the geometry of the surface formed by the partial differential equation. The Noether theorem states that there is a conserved quantity that corresponds to a symmetry, although it can be challenging to identify that quantity's physical characteristics. This theorem can be used to explain the conservation laws that result from a variational principle. Furthermore, these conservation laws describe the geometry of the surface formed by the solution of the partial differential equation.

5. Conservation Laws via Multiplier Approach

The study of conservation laws for the partial differential equations is significant because they explain the geometrical properties of the surface formed by solution and are simply mathematical objects. Conservation laws via multiplier approach are presented in this section. We follow the direct approach proposed by Anco and Bluman [6, 9] to find conservation laws. In this work, we use the multiplier approach to compute the conservation laws of the non-linear elastic wave equation. Multipliers are dependent on variables $\{x, t, u, u_x, u_t\}$ for which the corresponding conservation laws are formulated. Consider a multiplier $\mathcal{M} = \mathcal{M}(x, t, u, u_x, u_t)$ satisfying

$$\frac{\delta}{\delta u}[\mathcal{M}(x, t, u, u_x, u_t)(u_{tt} - \alpha u_{xx} - \beta u_x u_{xx})] = 0, \tag{5.1}$$

where $\frac{\delta}{\delta u}$ is the Euler operator. From (5.1), we obtain the following system of determining equations for the multipliers

$$\begin{aligned} \mathcal{M}_{u_t u_t} &= 0 \\ \mathcal{M}_{t u_t} &= 0 \\ \mathcal{M}_{tt} &= 0 \\ \mathcal{M}_{u_x} &= 0 \\ \mathcal{M}_{x} &= 0 \\ \mathcal{M}_{u} &= 0 \end{aligned} \tag{5.2}$$

The solution of the system of equations (5.2) gives

$$\mathcal{M} = c_1 t + c_2 u_t + c_3 \tag{5.3}$$

where $c_i,\ i = 1, 2, 3$ are constants. The multiplier $\mathcal{M}$ of the equation $E = u_{tt} - \alpha u_{xx} - \beta u_x u_{xx}$ has the property

$$D_i T^i = \mathcal{M} E \tag{5.4}$$

for arbitrary functions $u(x, t)$. From (5.3) and (5.4), we obtain the conservation laws presented in the form of conserved vectors given by

1. $T_1 = (T^t, T^x) = (tu_t - u, tu_t - u)$
2. $T_2 = (T^t, T^x) = (\frac{\beta}{6}u_x^3 + \frac{\alpha}{2}u_x^2 + \frac{1}{2}u_t^2, -[\frac{\beta}{2}u_x + \alpha]u_x u_t)$
3. $T_3 = (T^t, T^x) = (u_t, -\frac{1}{2}\beta u_x^2 - \alpha u_x)$

The local conservation laws are presented in the form of conserved vector $T = (T^t, T^x)$ corresponding to each multipliers such that it holds for all solutions of equation (1.6) and satisfy $D_tT^t + D_xT^x = 0$. The conservation laws presented in this section are obtained using the multiplier method that has led to some new conservation laws.

6. Non-linear Damped Elastic Wave Equation

In the study, Bokhari Et. al. [4] introduced the non-linear damped elastic wave equation by adding a damping element u_t to equation (1.6) given by

$$u_{tt} - (\alpha + \beta u_x)u_{xx} + \gamma u_t = 0 \tag{6.1}$$

The point symmetries of equation (6.1) forms a five dimensional algebra spanned by

$$\begin{aligned} X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial u}, \quad X_4 = e^{-\gamma t}\frac{\partial}{\partial u}, \\ X_5 = x\frac{\partial}{\partial x} + (3u + \frac{2\alpha}{\beta}x)\frac{\partial}{\partial u}. \end{aligned} \tag{6.2}$$

The commutator table of the Lie symmetry algebra is presented in Table (6) given by

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	$X_1 + \frac{2\alpha}{\beta}X_3$
X_2	0	0	0	$-\gamma X_4$	0
X_3	0	0	0	0	$3X_3$
X_4	0	γX_4	0	0	$3X_4$
X_5	$-X_1 - \frac{2\alpha}{\beta}X_3$	0	$-3X_3$	$-3X_4$	0

Table 3: Commutator Table

The adjoint representation values are presented in the Table (6) given by

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	X_1	X_2	X_3	X_4	$X_5 - \epsilon X_1 - \frac{2\alpha}{\beta}\epsilon X_3$
X_2	X_1	X_2	X_3	$e^{\gamma\epsilon}X_4$	X_5
X_3	X_1	X_2	X_3	X_4	$X_5 - 3\epsilon X_3$
X_4	X_1	$X_2 - \gamma\epsilon X_4$	X_3	X_4	$X_5 - 3\epsilon X_4$
X_5	$e^{\epsilon}X_1 + \frac{\alpha}{\beta}e^{\epsilon}(e^{2\epsilon} - 1)X_2$	X_2	$e^{3\epsilon}X_3$	$e^{3\epsilon}X_4$	X_5

Table 4: Adjoint Table

Theorem 6.1. *Let $\mathcal{L}^5$ be the Lie algebra of Lie symmetries with basis (60). Then the optimal system of one-dimensional subalgebras is given by the following set of generators:*

$$\begin{aligned} \mathcal{S}^1 &= cX_2 + X_5, \quad c \neq 0 \\ \mathcal{S}^2 &= X_5, \\ \mathcal{S}^3 &= X_1 \pm X_3, \\ \mathcal{S}^4 &= X_2 \pm X_3, \\ \mathcal{S}^5 &= X_1, \\ \mathcal{S}^6 &= X_2, \\ \mathcal{S}^7 &= X_1 \pm X_3 \pm X_4, \\ \mathcal{S}^8 &= X_3 \pm X_4, \\ \mathcal{S}^9 &= X_3, \\ \mathcal{S}^{10} &= X_1 \pm X_4, \\ \mathcal{S}^{11} &= X_4. \end{aligned} \tag{6.3}$$

Proof: Consider a general element $X \in \mathcal{L}^5$. We have,

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5. \tag{6.4}$$

Case 1: $a_5 \neq 0$. Under the adjoint action on X, we have

$$X' = Ad(e^{\epsilon}X_1)X = a_2X_2 + a_4X_4 + a_5X_5, \tag{6.5}$$

$$X'' = Ad(e^{\epsilon}X_4)X' = a_2X_2 + a_5X_5, \tag{6.6}$$

Take $a_5 = 1$

$$\mathcal{S}^1 = cX_2 + X_5. \quad c \neq 0 \tag{6.7}$$

Case 2: $a_5 \neq 0, a_2 = 0$.

$$X'' = a_5X_5, \tag{6.8}$$

$$\mathcal{S}^2 = X_5, \tag{6.9}$$

Case 3: $a_5 = 0, a_1 \neq 0, a_2 \neq 0, a_3 \neq 0$.

$$X' = Ad(e^{\epsilon}X_4)X = a_1X_1 + a_2X_2 + a_3X_3, \tag{6.10}$$

$$X'' = Ad(e^{\epsilon}X_5)X' = a_1e^{\epsilon}X_1 + a_3e^{3\epsilon}X_3, \tag{6.11}$$

$$X'' = a_1X_1 + a_3e^{2\epsilon}X_3, \tag{6.12}$$

We can scale $a_1 = 1$.

$$\mathcal{S}^3 = X_1 \pm X_3. \tag{6.13}$$

Case 4: $a_5 = 0, a_1 = 0, a_2 \neq 0, a_3 \neq 0$.

$$X' = Ad(e^{\epsilon}X_4)X = a_2X_2 + a_3X_3, \tag{6.14}$$

$$X'' = Ad(e^{\epsilon}X_5)X' = a_2X_2 + a_3e^{3\epsilon}X_3, \tag{6.15}$$

We can take $a_2 = 1$.

$$\mathcal{S}^4 = X_2 \pm X_3. \tag{6.16}$$

Case 5: $a_5 = 0, a_3 = 0, a_1 \neq 0, a_2 \neq 0$.

$$X' = Ad(e^{\epsilon}X_4)X = a_1X_1 + a_2X_2, \tag{6.17}$$

$$X'' = Ad(e^{\epsilon}X_5)X' = a_1X_1, \tag{6.18}$$

$$\mathcal{S}^5 = X_1. \tag{6.19}$$

Case 6: $a_5 = 0, a_1 = 0, a_3 = 0, a_2 \neq 0$.

$$X' = Ad(e^{\epsilon}X_4)X = a_2X_2, \tag{6.20}$$

$$\mathcal{S}^6 = X_2. \tag{6.21}$$

Case 7: $a_5 = 0, a_2 = 0, a_1 \neq 0, a_3 \neq 0$.

$$X' = Ad(e^{\epsilon}X_5)X = e^{\epsilon}a_1X_1 + e^{3\epsilon}a_3X_3 + e^{3\epsilon}a_4X_4, \tag{6.22}$$

$$X' = a_1X_1 + e^{2\epsilon}a_3X_3 + e^{2\epsilon}a_4X_4. \tag{6.23}$$

We can scale $a_1 = 1$.

$$\mathcal{S}^7 = X_1 \pm X_3 \pm X_4. \tag{6.24}$$

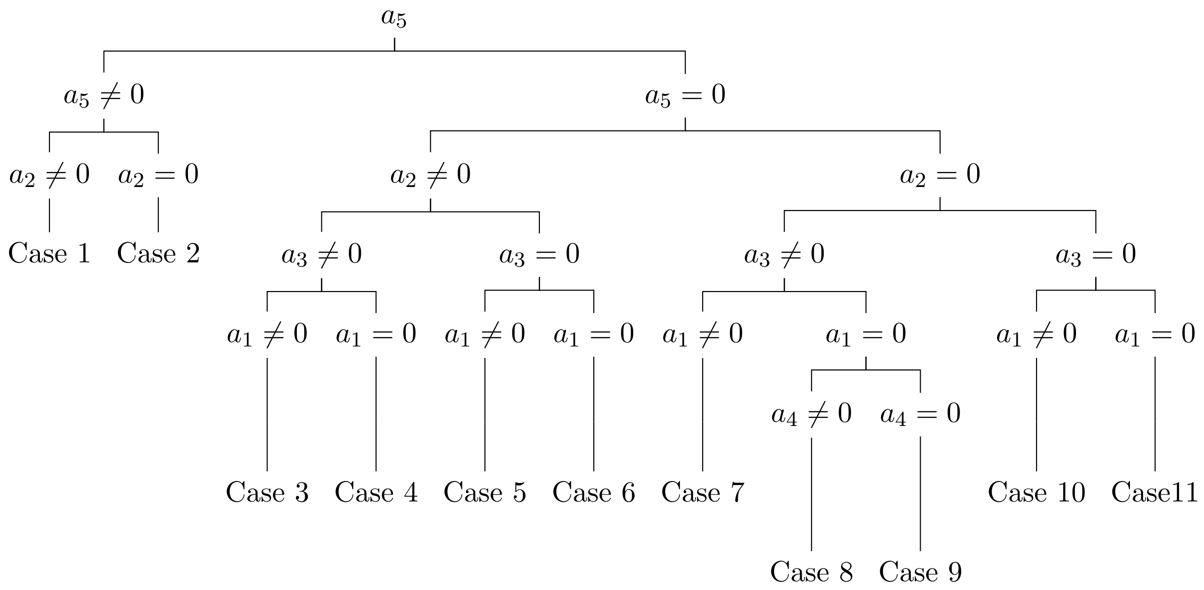

Figure 2: Tree leaf diagram of the case by case study of optimal system

Case 8: $a_5 = 0, a_2 = 0, a_1 = 0, a_3 \neq 0, a_4 \neq 0$.

$$X' = Ad(e^{\epsilon} X_2)X = a_3X_3 + e^{\gamma\epsilon}a_4X_4. \tag{6.25}$$

We can take $a_3 = 1$.

$$\mathcal{S}^8 = X_3 \pm X_4. \tag{6.26}$$

Case 9: $a_5 = 0, a_4 = 0, a_2 = 0, a_1 = 0, a_3 \neq 0$.

$$X = a_3X_3, \tag{6.27}$$

$$\mathcal{S}^9 = X_3. \tag{6.28}$$

Case 10: $a_5 = 0, a_3 = 0, a_2 = 0, a_1 \neq 0$.

$$X' = Ad(e^{\epsilon} X_2)X = a_1X_1 + e^{\gamma\epsilon}a_4X_4. \tag{6.29}$$

Take $a_1 = 1$.

$$\mathcal{S}^{10} = X_1 \pm X_4. \tag{6.30}$$

Case 11: $a_5 = 0, a_3 = 0, a_2 = 0, a_1 = 0$.

$$X = a_4X_4, \tag{6.31}$$

$$\mathcal{S}^{11} = X_4. \tag{6.32}$$

The case by case tree-leaf diagram is presented in Figure 2. The optimal system of sub algebras of non-linear damped elastic wave equation is presented in this section.

6.1. Reduction under Optimal System and Invariant Solutions

For X_1, the associated characteristic equation is given by

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}.$$

The invariant variables are $s(x,t) = t, \quad u = f(s)$. From these transformations equation (6.1) reduces to the ODE given by

$$f'' + \gamma f' = 0. \tag{6.33}$$

The equation $f''+\gamma f'=0$ yields $f(s)=c_1+c_2e^{-\gamma s}$. Consequently, the equation (6.1) has an exact solution that is invariant under X_1 given by

$$u(x,t)=c_1+c_2e^{-\gamma t}.$$

For X_2, we have

$$\frac{dx}{0}=\frac{dt}{1}=\frac{du}{0}.$$

The invariant variables are $s(x,t)=x, \quad u=f(s)$. From these transformations equation (6.1) reduces to the ODE given by

$$-f''(\alpha+\beta f')=0. \tag{6.34}$$

The equation $\alpha+\beta f'=0$ yields $f(s)=c_1-\frac{\alpha}{\beta}s$. Consequently, the equation (6.1) has an exact solution that is invariant under X_2 given by

$$u(x,t)=c_1-\frac{\alpha}{\beta}x.$$

For X_1+X_3, the associated characteristic equation is given by

$$\frac{dx}{1}=\frac{dt}{0}=\frac{du}{1}.$$

The invariant variables are $s(x,t)=t, \quad u=x+f(s)$. From these transformations equation (6.1) reduces to the ODE given by

$$f''+\gamma f'=0. \tag{6.35}$$

The equation $f''+\gamma f'=0$ yields $f(s)=x+c_1+c_2e^{-\gamma s}$. Consequently, the equation (6.1) has an exact solution that is invariant under X_1+X_3 given by

$$u(x,t)=x+c_1+c_2e^{-\gamma t}.$$

For X_1+X_4, the invariant variables are $s(x,t)=t, \quad u=xe^{-\gamma t}+f(s)$. From these invariant variables equation (6.1) reduces to the ODE given by

$$f''+\gamma f'=0. \tag{6.36}$$

The equation $f''+\gamma f'=0$ yields $f(s)=x+c_1+c_2e^{-\gamma s}$. Consequently, the equation (6.1) has an exact solution that is invariant under X_1+X_4 given by

$$u(x,t)=x+c_1+c_2e^{-\gamma t}.$$

For $X_1+X_3+X_4$, the invariant variables are $s(x,t)=t, \quad u=xe^{-\gamma t}+x+f(s)$. From these transformations equation (6.1) reduces to the ODE given by

$$f''+\gamma f'=0. \tag{6.37}$$

The equation $f''+\gamma f'=0$ yields $f(s)=x+c_1e^{-\gamma s}+c_2$. Consequently, the equation (6.1) has an exact solution that is invariant under $X_1+X_3+X_4$ given by

$$u(x,t)=(x+c_1)e^{-\gamma t}+x+c_2.$$

For X_2+X_3, the invariant variables are $s(x,t)=x, \quad u=t+f(s)$. From these invariant variables equation (6.1) reduces to the ODE given by

$$(-\alpha-\beta f')f''+\gamma=0. \tag{6.38}$$

The equation $(-\alpha-\beta f')f''+\gamma=0$ yields $f(s)=\frac{-(2\beta\gamma c_1+2\beta\gamma s+\alpha^2)^{\frac{3}{2}}}{3\beta^2\gamma}-\frac{\alpha}{\beta}s+c_2$. Consequently, the equation (6.1) has an exact solution that is invariant under X_2+X_3 given by:

$$u(x,t)=\frac{(-2\beta(c_1+x)\gamma-\alpha^2)}{3\beta^2\gamma}\sqrt{2\beta(c_1+x)\gamma+\alpha^2}-\frac{\alpha}{\beta}x+t+c_2.$$

For the non-linear damped wave equation, the reduced ODEs and invariant solutions are presented in this section.

6.2. Conservation Laws via Noether Approach

A non-trivial conservation law of the equation (6.1) exists, if there exist a vector (T^t, T^x) whose divergence $D_tT^t + D_xT^x$ vanishes on the solutions of the equation (6.1). The Lagrangian of the equation (6.1) is given by

$$\mathcal{L} = e^{\gamma t}(\frac{\alpha u_x^2}{2} + \frac{\beta u_x^3}{6} - \frac{u_t^2}{2}). \tag{6.39}$$

The Noether symmetry is presented in the form given by

$$\mathbf{X} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}, \tag{6.40}$$

is known as Noether symmetry of equation (6.1) associated with the Lagrangian (6.39) that satisfies the following identity given by

$$\mathbf{X}^{[\mathbf{1}]}\mathcal{L} + \mathcal{L}(D_x\xi + D_t\tau) = D_tG^1 + D_xG^2, \tag{6.41}$$

where $G^i(x,t,u), \quad i = 1,2$ are gauge terms. We obtain a set of determining equations from (6.41) given by

$$\begin{aligned}
\alpha e^{\gamma t}\eta_x - G_u^2 &= 0,\\
e^{\gamma t}\alpha(\eta_u - \xi_x) + \frac{\beta}{2}e^{\gamma t}\eta_x + \frac{\alpha}{2}e^{\gamma t}(\tau_t + \gamma\tau + \xi_x) &= 0,\\
\frac{\beta e^{\gamma t}}{6}(\tau_t + \gamma\tau + \xi_x) - \frac{\alpha}{2}\tau_t + \gamma\tau + \xi_x)\xi_u + \frac{\beta}{2}e^{\gamma t}(\eta_u - \xi_x) &= 0,\\
-e^{\gamma t}\eta_t &= G_u^1,\\
\frac{1}{2}e^{\gamma t}(\gamma\tau - \tau_t - \xi_x) - e^{\gamma t}(\eta_u - \xi_t) &= 0,\\
e^{\gamma t}\tau_u &= 0,\\
e^{\gamma t}(-\alpha\tau_x + \xi_t) &= 0,\\
e^{\gamma t}(\frac{-\alpha}{2}\tau_u - \frac{\beta}{2}\tau_x &= 0,\\
\frac{\beta}{3}e^{\gamma t}\xi_u &= 0,\\
G_x^1 + G_x^2 &= 0.
\end{aligned}$$

By solving the above set of determining equations, we get the following infinitesimals

$$\xi = c_3e^{\frac{\gamma}{5}t}, \quad \tau = c_1x + c_2, \quad \eta = \frac{\alpha}{\beta}c_1x + c_4t + \frac{c_1}{5}u + c_5,$$

and the corresponding gauge terms given by

$$G^1 = -c_4u + c_4t + c_6, \quad G^2 = \frac{\alpha^2}{\beta}c_1u - c_4x + c_5.$$

Noether symmetries of non-linear damped wave equation are given by

$$\begin{aligned}
X_1 &= x\frac{\partial}{\partial x} + (\frac{\alpha}{\beta}x + \frac{u}{5})\frac{\partial}{\partial u}; \quad G^1 = 0, G^2 = \frac{\alpha^2}{\beta}u\\
X_2 &= \frac{\partial}{\partial x}; \quad G^1 = 0, G^2 = 0\\
X_3 &= e^{\gamma t}\frac{\partial}{\partial t}; \quad G^1 = 0, G^2 = 0\\
X_4 &= t\frac{\partial}{\partial u}; \quad G^1 = t - u, G^2 = -x\\
X_5 &= \frac{\partial}{\partial u}. \quad G^1 = 0, G^2 = 1
\end{aligned}$$

From (4.4) we obtain the following set of conserved vectors given by

1. For $X_1 = x\frac{\partial}{\partial x} + (\frac{\alpha}{\beta}x + \frac{u}{5})\frac{\partial}{\partial u}$; $G^1 = 0, G^2 = \frac{\alpha^2}{\beta}u$
   $T^t = -e^{\gamma t}u_t(\frac{\alpha}{\beta}x + \frac{u}{5} - xu_x)$,
   $T^x = \frac{\alpha^2}{\beta}u - xe^{\gamma t}(\frac{\alpha}{2}u_x^2 + \frac{\beta}{6}u_x^3 - \frac{1}{2}u_t^2) + e^{\gamma t}(\frac{\alpha}{\beta}x + \frac{u}{5} - xu_x)(\alpha u_x + \frac{\beta}{2}u_x^2)$.
2. For $X_2 = \frac{\partial}{\partial x}$; $G^1 = 0, G^2 = 0$
   $T^t = e^{\gamma t}u_x u_t$,
   $T^x = -e^{\gamma t}(\frac{3}{2}\alpha u_x^2 + \frac{2}{3}\beta u_x^3 - \frac{1}{2}u_t^2)$.
3. For $X_3 = e^{\frac{\gamma}{5}t}\frac{\partial}{\partial t}$; $G^1 = 0, G^2 = 0$
   $T^t = -e^{\frac{6}{5}\gamma t}(\frac{\alpha}{2}u_x^2 + \frac{\beta}{6}u_x^3 - \frac{3}{2}u_t^2)$,
   $T^x = -e^{\frac{6}{5}\gamma t}u_t(\alpha u_x + \frac{\beta}{2}u_x^2)$.
4. For $X_4 = t\frac{\partial}{\partial u}$; $G^1 = t - u, G^2 = -x$
   $T^t = -u + t - te^{\gamma t}u_t$,
   $T^x = -x + te^{\gamma t}(\alpha u_x + \frac{\beta}{2}u_x^2)$.
5. For $X_5 = \frac{\partial}{\partial u}$; $G^1 = 0, G^2 = 1$
   $T^t = -e^{\gamma t}u_t$,
   $T^x = 1 + e^{\gamma t}(\alpha u_x + \frac{\beta}{2}u_x^2)$.

The Noether conservation laws are presented in the form of conserved vector $T = (T^t, T^x)$ corresponding to Noether symmetries such that it holds for all solutions of equation (6.1) and satisfy the divergence expression $D_tT^t + D_xT^x = 0$.

6.3. Conservation Laws via Multiplier Approach

By using the multiplier approach on equation (6.1) with the multiplier $\mathcal{M} = \mathcal{M}(x, t, u, u_x, u_t)$ depending on the variables (x, t, u, u_x, u_t), we have

$$\frac{\delta}{\delta u}[\mathcal{M}(x, t, u, u_x, u_t)(u_{tt} - \alpha u_{xx} - \beta u_x u_{xx} + \gamma u_t)] = 0, \tag{6.42}$$

where $\frac{\delta}{\delta u}$ represents the Euler operator. From (6.42), we obtain the following system of determining equations for the multipliers given by

$$\begin{aligned} \mathcal{M}_{tt} - \gamma\mathcal{M}_t &= 0 \\ \mathcal{M}_{u_t} &= 0 \\ \mathcal{M}_{u_x} &= 0 \\ \mathcal{M}_x &= 0 \\ \mathcal{M}_u &= 0 \end{aligned} \tag{6.43}$$

The solution of the system of equations (6.43) provides us the following multipliers given by

$$\mathcal{M} = c_1 + c_2e^{\gamma t}. \tag{6.44}$$

where $c_i,\ i = 1, 2$ are constants. The multiplier $\mathcal{M}$ of the equation $E = u_{tt} - \alpha u_{xx} - \beta u_x u_{xx} + \gamma u_t$ has the property

$$D_iT^i = \mathcal{M}E, \tag{6.45}$$

for arbitrary functions $u(x, t)$. From (6.44) and (6.45), we obtain the conserved vectors given by

1. $T^t = u_t + \gamma u$, $T^x = -\frac{1}{2}\beta u_x^2 - \alpha u_x$.
2. $T^t = e^{\gamma t}u_t$, $T^x = -(\frac{\beta}{2}u_x + \alpha)e^{\gamma t}u_x$.

The local conservation laws are presented in the form of conserved vector $T = (T^t, T^x)$ corresponding to each multipliers such that it holds for all solutions of equation (6.1) and satisfy $D_tT^t + D_xT^x = 0$. The conservation laws presented in this section are obtained using the multiplier method that has led to some new conservation laws for the non-linear damped elastic wave equation.

7. Conclusion

The Lie symmetry classification of non-linear elastic wave equation (1.6) and non-linear damped elastic wave equation (6.1) is reviewed which was already done by Bokhari et. al. [4]. Since there exist infinite many linear combinations of Lie symmetry generators so the classification upto non-equivalent classes of symmetry generators is necessary. This can be obtained by constructing the optimal system of sub algebras of their Lie algebra. The optimal systems of non-linear elastic wave equation (1.6) and non-linear damped elastic wave equation (6.1) are presented in section (2) and section (6) respectively. The discussion of the one-dimensional optimal system is used to categorise and divide the Lie symmetries generators into disjoint, non-equivalent classes. These optimal systems further categorise group invariant solutions. Complete set of reductions by similarity variables, exact invariant solutions under the optimal systems are presented in this study for both equations (1.6) and (6.1). Explicit new solutions are also obtained. Variational symmetries of equation (1.6) and equation (6.1) are studied and conservation laws are derived using Noether approach. Furthermore, the multiplier approach is used to study the conservation laws of equation (1.6) and equation (6.1). The conservation laws obtained using the multiplier approach has led us to some new conservation laws.

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