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On f-Derivations in Residuated Lattices

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Abstract

In this paper, as a generalization of derivation in a residuated lattice, the notion of f-derivation for a residuated lattice is introduced and some related properties of isotone (resp. contractive) f-derivations and ideal f-derivations are investigated. Also, we define principal f-derivation and their properties. Finally, we define the notion of fixed point. In particular, as an application of ideal f-derivation in Heyting algebras, we obtain that the fixed point set is still a residuated lattice.

Keywords: Residuated Lattice, f-derivation, ideal f-derivation, fixed point set.
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1. Introduction

It is well known that certain information processing is based on the classical two-valued logic (Boolean logic). Naturally, it is necessary to establish some rational logic systems as the logical foundation for uncertain information processing. For this reason, various kinds of non-classical logic systems have been extensively proposed and researched, for example, BL-algebras [19], MV-algebras [3], MTV-algebras [7] and so on. Residuated lattice are very basic and important as an algebraic structure.

The notion of derivation is a very interesting and important area of research, because it is helpful in studying structures and properties in algebraic systems. In 1957, Posner [15] introduced the notion of derivation in a prim ring $(R, +, .)$. In 2004, Jun and Xin [11] applied the notion of derivations to BCI-algebras. In 2005, Zhan and Liu [20] examined the notion of f-derivation of BCI-algebras. In 2008, Xin et al. [22] proposed the concept of a derivation on a lattice $(L, \wedge, \vee)$. In the same year, Çeven and Özturk [23] studied the notion of an f-derivation on a lattice. In 2016, He et al. [9] introdused the concept of derivation in a residuated lattice, and they characterized some special types of residuated lattices in terms of derivations. In 2018, Rachunek and Salunova [18] have introduced the concept of derivations and a complete description

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of all derivations on a non-commutative generalization of MV-algebras. In the same year, Liang et al [13] presented the notions of derivations on EQ-algebras and obtained many special types of them. In addition, Wang et al. [25] introduced the notion of derivations of commutative multiplicative semilattices, they investigated the related properties of some special derivations and gave some characterizations.In 2019, Wang et al [26] gave some representations of MV -algebras in terms of derivations. Rasheed and Majeed [17] studied some results of (α, β)-derivations on prime seeding. Dey et al [6] considered generalized orthogonal derivations of semiprimary rings. Ciungu [5] studied the properties of implicit derivations in pseudo-BCI-algebras. Chaudhuri [4] discussed (σ, τ)-derivations of group rings. In 2020, Guven [8] proposed the notion of (σ, τ)-derivations generalized on rings and discussed some related aspects. Hosseini and Fosner [10] studied the image of left Jordan derivations on algebras. Ali and Rahaman [1] studied a pair of generalized derivations in rings. Zhu et al. [21] introduced the notion of a generalized derivation and investigated some related properties of them. In 2021, Ling and Zhu [14] proposed a generalization of a derivation in a residuated lattice and some related properties are investigated.

Motivated by the above research, this paper, introduced the notion of multiplicative f-derivation d_f, as a generalization of a derivation in a residuated lattice, determined by a function f from L to L. More precisely, for any $x, y \in L$, we propose the following formula: $d_f(x \otimes y) = (d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y))$. At the same time, we discuss and investigate some related properties.

This paper is organized as follows. In section 2, we recall some concepts and results on residuated lattices. In section 3, we propose the notion of multiplicative f-derivation in residuated lattices and investigate some related properties of isotone, contractive, ideal and good commutative f-derivation. Moreover, we define principal f-derivation and their properties. finally, we define the notion of fixed point. In particular, as an application of ideal f-derivation in Heyting algebras, we obtain that the fixed point set is still a residuated lattice.

2. Preliminaries

We assume that the reader is familiar with the classical results concerning residuated lattices, but to make this work more self-contained, we briefly introduce some basic notions used in the rest of the work.

Definition 1. *[24] An algebraic structure* $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *is called a bounded commutative residuated lattice (simply called a residuated lattice) if:*

1. $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice,*
2. $(L, \otimes, 1)$ *is a monoid with unit element* 1*,*
3. *For all* $x, y, z \in L$, $x \otimes y \leq z$ *if and only if* $x \leq y \rightarrow z$*.*

In what follows, we denote by L a residuated lattice $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$.
For any $x \in L$ and a natural number n, we define $x^{'} = x \rightarrow 0$, which is a negation in a sense. $x^{''} = (x^{'})^{'}$, $x^0 = 1$, $x^n = x^{n-1} \otimes x$ for all $n \geq 1$.

Proposition 1. *[24] For all* $x, y, z, w \in L$*, we have:*

1. $1 \rightarrow x = x$, $x \rightarrow 1 = 1$,
2. $x \leq y$ *if and only if* $x \rightarrow y = 1$,
3. *If* $x \leq y$, *then* $z \rightarrow x \leq z \rightarrow y$ *and* $y \rightarrow z \leq x \rightarrow z$,
4. *If* $x \leq y$ *and* $z \leq w$ *then* $x \otimes z \leq y \otimes w$,
5. $x \otimes y \leq x \wedge y$,
6. $0^{'} = 1$, $1^{'} = 0$, $x \leq x^{''}$,
7. $x \otimes y = 0$ *if and only if* $x \leq y^{'}$,
8. $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$,
9. $x \rightarrow (y \rightarrow z) = (x \otimes y) \rightarrow z = y \rightarrow (x \rightarrow z)$.

An element $x \in L$ is called complemented if there exists an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$. By $B(L)$, we mean the set of all complemented elements of L, i.e.,

$$B(L) = \{x \in L : \exists y \in L, \ x \wedge y = 0, \ x \vee y = 1\}.$$

Proposition 2. *[12] For a residuated lattice L we have:*

1. *$x \in B(L)$ if and only if $x \vee x^{'} = 1$,*
2. *If $x \in B(L)$, then $x \wedge y = x \otimes y$ for all $y \in L$,*
3. *If $x \in B(L)$, then $x \otimes x = x$.*

In what follows, we recall the structure of Heyting algebras.

Definition 2. *[2] A lattice $(L, \vee, \wedge)$ is called to be a Heyting algebra if for any $x, y \in L$, there exists $x \rightarrow y \in L$ such that $z \leq x \rightarrow y$ if and only if $z \wedge x \leq y$ for all $z \in L$.*

Theorem 1. *[16] Let $(L, \vee, \wedge, \otimes, 0, 1)$ be a residuated lattice. Then, the following statements are equivalent:*

1. *L is a Heyting algebra,*
2. *$x \otimes y = x \wedge y = x \otimes (x \rightarrow y)$ for all $x, y \in L$.*

At the end of this section, we give the notion of multiplicative derivation in a residuated lattice L as follows.

Definition 3. *[9] A mapping d: $L \longrightarrow L$ is called a multiplicative derivation on L if it satisfies the following conditions: for any $x, y \in L$,*

$$d(x \otimes y) = (d(x) \otimes y) \vee (x \otimes d(y)).$$

3. *f*-derivations in Residuated Lattices

In this section, as a generalization of a derivation on a residuated lattice, the notion of f-derivation for a residuated lattice is introduced and some related properties are investigated. Firstly, we give the concept of f-derivation in a residuated lattice as follows.

Definition 4. *Let L be a residuated lattice. A map d_f: $L \longrightarrow L$ is called a multiplicative f-derivation on L if there exists a function f: $L \longrightarrow L$ such that*

$$d_f(x \otimes y) = (d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y))$$

for any $x, y \in L$.

Remark 1. *If f is an identity function then d_f is a derivation on a residuated lattice L [9].*

In what follows, unless otherwise stated, a multiplicative f-derivation on L is called a f-derivation on L. The following example, showed that an f-derivation is not a derivation in general.

Example 1. *Let $L = \{0, a, b, 1\}$ be a chain and the operations $\otimes$, $\rightarrow$ be defined as follows:*

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then it is easy to verify that L is a residuated lattice, where $x \wedge y = min\{x, y\}$ and $x \vee y = max\{x, y\}$. We define a mapping $d : L \longrightarrow L$ by $d0 = 0$, $da = b$, $db = a$, $d1 = a$. Since $d(a \otimes a) = 0$ and $(d(a) \otimes a) \vee (a \otimes d(a)) = a$. Then d is not a derivation on L. Based on d, we define a mapping f by $f0 = 0$, $fa = 0$, $fb = b$, $d1 = 1$. Then d satisfies the equation $d(x \otimes y) = (d(x) \otimes f(y)) \vee (f(x) \otimes d(y))$ for any $x, y \in L$, so d is a f-derivation on L.

Proposition 3. *Let d_f be a f-derivation on L. Then the following statements hold.*

1. *If $f(0) = 0$, then $d_f(0) = 0$,*
2. *$f(x) \otimes d_f(1) \leq d_f(x)$ for all $x \in L$,*
3. *If f is an homomorphism under $\otimes$, then $d_f(x^n) = f^{n-1}(x) \otimes d_f(x)$ for all $x \in L$,*
4. *If $x \leq y^{'}$ and $f(0) = 0$, then $d_f(y) \leq (f(x))^{'}$ and $d_f(x) \leq (f(y))^{'}$ for all $x, y \in L$,*
5. *If $(f(x))^{'} \leq f(x^{'})$, then $d_f(x^{'}) \leq (d_f(x))^{'}$ for all $x \in L$.*

Proof. (1) It follows from Definition 4 that $d_f(0) = d_f(0) \otimes f(0)$. Then $d_f(0) = 0$.

(2) Let $x \in L$. Then we have $d_f(x) = d_f(1 \otimes x) = (d_f(1) \otimes f(x)) \vee (f(1) \otimes d_f(x))$, which implies $f(x) \otimes d_f(1) \leq d_f(x)$.

(3) Let $x \in L$. Then we have $d_f(x^2) = d_f(x \otimes x) = f(x) \otimes d_f(x)$. By induction, we have $d_f(x^n) = d_f(x^{n-1} \otimes x) = (d_f(x^{n-1}) \otimes f(x)) \vee (f(x^{n-1}) \otimes d_f(x)) = ((f^{n-2}(x) \otimes d_f(x) \otimes f(x))) \vee (f^{n-1}(x) \otimes d_f(x)) = f^{n-1}(x) \otimes d_f(x)$.

(4) Let $x, y \in L$ and $x \leq y^{'}$ then $x \otimes y = 0$. Thus $d_f(x \otimes y) = d_f(0) = 0$, then $(d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y)) = 0$, which implies $d_f(x) \otimes f(y) = 0$ and $f(x) \otimes d_f(y) = 0$. Therefore, $d_f(y) \leq (f(x))^{'}$ and $d_f(x) \leq (f(y))^{'}$.

(5) Let $x \in L$, and $(f(x))^{'} \leq f(x^{'})$. Then it follows from Proposition 1 that $x \leq x^{''}$ then $x \leq (x^{'})^{'}$. From 4. we have $d_f(x) \leq (f(x^{'}))^{'}$. Then $d_f(x^{'}) \leq (f(x^{''}))^{'}$. Also we have $f(x^{'})^{''} \leq (d_f(x))^{'}$. Thus $d_f(x^{'}) \leq (f(x^{''}))^{'} \leq f(x^{'})^{''} \leq (d_f(x))^{'}$. Therefore, $d_f(x^{'}) \leq (d_f(x))^{'}$. □

Definition 5. *Let d_f be a f-derivation on L. Then for all $x, y \in L$,*

1. *If $x \leq y$ implies $d_f(x) \leq d_f(y)$, we call d_f an isotone f-derivation,*
2. *If $d_f(x) \leq f(x)$, we call d_f a contractive f-derivation.*

In particular, if d_f is both isotone and contractive, then we call d_f an ideal f-derivation.

Example 2. *Let $L = \{0, a, 1\}$ with $0 < a < 1$. The lattice L be a residuated lattice if we define $x \otimes y = x \wedge y$ and*

$$x \to y = \begin{cases} 1 & if \quad x \leq y, \\ y & otherwise. \end{cases}$$

Define a map d_a by $d_a(x) = x \wedge a$ for all $x \in L$ and a mapping f by $f0 = 0$, $fa = a$, $f1 = 1$. It is easy to verify that d_f is an ideal f-derivation on L.

Now, some properties of isotone and contractive f derivation are investigated.

Proposition 4. *Let d_f be an isotone f-derivation on L. Then the following statements hold.*

1. *if $z \leq x \to y$, then $f(z) \leq d_f(x) \to d_f(y)$ and $f(x) \leq d_f(z) \to d_f(y)$ for all $x, y, z \in L$,*
2. *$f(x \to y) \leq d_f(x) \to d_f(y)$ and $d_f(x \to y) \leq f(x) \to d_f(y)$ for all $x, y \in L$,*
3. *$f(x) \leq d_f(y) \to d_f(x)$ and $f(y) \leq d_f(x) \to d_f(y)$ for all $x, y \in L$.*

Proof. (1) Let $x, y, z \in L$ and $z \leq x \to y$. Then $x \otimes z \leq y$. Since d_f is an isotone f-derivation on L, we have $(d_f(x) \otimes f(z)) \vee (f(x) \otimes d_f(z)) \leq d_f(y)$. Then $f(z) \otimes d_f(x) \leq d_f(y)$ and $f(x) \otimes d_f(z) \leq d_f(y)$. Therefore $f(z) \leq d_f(x) \to d_f(y)$ and $f(x) \leq d_f(z) \to d_f(y)$.

(2) Since $x \otimes (x \to y) \leq y$, because $x \otimes (x \to y) \leq x \wedge y$ for all $x, y \in L$, we have $d_f(x \otimes (x \to y)) \leq d_f(y)$. It follows that $(d_f(x) \otimes f(x \to y)) \vee (f(x) \otimes d_f(x \to y)) \leq d_f(y)$, which implies $f(x \to y) \otimes d_f(x) \leq d_f(y)$ and $d_f(x \to y) \otimes f(x) \leq d_f(y)$, Therefore $f(x \to y) \leq d_f(x) \to d_f(y)$ and $d_f(x \to y) \leq f(x) \to d_f(y)$ for all $x, y \in L$.

(3) Let $x, y \in L$. Since $x \otimes y \leq x$, we have $d_f(x \otimes y) \leq d_f(x)$. It follows from definition 4 that $d_f(y) \otimes f(x) \leq d_f(x \otimes y) \leq d_f(x)$. Thus, $f(x) \leq d_f(y) \to d_f(x)$. In the similar way, we have $f(y) \leq d_f(x) \to d_f(y)$. □

Proposition 5. *Let d_f be a contractive f-derivation on L. Then the following statements hold.*

1. $d_f(x) \otimes d_f(y) \leq d_f(x \otimes y) \leq d_f(x) \vee d_f(y)$ *for all* $x, y \in L$,
2. *If* $d_f(1) = 1$, *then* $d_f(x) = f(x)$ *for all* $x \in L$,
3. *If* $d_f(1) = 1$, *then* $f(x) \otimes f(y) \leq d_f(x \otimes y)$ *for all* $x, y \in L$.

Proof. (1) Let $x, y \in L$. Since $d_f(x) \leq f(x)$ and $d_f(y) \leq f(y)$. we have, $d_f(y) \otimes d_f(x) \leq d_f(y) \otimes f(x)$ and $d_f(x) \otimes d_f(y) \leq d_f(x) \otimes f(y)$. Thus, $d_f(x) \otimes d_f(y) \leq (d_f(x) \otimes f(y)) \vee (d_f(y) \otimes f(x))$. On the other hand, since $f(y) \leq 1$ and $f(x) \leq 1$. We have, $d_f(x) \otimes f(y) \leq d_f(x)$ and $d_f(y) \otimes f(x) \leq d_f(y)$. Thus, $d_f(x \otimes y) \leq d_f(x) \vee d_f(y)$. Finally, $d_f(x) \otimes d_f(y) \leq d_f(x \otimes y) \leq d_f(x) \vee d_f(y)$ for all $x, y \in L$.

(2) Let $x \in L$. From Proposition 3 it follows that $f(x) \otimes d_f(1) \leq d_f(x)$. Then $f(x) \leq d_f(x)$. Since $d_f(x) \leq f(x)$, we get $d_f(x) = f(x)$.

(3) Let $x, y \in L$. It follows from Definition 4 that $d_f(x) \otimes f(y) \leq d_f(x \otimes y)$. Since $d_f(1) = 1$ then $f(x) \leq d_f(x)$. Therefore, $f(x) \otimes f(y) \leq d_f(x \otimes y)$. □

Proposition 6. *Let d_f be an ideal f-derivation on L. Then* $d_f(x \to y) \leq d_f(x) \to d_f(y) \leq d_f(x) \to f(y)$.

Proof. Let $x, y \in L$. Since $x \otimes (x \to y) \leq y$, we have $d_f(x \otimes (x \to y)) \leq d_f(y)$. Then, $d_f(x) \otimes d_f(x \to y) \leq d_f(y)$. Thus, $d_f(x \to y) \leq d_f(x) \to d_f(y)$. On the other hand, Since $d_f(y) \leq f(y)$, we have $d_f(x) \to d_f(y) \leq d_f(x) \to f(y)$. Finally, $d_f(x \to y) \leq d_f(x) \to d_f(y) \leq d_f(x) \to f(y)$. □

Proposition 7. *Let d_f be an f-derivation on L and f is an increasing function. If d_f satisfies* $d_f(x) \to d_f(y) = d_f(x) \to f(y)$ *for all* $x, y \in L$, *then d_f is an ideal f-derivation on L.*

Proof. Let $d_f(x) \to d_f(y) = d_f(x) \to f(y)$ for all $x, y \in L$. Since $d_f(x) \otimes 1 \leq d_f(x)$, we have $1 \leq d_f(x) \to d_f(x) = d_f(x) \to f(x)$. Thus, $d_f(x) \otimes 1 \leq f(x)$, which implies $d_f(x) \leq f(x)$ for all $x \in L$ then, d_f is contractive. On the other hand, let $x, y \in L$ and $x \leq y$. Since f is an increasing function, we have $f(x) \leq f(y)$. Thus, $d_f(x) \otimes 1 \leq d_f(x) \leq f(x) \leq f(y)$. Then, $d_f(x) \otimes 1 \leq f(y)$, which implies $1 \leq d_f(x) \to f(y) = d_f(x) \to d_f(y)$. Then, $1 \otimes d_f(x) \leq d_f(y)$, which implies d_f is isotone. Therefore, d_f is an ideal f-derivation on L. □

An ideal f-derivation is said to be good if $d_f(1) \in B(L)$.

Proposition 8. *Let d_f be a good ideal f-derivation on L, then the following statements hold.*

1. $d_f(x) = f(x) \otimes d_f(1)$ *for all* $x \in L$,
2. *If* $d_f(1) = 1$ *then* $d_f(x) = f(x)$ *and* $f(x \otimes y) = f(x) \otimes f(y)$ *for all* $x, y \in L$.

Proof. (1) Let $x \in L$. We have $f(x) \otimes d_f(1) \leq d_f(x)$ from Proposition 3. On the other hand, since $d_f(x) \leq d_f(1)$ and $d_f(x) \leq f(x)$, we have $d_f(x) \leq d_f(1) \wedge f(x) = d_f(1) \otimes f(x)$, which implies $d_f(x) = f(x) \otimes d_f(1)$.

(2) If $d_f(1) = 1$. Then, $d_f(x) = f(x) \otimes d_f(1) = f(x)$ and $f(x \otimes y) = d_f(x \otimes y) = (d_f(x) \otimes f(y)) \vee (f(x) \otimes d_f(y)) = (f(x) \otimes f(y)) \vee (f(x) \otimes f(y)) = f(x) \otimes f(y)$, which implies $f(x \otimes y) = f(x) \otimes f(y)$. □

Definition 6. *Let $a \in L$. We define a principal multiplicative mapping $d_{(a,f)} : L \longrightarrow L$ as follows:* $d_{(a,f)}(x) = a \otimes f(x)$ *for all* $x \in L$.

Proposition 9. *Let d_f be a good ideal derivation on L, then d_f is a principal multiplicative mapping and* $d_f = d_{(d_f(1),f)}$

Proof. Easy, since $d_f(x) = f(x) \otimes d_f(1)$ for all $x \in L$. □

Proposition 10. *Let $d_{(a,f)}$ be a principal multiplicative mapping and $f(x \otimes y) = f(x) \otimes f(y)$ for all $x, y \in L$, then the following statements hold.*

1. $d_{(a,f)}$ *is an f-derivation;*
2. *If f is an increasing function. Then, $d_{(a,f)}$ is an ideal f-derivation on L.*

Proof. (1)Let $x, y \in L$, then

$$\begin{aligned} d_{(a,f)}(x \otimes y) &= a \otimes f(x \otimes y) \\ &= (a \otimes f(x \otimes y)) \vee (a \otimes f(x \otimes y)) \\ &= (a \otimes f(x) \otimes f(y)) \vee (a \otimes f(x) \otimes f(y)) \\ &= (d_{(a,f)}(x) \otimes f(y)) \vee (f(x) \otimes d_{(a,f)}(y)). \end{aligned}$$

Then, $d_{(a,f)}$ is an f-derivation.

(2)Let $x \leq y$. Since f is an increasing function, we have $f(x) \leq f(y)$. Thus, $d_{(a,f)}(x) = a \otimes f(x) \leq a \otimes f(y) = d_{(a,f)}(y)$, which implies that $d_{(a,f)}$ is isotone. Moreover, since $a \leq 1$, we have $d_{(a,f)}(x) = a \otimes f(x) \leq f(x)$ for all $x \in L$, which implies that $d_{(a,f)}$ is contractive. Therefore, $d_{(a,f)}$ is an ideal f-derivation on L. □

Next, we discuss the structures and properties of the fixed point set of ideal f-derivation. Firstly, we give the concept of the fixed point set of a f-derivation in residuated lattice as follows.

Definition 7. *Let d_f be an ideal f-derivation on L. Define a set $Fix_{d_f}(L) = \{x \in L : \quad d_f(x) = x\}$. Fix_{d_f} is called the set of fixed elements of L for d_f.*

Now, we investigate some operations of $Fix_{d_f}(L)$.

Proposition 11. *Let d_f be an ideal f-derivation on L and $f(x) \leq x$ for all $x \in L$. Then we have: for all $x, y \in Fix_{d_f}(L)$: $x \otimes y,\ x \vee y\ \in Fix_{d_f}(L)$.*

Proof. Let $x, y \in Fix_{d_f}(L)$, we have $d_f(x) = x$ and $d_f(y) = y$. Then, $x \otimes y = d_f(x) \otimes d_f(y) \leq d_f(x) \otimes f(y) \leq d_f(x \otimes y)$. On the other hand, since d_f is an ideal f-derivation on L, we have $d_f(x \otimes y) \leq f(x \otimes y) \leq x \otimes y$, which implies $d_f(x \otimes y) = x \otimes y$. Therefore, $x \otimes y \in Fix_{d_f}(L)$. Moreover, since d_f is an ideal f-derivation on L, we have $x \vee y = d_f(x) \vee d_f(y) \leq d_f(x \vee y) \leq f(x \vee y) \leq x \vee y$, then we have $d_f(x \vee y) = x \vee y$, which implies that $x \vee y \in Fix_{d_f}(L)$. □

Theorem 2. *Let L be a Heyting algebra, d_f an ideal f-derivation on L and $f(x) \leq x$ for all $x \in L$. Then $(Fix_{d_f}(L), \wedge, \vee, \otimes, \longmapsto, 0, \overline{1})$ is a residuated lattice, where $x \longmapsto y = d_f(x \longrightarrow y)$ and $\overline{1} = d_f(1)$ for all $x, y \in L$.*

Proof. We complete the proof by three steps.

1. First, we show that $(Fix_{d_f}(L), \wedge, \vee, \otimes, \longmapsto, 0, \overline{1})$ is a bounded lattice with 0 as the smallest element and $\overline{1}$ as the greatest element. From Proposition 11 and Theorem 1, we have $Fix_{d_f}(L)$ is closed under $\vee$ and $\wedge$. Therefore, $(Fix_{d_f}(L), \wedge, \vee)$ is a lattice. Let $x \in Fix_{d_f}(L)$, we have, $x \wedge 0 = 0$ and

$$\begin{aligned} x \vee d_f(1) &= d_f(x) \vee d_f(1) \\ &= d_f(1). \end{aligned}$$

Therefore, 0 the smallest element and $\overline{1} = d_f(1)$ is the greatest element in $Fix_{d_f}(L)$.

2. Next,we prove that $(Fix_{d_f}(L), \otimes, \overline{1})$ is a commutative monoid with $\overline{1} = d_f(1)$ as neutral element. It follows from Proposition 11 that $(Fix_{d_f}(L), \otimes)$ is closed under $\otimes$, and easy to show that it satisfies associative laws. Thus, $(Fix_{d_f}(L), \otimes)$ is a commutative semigroup. Let $x \in Fix_{d_f}(L)$, since d_f is contractive and $f(x) \leq x$ we get $d_f(x) = f(x)$ by this fact, we obtain

$$\begin{aligned} x \otimes \overline{1} &= d_f(x) \otimes d_f(1) \\ &= d_f(x \otimes 1) \\ &= d_f(x) \\ &= x, \end{aligned}$$

which implies $\overline{1} = d_f(1)$ is unit element.

3. Finally, we show that $x \otimes y \leq z$ if and only if $y \leq x \mapsto z$ for all $x, y \in Fix_{d_f}(L)$. We have for all $x, y, z \in Fix_{d_f}(L)$

$$\begin{aligned} x \otimes y \leq z &\Leftrightarrow y \leq x \rightarrow z \\ &\Leftrightarrow d_f(y) \leq d_f(x \rightarrow z) \\ &\Leftrightarrow d_f(y) \leq x \mapsto z \\ &\Leftrightarrow y \leq x \mapsto z. \end{aligned}$$

Therefore, $(Fix_{d_f}(L), \wedge, \vee, \otimes, \longmapsto, 0, \overline{1})$ is a residuated lattice. □

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References

[1] A. Ali, M.H. Rahaman, "On pair of generalized derivations in rings," *Khayyam J. Math.*, vol. 6, no. 1, pp. 87-94, 2020. 1
[2] T.S. Blyth, *Lattices and ordered algebraic structures*, Springer, London: 2005. 2
[3] C.C. Chang, "Algebraic analysis of many-valued logic," *Trans. Am.Math. Soc.*, vol. 88, pp. 467-490, 1958. 1
[4] D. Chaudhuri, "(σ, τ)-derivations of group rings," *Commun. Algebra*, vol. 47, no. 9, pp. 3800-3807, 2019. 1
[5] L.C. Ciungu, "Pseudo BCI-algebras with derivations," *arXiv preprint arXiv: 1902.09895*, 2019. 1
[6] K.K. Dey, S.K. Saha, A.C. Paul, "On orthogonal generalized derivations of semiprime-rings," *GANIT: J. Bangladesh Math. Soc.*, vol. 39, pp. 63-70, 2019. 1
[7] F. Esteva, L. Godo, "Monoidal t-norm-based logic: towards a logic forleft-continuuous t-norm," *Fuzzy Sets Syst.*, vol. 124, pp. 271-288, 2001. 1
[8] E. Guven, "One sided generalized (σ, τ)-derivations on rings," *Bole.da Soc. Par. de Mat.*, vol. 38, no. 2, pp. 41-50, 2020. 1
[9] P. He, X. Xin, J. Zhan, "On derivations and their fixed point sets in residuated lattices," *Fuzzy Sets Syst.*, 303, pp. 97-113, 2016. 1, 3, 1
[10] A. Hosseini, A. Fosner, "The image of Jordan left derivations on algebras," *Bole. da Soc. Par. de Mat.*, vol. 38, no. 6, pp. 53-61, 2020. 1
[11] Y.B. Jun, X.L. Xin, "On derivations of BCI-algebras," *Inform. Sci.*, vol. 159, pp. 167-176, 2004. 1
[12] T. Kowalski, H. Ono, *Residuated Lattices: An Algebraic Glimpse at Logic Without Contraction*, 2001. 2
[13] J. Liang, X.L. Xin, J.T. Wang, "On derivations of EQ-algebras", *J. Intell. Fuzzy Syst.*, 35, pp. 5573-5583, 2018. 1
[14] D. Ling, K. Zhu, "On a new class of derivations on residuated lattices," *Ital. J. Pure Appl. Math.*, N. 45, pp. 843-857, 2021. 1
[15] E. Posner, "Derivations in prime rings," *Proc. Am. Math. Soc.*, vol. 8, pp. 1093-1100, 1957. 1
[16] D. Piciu, *Algebras of fuzzy logic*, Ed. Universitaria, Craiova, 2007. 1
[17] M.K. Rasheed, A.H. Majeed, "Some results of (α, β)-derivations on prime semirings," *Iraqi J. Sci.*, vol. 60, no.5, pp. 1154-1160, 2019. 1
[18] J. Rachunek, D. Salounova, "Derivations on algebras of a non-commutative generalization of the Lukasiewicz logic," *Fuzzy Sets Syst.*, 333 , pp. 11-16, 2018. 1
[19] E. Turumen, *Mathematics behind fuzzy logic*, Physica-Verlag, 1999. 1
[20] J. Zhan, Y.L. Liu, "On f-derivations of BCI-algebras," *Int. J. Math. Math.Sci.*, 25 , pp. 1675-1684, 2005. 1
[21] K. Zhu, J. Wang and Y. Yang, "On Generalized Derivations in Residuated Lattices," *IAENG International Journal of Applied Mathematics*, 50:2, IJAM-50-2-12. 1
[22] X.L. Xin, T.Y. Li, "J.H. Lu, On derivations of lattices," *Inf. Sci.*, 1778, pp. 307-316, 2008. 1
[23] Y. Çeven, M.A. Özturk, "On f-derivations of lattices," *Bull. Korean Math.Soc.*, 45, pp. 701-707, 2008. 1
[24] P. Ward, R.M. Dilworth, "Residuatd lattice," *Trans. Amer. Math. Soc.*, vol. 45, pp. 335-354, 1939. 1, 1
[25] J.T. Wang, A.B. Saeid, M. Wang, "On derivations of commutative multiplicative semilattices", *J. Intell. Fuzzy Syst.*, 35, pp. 957-966, 2018. 1
[26] J.T. Wang, Y.H. She, T. Qian, "Study of MV -algebras via derivations," *An. St. Univ. Ovidius Constantta*, vol. 27, no. 3, pp. 259-278, 2019. 1