Milnor Fibrations and Regularity Conditions for Real Analytic Mappings

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Abstract

When $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is a surjective real analytic map with isolated critical value, we prove that the (m)-regularity condition (in a sense we define) ensures that $\frac{f}{||f||}$ is a fibration on small spheres, f induces a fibration on the tubes and these fibrations are equivalent.

In particular, we make the statement of [12] more precise in the case of an isolated critical point and we extend it to the case of an isolated critical value.

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Introduction

Milnor's fibration theorem for holomorphic maps is a key-stone in singularity theory; it says that for a given holomorphic map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated critical point at the origin, the map $\frac{f}{||f||}$ is the projection map of a locally trivial bundle

$$\frac{f}{||f||} : \mathbb{S}_\epsilon^{2n-1} \setminus f^{-1}(0) \to \mathbb{S}^1$$

for a sufficiently small sphere $\mathbb{S}_\epsilon^{2n-1}$ centered at the origin.

Extensions of this result in different directions became an important theme of study. In particular the extension to the case of real analytic maps has been considered by Milnor himself in [11]. He proved that if $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with $n \geq p \geq 2$ is real analytic with an isolated critical point at the origin, then

$$\frac{f}{||f||} : \mathbb{S}_\epsilon^{n-1} \setminus N(f^{-1}(0)) \to \mathbb{S}^1$$

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is a fibration on the complement of a tubular neighborhood of $f^{-1}(0)$ in sufficiently small spheres. Furthermore, he proved that this fibration can be extended to $\mathbb{S}_\epsilon^{n-1} \setminus f^{-1}(0)$. However this extension may not be given by the natural map $\frac{f}{||f||}$. He actually gave an example where this natural map does not induce a fibration in the sphere.

Many authors started investigating conditions to ensure that a real analytic map induces a fibration on small spheres.

In this work we will be concerned with the (c) and (m)-regularity conditions introduced by K. Bekka in [3].

M. Ruas and R.A. dos Santos, proved in [12], that if a real analytic map $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ with an isolated critical point is (c)-regular (in a certain sense they define), then $\frac{f}{||f||} : \mathbb{S}_\epsilon^{n-1} \setminus f^{-1}(0) \to \mathbb{S}^1$ is a fibration. However they noticed that the (c) condition is too strong.

We prove here that for a real analytic map $f : (R^n, 0) \to (\mathbb{R}^p, 0)$ with an isolated critical point and $p \leq n$, the map $\frac{f}{||f||}$ is a fibration on small spheres if and only if the map f satisfies the weaker regularity condition (m).

We also prove that, for a surjective real analytic map $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with an isolated critical value, the (m)-regularity condition (in a certain sense we make precise here) ensures not only that $\frac{f}{||f||}$ is a fibration on small spheres but also that f induces a fibration on the tube

$$f : \mathbb{B}_\epsilon \cap f^{-1}(\partial \mathbb{D}_\delta \setminus \{0\}) \to \partial \mathbb{D}_\delta \setminus \{0\}$$

where δ is sufficiently small with respect to a sufficiently small ϵ; $\mathbb{B}_\epsilon$ and $\mathbb{D}_\delta$ being respectively balls in $\mathbb{R}^n$ and $\mathbb{R}^p$. In this case these two fibrations will be equivalent.

Actually what we prove here is that the (m)-regularity condition with control function *distance to the origin* is equivalent to (d)-regularity, and (m)-regularity with control function *distance to the special fiber* is equivalent to the transversality of fibers, near the special fiber, with sufficiently small spheres. In other words, we show that the arguments behind (m)-regularity that ensure Milnor fibrations are the ones introduced in [5], namely (d)-regularity and transversality of fibers near the special one with small spheres.

1. A view of some classical regularity conditions

Many of the regularity criteria we use in this work, include some stratification condition.

Let us state, very briefly, the main stratification concepts we use.

Consider an analytic or semi-analytic subspace $X \subset U \subset \mathbb{R}^n$, where U is an open set of $\mathbb{R}^n$. A semi-analytic stratification of X (resp. U), is a locally finite partition of X (resp. U) into locally closed, connected, non singular, semi-analytic spaces X_α (resp. U_α). Such a stratification of U is adapted to X if X is a union of strata.

A stratification $X = \cup X_\alpha$ satisfies the frontier condition when for each pair of indices α and β we have $\overline{\mathcal{X}}_\alpha \cap X_\beta \neq \emptyset \Rightarrow X_\beta \subset \overline{X}_\alpha$.

Whitney's conditons a and b: For a locally finite stratification satisfying the frontier condition, H. Whitney introduced the following two conditions:

Definition 1.1. Whitney (a)-condition: A pair (X_α, X_β) with $X_\beta \subset \overline{X}_\alpha$ is (a)-regular at $y \in X_\beta$ if, whenever $\{x_i\} \in X_\alpha$ is a sequence converging to y such that the sequence of tangent spaces $\{T_{x_i} X_\alpha\}$ has a limit T, then $T_y X_\beta \subset T$.

Whitney (b)-condition: The pair (X_α, X_β), with $X_\beta \subset \overline{X}_\alpha$, is (b)-regular at $y \in X_\beta$, if whenever we have sequences of points $\{x_i\} \in X_\alpha$ and $\{y_i\} \in X_\beta$ both converging to y and such that $\{T_{x_i} X_\alpha\}$ has a limit T, and the sequence of lines $\overline{x_i y_i}$ has a limit ℓ, then $\ell \subset T$.

It is well known that condition (b) implies condition (a), see for example [16].

A stratification is called a *Whitney stratification*, or Whitney regular, if it satisfies condition (b) for all pairs of strata at every point of the small stratum.

Given an arbitrary (U, X) as above, there exist Whitney stratifications of U adapted to X. This was proved by Whitney in [17] in the complex analytic setting, and by Hironaka in [8] in general.

An important consequence of Whitney conditions is topological triviality. In fact, if (X_β, X_α) are two regular strata with $X_\alpha \subset \overline{X}_\beta$, then X_β is homeomorphic to a product of X_α with a section of X_β of suitable dimension.

We now let $f : X \to Y$ be an analytic map between analytic spaces. The map f is said to be stratified if there exist Whitney stratifications $\Sigma = \{S_\alpha\}$ and $\sigma = \{s_\beta\}$ of X and Y respectively, such that f induces an analytic submersion on each stratum of Σ into a stratum of σ.

Thom's first isotopy lemma says that a proper smooth stratified map is the projection of a smooth fiber bundle (see for instance [10]). This generalizes to the case of singular varieties, the classical fibration theorem of Ehresmann.

Thom's a_f-condition: Assume now that $f : (U, 0) \subset (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $n > p$, is a real analytic map. A point $x \in U$ is critical for f when the derivative of f at x has rank less than p. A point $y \in \mathbb{R}^p$ is a critical value of f when it is the image of a critical point.

Definition 1.2. Let $f : X \to Y$ be a stratified real analytic map between real analytic Whitney stratified spaces $X = \cup X_\alpha$ and $Y = \cup Y_\beta$. Let $x \in X_{\alpha_1} \subset \overline{X}_{\alpha_2}$.

The map f satisfies the *Thom a_f-condition* at x, if for every sequence of points $x_n \in X_{\alpha_2}$ converging to x, for which the sequence of tangent spaces $T_{x_n}(f^{-1}(f(x_n)) \cap X_{\alpha_2})$ converges to a space T, we have $T_x(f^{-1}(f(x)) \cap X_{\alpha_1}) \subset T$.

The map f is said to satisfy the *Thom a_f-condition*, if it satisfies it at each point $x \in X_\alpha$ for any pair of strata $X_\alpha \subset \overline{X}_\beta$.

We say that a real analytic map $f : X \to Y$ has the *Thom property* when there exist Whitney stratifications of X and Y, for which f satisfies Thom's a_f-condition.

In particular, let $f : (U, 0) \subset (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a real analytic map with an isolated critical value at the origin. Let $V = f^{-1}(0)$ and consider a Whitney stratification $V = \cup V_\alpha$. The stratification $(U \setminus V) \bigcup_\alpha V_\alpha$ is a Whitney regular stratification of U adapted to V. In order to check Thom's a_f-property for f with respect to that stratification, it is enough to check it for the pairs of strata of the form $(U \setminus V, V_\alpha)$.

We know by the conical structure theorem ([4, lemma 3.2]) that for a sufficiently small positive integer ϵ, the sphere $\mathbb{S}_\epsilon$ centered at the origin with radius ϵ is transversal to every stratum of a Whitney regular stratification of V. So Thom's a_f condition ensures the following property:

Proposition 1.3. *Let $f : (U, 0) \subset (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be an analytic map-germ as above with the Thom property.*

There exists $\epsilon_0 > 0$ such that for every $0 < \epsilon \le \epsilon_0$, there exists $\delta > 0$ such that for every $t \in \mathbb{R}^p$ with $\|t\| \le \delta$, the fiber $f^{-1}(t)$ intersects the sphere $\mathbb{S}_\epsilon$ transversally.

We summarize this property by saying that in a sufficiently small ball, every sphere is transversal to fibers sufficiently close to the special fiber.

Bekka's c-regularity condition: Consider a smooth analytic space M and a pair of strata (X, Y) with $Y \subset \overline{X}$. Let $\rho : M \to \mathbb{R}_+$ be a non-negative smooth function with $\rho^{-1}(0) = Y$. We say that ρ is a control function for the pair in M with respect to Y.

Definition 1.4 ((c)-regularity)**.** The pair (X, Y) is (c)-regular at $y \in Y$ with respect to the control function ρ if, for any sequence of points in X, $\{x_i\} \to y$ such that the sequence of planes $\{ker(d\rho(x_i)) \bigcap T_{x_i} X\}$ converges to a plane T in the Grassmann space of $(dim X - 1)$-planes, then $T_y Y \subset T$.

The pair (X, Y) is (c)-regular with respect to the control function ρ if it is (c)-regular for every point $y \in Y$ with respect to the function ρ.

The c-regularity condition for a pair of strata (X, Y) with respect to a control function ρ is equivalent to Thom's a_ρ condition for these strata.

Bekka proved in [3, theorem 1], that the (c)-regularity condition ensures topological triviality. This makes it a weaker condition than the (b)-condition for getting topological triviality. This is the main reason for using it in [12].

The m-regularity condition: Assume that the manifold M has a Riemannian metric. Consider a pair of strata (X, Y) with $Y \subset \overline{X}$.

Let (T_Y, π, ρ) be a tubular neighborhood of Y in M together with a projection $\pi : T_Y \to Y$, associated to a smooth non-negative control function ρ such that $\rho^{-1}(0) = Y$ and $\nabla \rho(x) \in ker(d\pi(x))$.

Definition 1.5 ((m)-regularity)**.** The pair of strata (X, Y) satisfies **condition** (m) if there exists a positive real number $\epsilon > 0$ such that

$$(\pi, \rho) \mid_{X \cap T_Y^\epsilon} : X \cap T_Y^\epsilon \to Y \times [0, \epsilon)$$

$$x \longmapsto (\pi(x), \rho(x))$$

is a submersion, where $T_Y^\epsilon := \{x \in T_Y, \rho(x) < \epsilon\}$. When this happens we say that (X, Y) is (m)-regular with respect to the control function ρ.

The following can be found in [3].

Proposition 1.6. *If a pair (X, Y) is (c)-regular with respect to a control function ρ, then this pair is Whitney (a) regular and (m)-regular with respect to ρ.*

If (X, Y) is (a) regular and also (m)-regular with respect to a control function ρ, it is not yet known whether this implies (X, Y) is (c)-regular.

The (d)-regularity for map-germs:

Consider an analytic map $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$.

Following [14, 13, 12, 5], we associate to f a family of varieties: For each line $\mathcal{L}$ through the origin in $\mathbb{R}^p$ we define:

$$X_{\mathcal{L}} := \{x \in \mathbb{R}^n \mid f(x) \in \mathcal{L}\}.$$

Definition 1.7. The family of analytic spaces $X_{\mathcal{L}}$, as $\mathcal{L}$ varies in $\mathbb{R}P^{p-1}$, is called the *canonical pencil* associated to f.

This pencil was first introduced in [14] for a certain family of maps into $\mathbb{R}^2$, to show that these have Milnor open-book fibrations. This was later used in [12] in relation with (c)-regularity (see Theorem 3.2 below).

The union of all $X_{\mathcal{L}}$ is the whole $\mathbb{R}^n$ and their intersection is V. It is in this sense that this family forms a pencil with axis V. In particular, when f has an isolated critical value, these analytic varieties are all smooth away from V.

The following concept was introduced in [6, definition 5.4] and [5, definition 2.4].

Definition 1.8. Let $f : U \to \mathbb{R}^p$ be a locally surjective real analytic map with isolated critical value at 0. We say that f is **d-regular** if there exists $\epsilon_0 > 0$ such that for any $\epsilon \leq \epsilon_0$ and for any line $0 \in \mathcal{L} \subset \mathbb{R}^p$ the sphere S_ϵ^{n-1} and the space $X_{\mathcal{L}}$ are transverse.

In other words, $d-$regularity means transversality of sufficiently small spheres with every element of the canonical pencil.

The following important characterization of d-regularity comes from [5, proposition 3.2]:

Proposition 1.9. *The following statements are equivalent:*

- *The map-germ f is d-regular.*

- *The map $\phi = \frac{f}{\|f\|} : \mathbb{S}_\eta^{n-1} \setminus K_\eta \longrightarrow \mathbb{S}^{p-1}$ is a submersion for every sufficiently small sphere $\mathbb{S}_\eta^{n-1}$.*

2. Milnor fibrations

Consider a real analytic map $f : \mathbb{R}^n \to \mathbb{R}^p$, $n \geq p$, with an isolated critical value. The critical points of f are then concentrated in the fiber $f^{-1}(0)$. a natural question is to ask to what extend does f induce a fibration outside $f^{-1}(0)$.

Following J. Milnor's pioneer work in [11], one looks for two possible types of fibrations:

Fibration in the tube: Intuitively, we refer to the situation where the map f induces a fibration in the space made of nearby fibers to $f^{-1}(0)$ still in a small neighborhood of the origin. More precisely:

Definition 2.1. We say that $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ induces a fibration in the tube if there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$ there exists $0 < \delta$ such that the restriction map:

$$f : \mathbb{B}_\epsilon \cap f^{-1}(\mathbb{B}_\delta^p \setminus \{0\}) \to \mathbb{B}_\delta^p \setminus \{0\}$$

is a fibration, where $\mathbb{B}_\epsilon$ (resp. $\mathbb{B}_\delta^p$) is the open ball of $\mathbb{R}^n$ (resp. of $\mathbb{R}^p$) centered at the origin with radius ϵ (resp. δ).

When f has an isolated critical point, Milnor showed in [11] that one always has a fibration in a tube. Another classical case is when $p = 2$, n is even and f is holomorphic. In this situation, we have an isolated critical value and D.T. Lê proved in [15] that such an f induces a fibration in the tube.

In the general setting, J.L. Cisneros-Molina, together with the first and third authors, proved in [5, Proposition 5.1 and Remark 5.7], the following:

Theorem 2.2. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $n \geq p$, be a locally surjective real analytic map with isolated critical value. If there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$ there exists $\delta > 0$ such that for any $y \in \mathbb{R}^p$ with $0 < \|y\| < \delta$ one has $f^{-1}(y)$ is transversal to $\mathbb{S}_\epsilon$, then the map f induces a Milnor fibration in a tube; where $\mathbb{S}_\epsilon \subset \mathbb{R}^n$ is the sphere centered at the origin with radius ϵ.*

In particular, one obtains from proposition 1.3, that a locally surjective map germ with isolated critical value that satisfies Thom property induces a fibration in a tube.

Fibration in the sphere: This comes from the holomorphic case of maps from $\mathbb{C}^n \to \mathbb{C}$, and it refers to the fact that the argument of f restricted to a sufficiently small sphere of $\mathbb{R}^{2n}$ is a fibration. More precisely and in a more general setting we mean:

Definition 2.3. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a locally surjective real analytic map with an isolated critical value, with $n \geq p$. We say that f induces a fibration in the sphere if there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon \leq \epsilon_0$ the map

$$\frac{f}{\|f\|} : \mathbb{S}_\epsilon \setminus f^{-1}(0) \to \mathbb{S}^{p-1}$$

is a fibration.

For holomorphic functions with isolated critical points, J. Milnor proved that they always induce a fibration in a sphere. This was extended to holomorphic functions on a singular variety $X \to \mathbb{C}$ with isolated critical value in [6].

Many authors got interested in the general case of real analytic maps, see for example [9, 7, 14, 1, 2].

In particular, Ruas and Dos Santos, in [12], related the (c)-regularity condition to the fibration in the sphere in a particular case; this will be discussed in the next section.

In [5, Theorem 5.3 and Remark 5.7], the following result is achieved:

Theorem 2.4. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a locally surjective analytic map with an isolated critical value with $n \geq p$. Assume f is d-regular and there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$ there exists $\delta > 0$ such that for any $y \in \mathbb{R}^p$ with $\|y\| < \delta$ the fiber $f^{-1}(y)$ is transversal to the sphere $\mathbb{S}_\epsilon$. Then the map f induces a fibration in the sphere and a fibration in the tube, and these are equivalent.*

In the particular case of a map with an isolated critical point, since the transversality of the nearby fibers with the spheres is always achieved, theorem 2.4 can be reduced to the following:

Corollary 2.5. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be an analytic map with an isolated critical point, $n \geq p$. It induces a fibration in the sphere if and only if it is d-regular.*

3. Milnor fibrations and c-regularity

In this section we explain briefly the main result in [12] relating *c*-regularity to Milnor fibrations for real analytic map-germs. The statement and proof in [12] use something similar to what we call the canonical pencil.

The situation considered in [12] is the one of a germ of real analytic map from $\mathbb{R}^n$ to $\mathbb{R}^2$ with an isolated critical point. Since our goal is to make their statement more precise and to extend it to a more general situation, we will start in the following setting.

Consider an analytic map

$$\begin{aligned} f : (\mathbb{R}^n, 0) &\to (\mathbb{R}^p, 0) \\ x = (x_1, \cdots, x_n) &\mapsto (f_1(x), \cdots, f_n(x)) \end{aligned}$$

with an isolated critical value, $n \geq p$.

Call $V := f^{-1}(0)$ the special fiber of f. The critical points of f are then all contained in V.

Consider the map

$$\begin{aligned} \phi : \mathbb{R}^n \setminus V &\to \mathbb{R}\mathbb{P}^{p-1} \\ x &\mapsto (f_1(x) : \cdots : f_p(x)) \end{aligned}$$

Note that the fibers of the map ϕ together with the subspace V are the elements of the canonical pencil associated to f.

We define the blow-up of V in $\mathbb{R}^n$, or more precisely, of the ideal $(f_1, \cdots, f_p)$ as follows:

Consider the subspace $X \subset \mathbb{R}^n \times \mathbb{R}\mathbb{P}^{p-1}$ defined by the equations

$$f_i t_j - f_j t_i = 0, \, 0 < i < j \leq p$$

where $(t_1 : \cdots : t_p)$ is a system of homogeneous coordinates in $\mathbb{R}\mathbb{P}^{p-1}$.

The restriction of the first projection to X induces an analytic map

$$e : X \to \mathbb{R}^n$$

that we will call the blow-up of V in $\mathbb{R}^n$.

The map e induces an isomorphism $X \setminus e^{-1}(V) \cong \mathbb{R}^n \setminus V$.

The inverse image of V by e is $e^{-1}(V) = V \times \mathbb{R}\mathbb{P}^{p-1}$.

The second projection of $\mathbb{R}^n \times \mathbb{R}\mathbb{P}^{p-1}$ induces a map

$$\psi : X \to \mathbb{R}\mathbb{P}^{p-1}$$

that extends ϕ in the sense that $\phi \circ e = \psi$.

We want to consider a stratification of X compatible with $Y := V \times \mathbb{R}\mathbb{P}^{p-1}$.

Consider a decomposition $V = \cup_\alpha V_\alpha$ into semi-analytic smooth connected varieties all adherent to 0. The corresponding stratification of Y is given by $Y = \cup_\alpha Y_\alpha$, where $Y_\alpha = V_\alpha \times \mathbb{R}\mathbb{P}^{p-1}$.

Similarly, consider a decomposition $X \setminus Y = \cup_\beta X_\beta$ into semi-analytic smooth and connected varieties all adherent to 0. Note that $X \setminus Y$ being smooth, this decomposition can be taken as the decomposition of $X \setminus Y$ into connected components.

We may choose these decompositions minimal with that property.

Notice that some strata V_α may not be contained in the closure of X. The reason is that the blow-up of V may not coincide with the closure of the graph of the map ϕ in $\mathbb{R}^n \times \mathbb{RP}^{p-1}$.

Example 3.1. Consider the map

$$\begin{aligned} f : \mathbb{R}^3 &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto (xy, xz) \end{aligned}$$

$V = (x = 0) \cup (y = z = 0)$. The component defined by the hyperplane $(x = 0) \times \mathbb{RP}^1$ in not in the closure of X which has the same dimension.

By abuse of language, and in order to simplify the statements, in what follows we will call a pair of strata (X, Y) any pair of strata (X_β, Y_α) with the frontier condition: $Y_\alpha \subset \overline{X}_\beta$.

Let us now go toward (c)-regularity.

Define the control function

$$\begin{aligned} \rho_0 : X &\to \mathbb{R}_+ \\ (x_1, \cdots, x_n, t_1 : \cdots : t_p) &\mapsto \sum_{i=1}^{n} x_i^2 \end{aligned}$$

The inverse image $\rho_0^{-1}(0) = \{0\} \times \mathbb{RP}^{p-1}$. We can admit it as a stratum in Y. Let us call it Y_0. We have $Y_0 \subset \overline{X}_\beta$, for any β. We can then consider them as admissible strata.

Theorem 3.2. *[12, Theorem 3]*

Assume f has an isolated critical point at 0 and suppose $p = 2$. If the above pair $(X_\beta \setminus Y_0, Y_0)$ is (c)-regular with respect to the control function ρ_0 (distance to Y_0), then the map f induces a Milnor fibration in the sphere.

In the same work, M. Ruas and R.A. dos Santos gave an example showing that (c)-regularity in this context is strictly stronger than Milnor fibration in the sphere. More precisely, their example consists in a map for which the considered strata do not satisfy Whitney's condition (a) and however the map induces a fibration in the sphere. In the following section we will clarify this situation.

4. Milnor fibrations and m-regularity

Let $f = (f_1, \cdots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$ be a real analytic map with isolated critical value. Define V, X, Y, Y_0 and ρ_0 as in the previous section.

Definition 4.1. We say that the map f is m_0-regular if the strata $(X_\alpha \setminus Y_0, Y_0)$ are m-regular with respect to the control function ρ_0.

In this case the contraction π is the natural map $\mathbb{R}^n \times \mathbb{RP}^{p-1} \to \{0\} \times \mathbb{RP}^{p-1}$.

Lemma 4.2. *The map f is m_0-regular if and only if it is (d)-regular.*

Proof. Recall that $Y_0 := \{0\} \times \mathbb{RP}^{p-1}$. A tubular neighborhood of Y_0 in the ambient space $\mathbb{R}^n \times \mathbb{RP}^{p-1}$ is of the form $T_{Y_0}^\epsilon := B_{(0,\epsilon)}^n \times \mathbb{RP}^{p-1}$ where $B_{(0,\epsilon)}^n$ is the open ball of $\mathbb{R}^n$ centered at the origin with radius ϵ.

Consider the map

$$\begin{aligned} (\pi_0, \rho_0) : T_{Y_0}^\epsilon &\to Y_0 \times [0, \epsilon) \\ (x_1, \cdots, x_n, t_1 : \cdots : t_p) &\mapsto ((0, t_1 : \cdots : t_p), \rho_0(x_1, \cdots, x_n)) \end{aligned}$$

Following definition 1.5, m_0-regularity means that the restriction of (π_0, ρ_0) to $X \cap T_{Y_0}^\epsilon$ is a submersion.

Since the map (π_0, ρ_0) is a submersion, m_0-regularity is equivalent to transversality of the stratum X_α with the fibers of (π_0, ρ_0).

Consider the projection map:

$$\psi : X \to \mathbb{RP}^{p-1}$$

induced by the second factor projection on $\mathbb{R}^n \times \mathbb{RP}^{p-1}$. The fibers of the map ψ are precisely the elements $X_{\mathcal{L}}$ of the canonical pencil associated to f, expanded along the projective space $\mathbb{RP}^{p-1}$.

Let $(0, \xi, \delta) \in \{0\} \times \mathbb{RP}^{p-1} \times [0, \epsilon)$. The fiber $(\pi_0, \rho_0)^{-1}(0, \xi, \delta)$ is $\mathbb{S}_\delta \times \{\xi\}$. This fiber is transversal to X_α at a point of the form $(x, \xi) \in \mathbb{R}^n \times \mathbb{RP}^{p-1}$ if and only if the fiber $\psi^{-1}(\xi)$ is transversal to $\mathbb{S}_\delta \times \{\xi\}$; equivalently, the sphere $\mathbb{S}_\delta$ is transversal to the elements of the canonical pencil. This last statement is the d-regularity condition. $\qquad\square$

In [12, Theorem 3], it is proved that, for maps with isolated critical points, (c)-regularity implies Milnor Fibration in the sphere. We know that (c)-regularity implies (a) and (m)-regularity. We also know from [5] that for such maps, Milnor fibration in the sphere is equivalent to d-regularity. Therefore, Lemma 4.2 makes Ruas and dos Santos's result more precise:

Corollary 4.3. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be analytic with an isolated critical point. The map f induces Milnor Fibration in the sphere if and only if it is m_0-regular.*

Let us now explore the isolated critical value situation.

Let f be as in the beginning of this section. Recall that $V = f^{-1}(0)$ and $Y = V \times \mathbb{RP}^{p-1}$.

V may have many irreducible components, each of them can be expressed as a union of strata. since all the strata are adherent to the origin, any tubular neighborhood of any of these strata will intersect all the other strata. In the definition of m-regularity, we need a control function whose zero set inside a tubular neighborhood of a stratum Y_α is precisely Y_α. If we keep this definition we will not be able to use it in the case where V has more than one irreducible component. That is why we need to modify slightly the definition of m-regularity, making it point-wise.

Definition 4.4. Let (X_β, Y_α) be a pair of strata with $Y_\alpha \subset Y$, $X_\beta \subset X \setminus Y$ and $Y_\alpha \subset \bar{X}_\beta$. Let T_α be a tubular neighborhood of Y_α and $\pi : T_\alpha \to Y_\alpha$ a $\mathcal{C}^1$ retraction. Consider a non-negative function $\rho : T_\alpha \to \mathbb{R}_+$. Let $y_0 \in Y_\alpha$.

We say that the pair (X_β, Y_α) is m-regular at y_0 if there exist a neighborhood U of y_0 in the ambient space, and a positive number δ_0 such that for any $\delta \leq \delta_0$ we have:

- $\rho^{-1}(0) \cap U = Y_\alpha \cap U$
- and the restriction of $(\pi, \rho) : T_\alpha^\delta \to Y_\alpha \times [0, \delta)$ to $X_\beta \cap T_\alpha^\delta$ is a submersion

where $T_\alpha^\delta = T_\alpha \cap \rho^{-1}([0, \delta))$.

We will say that the pair (X_β, Y_α) is m-regular if it is m-regular at every point of Y_α.

Note that if a pair of strata is m-regular in the sense of definition 1.5, then it is still m-regular in the sense of definition 4.4. When the small stratum is just a point, they obviously coincide.

The difference between these two definitions appears clearly in the case of different strata adherent to the same one, and all of them in $\rho^{-1}(0)$.

From now on, when we refer to m-regularity, we mean definition 4.4

Let us consider a stratum V_α of V and call $Y_\alpha = V_\alpha \times \mathbb{RP}^{p-1}$. We will say that Y_α is a maximal stratum of Y if it is not contained in the closure of any other stratum $V_\beta \times \mathbb{RP}^{p-1}$. Let T_α be a tubular neighborhood of Y_α in $\mathbb{R}^n \times \mathbb{RP}^{p-1}$. Let $\pi : T_\alpha \to Y_\alpha$ be a $\mathcal{C}^1$ retraction and

$$\rho : T_\alpha \to \mathbb{R}_+$$

$$\rho(x_1, \cdots, x_n, t_1 : \cdots : t_p) = \sum_{i=1}^p f_i^2(x_1, \cdots, x_n)$$

a control function such that $\nabla\rho_p \in T_p\text{Ker}\pi$ at any point p. Recall that X is the real blow-up of V in $\mathbb{R}^n$ and (X_α) is a stratification of $X \setminus Y$.

Theorem 4.5. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a surjective analytic map, with isolated critical value and $n \geq p$. and let $Y_\alpha = V_\alpha \times \mathbb{RP}^{p-1}$ be any maximal stratum of Y and X_β a stratum of $X \setminus Y$ adherent to Y_α. If the pair (X_β, Y_α) is m-regular for any $\mathcal{C}^1$ the retraction $\pi : T_\alpha \to Y_\alpha$ and for the previously defined control function ρ, then the map f induces a Milnor fibration on the tube.*

Remark 4.6. In case of non maximal strata, it does not make sense to talk about m-regularity because, a tubular neighborhood of such a stratum will necessarily intersect another stratum of $V \times \mathbb{RP}^{p-1}$.

Proof. Following theorem 2.2, we are going to prove that for sufficiently small spheres, a fiber over a point with sufficiently small norm is transversal to the given sphere. We claim that the m-regularity statement, for different retraction maps, implies this property.

In order to prove that, let us explore the meaning of m-regularity on strata X_β and Y_α of our situation.

First of all, since $Y_\alpha = V_\alpha \times \mathbb{RP}^{p-1}$, a tubular neighborhood T_α in $\mathbb{R}^n \times \mathbb{RP}^{p-1}$ is of the form $\mathcal{T}_\alpha \times \mathbb{RP}^{p-1}$, where $\mathcal{T}_\alpha$ is a tubular neighborhood of V_α in $\mathbb{R}^n$. A retraction $\pi : T_\alpha \to Y_\alpha$ decomposes into $(\tau, Id) : \mathcal{T}_\alpha \times \mathbb{RP}^{p-1} \to V_\alpha \times \mathbb{RP}^{p-1}$.

Consider a positive real number δ_0 such that

$$(\pi, \rho) : X \cap T_\alpha^{\delta_0} \to Y_\alpha \times [0, \delta_0)$$

is a submersion near a point $y \in Y_\alpha$. This is equivalent to saying that the fibers of (π, ρ), before intersecting with X, are transversal to X.

So let $y = (v, t) \in Y = V \times \mathbb{RP}^{p-1}$. Let $\delta \in [0, \delta_0)$. Recall that $\rho = \sum f_i^2$. The fiber $(\pi, \rho)^{-1}(y, \delta)$ is $(f^{-1}(\partial\mathbb{D}_{\delta^{1/2}}) \cap \tau^{-1}(v)) \times \{t\}$. This fiber is transversal to X if and only if, for every $z \in \mathbb{R}^p$ with $\|z\|^2 = \delta$, the fiber $f^{-1}(z)$ is transversal to $\tau^{-1}(v)$ in $\mathbb{R}^n$.

Consider a sufficiently small real number $0 < \epsilon$ and $\mathbb{S}_\epsilon \subset \mathbb{R}^n$ the sphere centered at the origin with radius ϵ. For every point in $\mathbb{S}_\epsilon \cap V_\alpha$ and for every stratum V_α, we have a positive integer δ_0 given by the m-regularity statement. Since the intersection $V \cap \mathbb{S}_\epsilon$ is compact and the number of strata of V is finite, we can choose a value δ_0 valid for all the intersection points.

Consider $0 < \delta < \delta_0$ and $z \in \mathbb{R}^p$ with $\|z\| = \delta$. Let $x \in f^{-1}(z) \cap \mathbb{S}_\epsilon$. We are going to prove that this last intersection is transversal.

We can choose a $\mathcal{C}^1$ retraction for which the tangent space $T_x\pi^{-1}(\pi(x))$ is contained in the tangent space $T_x\mathbb{S}_\epsilon$ to the sphere. The transversality property between $f^{-1}(z)$ and $\pi^{-1}(\pi(x))$, consequence of the m-regularity, ensures transversality between the fiber $f^{-1}(z)$ and the sphere $\mathbb{S}_\epsilon$ at the point x.

This is precisely the condition required in theorem 2.2, to have a fibration in the tube.

$\square$

Combining Theorem 4.5, Lemma 4.2 with Theorem 2.4, we obtain:

Theorem 4.7. *Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be a locally surjective real analytic map with an isolated critical value. If f is m_0-regular and the pair (X_β, Y_α) is m-regular for any $\mathcal{C}^1$ retraction $\pi : T_\alpha \to Y_\alpha$, for any maximal stratum Y_α of $V = f^{-1}(0)$, any adherent stratum X_β of $X \setminus Y$, and for the control function $\rho = \sum_i f_i^2$, then the map f induces a Milnor fibration on the tube and on the sphere, and these fibrations are equivalent.*

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