Available online at http://jprm.sms.edu.pk/
Journal of Prime Research in Mathematics, 19(2) (2023), 37–59

Journal of Prime Research in Mathematics

A solvable three dimensional system of difference equations of second order with arbitrary powers

M. C. Benkara Mostefa[a,*], A. Çete[b], N. Touafek[c], Y. Yazlik[d]

[a]Department of Mathematics, Faculty of Exact Sciences, University of Mentouri Constantine 1, Constantine, Algeria.
[b]Nevsehir Haci Bektas Veli University, Faculty of Science and Art, Department of Mathematics, Nevsehir, Turkey.
[c]LMAM Laboratory, Faculty of Exact Sciences and Informatics, University of Jijel, 18000 Jijel, Algeria.
[d]Nevsehir Haci Bektas Veli University, Faculty of Science and Art, Department of Mathematics, Nevsehir, Turkey.

Abstract

The solvability in a closed form of the following three-dimensional system of difference equations of second order with arbitrary powers

$$x_{n+1} = \frac{y_n y_{n-1}^q}{x_n^p(a + b y_n y_{n-1}^q)},\ y_{n+1} = \frac{z_n z_{n-1}^r}{y_n^q(c + d z_n z_{n-1}^r)},\ z_{n+1} = \frac{x_n x_{n-1}^p}{z_n^r(h + k x_n x_{n-1}^p)},\ n,\ p,\ q,\ r \in \mathbb{N}_0$$

where the initial values x_{-i}, y_{-i}, z_i, $i = 0, 1$ are non-zero real numbers and the parameters a, b, c, d, h, are real numbers, will be the subject of the present work. We will also provide the behavior of the solutions of some particular cases of our system.

Keywords: Systems of difference equations, form of the solutions, limiting behavior of the solutions, periodic solutions.
2010 MSC: 39A10, 39A23, 40A05.

1. Introduction

The study of solvable systems of difference equations and systems continues to attract with great interest researchers [1]-[18]. In particular some models of systems of difference equations with powers of variables are the goal of some recently published papers. Here are some examples.
In [12], the authors studied the following higher order system of difference equations

$$x_{n+1} = \frac{x_{n-k+1}^p y_n}{a y_{n-k}^p + b y_n},\ y_{n+1} = \frac{y_{n-k+1}^p x_n}{\alpha x_{n-k}^p + \beta x_n}$$

*Corresponding author
Email addresses: che.benkara@gmail.com (M. C. Benkara Mostefa), ayferdoganc@gmail.com (A. Çete), ntouafek@gmail.com (N. Touafek), yyazlik@gmail.com (Y. Yazlik)

Received : 10 May 2023*; Accepted:* 28 November 2023*; Published Online:* 07 December 2023.

this system is a generalization of the following systems investigated in [17]

$$x_{n+1}=\frac{x_{n-1}y_n}{y_{n-2}\pm by_n},\ y_{n+1}=\frac{y_{n-1}x_n}{x_{n-2}\pm x_n},\ n\in\mathbb{N}_0.$$

As a generalization of the work of [12], Akrour et al. in [2], studied the following three dimensional system

$$x_{n+1}=\frac{x^p_{n-k+1}y_n}{\alpha y^p_{n-k}+\beta y_n},\ y_{n+1}=\frac{y^p_{n-k+1}z_n}{\alpha z^p_{n-k}+\beta z_n},\ z_{n+1}=\frac{z^p_{n-k+1}x_n}{Ax^p_{n-k}+Bx_n}.$$

Hamioud et al. in [14], solved in closed form the following third order system

$$x_{n+1}=\frac{y_ny_{n-1}x^p_{n-1}}{x_n(a_ny^q_{n-2}+b_ny_ny_{n-1})},\ y_{n+1}=\frac{x_nx_{n-1}y^q_{n-1}}{y_n(c_nx^p_{n-2}+d_nx_nx_{n-1})},\ ,n\in\mathbb{N}_0,\ p,\ q\in\mathbb{N}$$

and as a generalization of this system, a more general system defined by one to one functions was also presented.
Alzahrani et al. in [3], presented the form of the solutions of the following second order system of difference equations

$$x_{n+1}=\frac{y_ny_{n-1}}{x_n(\pm 1\pm y_ny_{n-1})},\ y_{n+1}=\frac{x_nx_{n-1}}{y_n(\pm 1\pm x_nx_{n-1})},\ n\in\mathbb{N}_0 \tag{1.1}$$

this system was generalized to the more general system with variable coefficients

$$x_{n+1}=\frac{y_ny_{n-1}}{x_n(a_n+b_ny_ny_{n-1})},\ y_{n+1}=\frac{x_nx_{n-1}}{y_n(\alpha_n+\beta_nx_nx_{n-1})},\ n\in\mathbb{N}_0 \tag{1.2}$$

by Stevic et al. in [16].
Motivated by all the above mentioned works, we will solve in a closed form the following three-dimensional system of difference equations of second order with arbitrary powers

$$x_{n+1}=\frac{y_ny^q_{n-1}}{x^p_n(a+by_ny^q_{n-1})},\ y_{n+1}=\frac{z_nz^r_{n-1}}{y^q_n(c+dz_nz^r_{n-1})},\ z_{n+1}=\frac{x_nx^p_{n-1}}{z^r_n(h+kx_nx^p_{n-1})},\ n\in\mathbb{N}_0 \tag{1.3}$$

where the initial values $x_{-i},\ y_{-i},\ z_i,\ i=0,1$ are non-zero real numbers, the parameters $a,\ b,\ c,\ d,\ h,$ are real numbers and $p,\ q\ ,r\in\mathbb{N}_0$. The behavior of the solutions of the two particular cases of System (1.3) when $p=q=r=0$ and $p=q=r=1$ will be also showed. It is not hard to see that our system is a three dimensional generalization Systems (1.1) and System (1.2) when the coefficients are constant.

Definition 1.1. By a well defined solution of System (1.3), we mean a solution such that

$$x^p_ny^q_nz^r_n(a+by_ny^q_{n-1})(c+dz_nz^r_{n-1})(h+kx_nx^p_{n-1})\neq 0,\ n=0,1,\cdots.$$

Arguing as in [16] it is very easy to see that

$$x^p_ny^q_nz^r_n\neq 0,\ n=1,2,\cdots.\Leftrightarrow x_{-i}y_{-i}z_i\neq 0,\ i=0,1$$

and this explain why we have to choose the initial values non-zero, otherwise solutions of System (1.3) will be not defined. For the rest of a paper by a solution of System (1.3) we mean a well-defined solution.

To solve our system, we first transform it to solvable linear system and then we deduce the solutions of System (1.3) from those of the linear one.
Now, we recall the following well-known result.

Lemma 1.2. *The solutions of the following third order linear difference equation*

$$y_{n+3}=\alpha y_n+\beta,\ n\in\mathbb{N}_0,\alpha,\ \beta\in\mathbb{R}$$

are given for $i=0,1,2$ by

$$y_{3n+i}=y_i+\beta n,\ \textit{if}\ \alpha=1,\ \textit{and},\ y_{3n+i}=\alpha^ny_i+(\frac{1-\alpha^n}{1-\alpha})\beta,\ \textit{if}\ \alpha\neq 1$$

2. Explicit formulas for the solutions of System (1.3)

Assume that $(x_n, y_n, z_n)_{n\geq -1}$ is a solution of System (1.3). Multiplying the first equation in (1.3) by x_n^p, the second one by y_n^q, and the third equation by z_n^r, and let

$$u_n = \frac{1}{x_n x_{n-1}^p}, \quad v_n = \frac{1}{y_n y_{n-1}^q}, \quad w_n = \frac{1}{z_n z_{n-1}^r}, \; n \in \mathbb{N}_0, \tag{2.1}$$

The changes of variables (2.1) are always defined as we have

$$x_n y_n z_n \neq 0, n = -1, 0, \cdots .$$

Using (2.1), System (1.3) is transformed in following linear one:

$$u_{n+1} = av_n + b, \quad v_{n+1} = cw_n + d \quad w_{n+1} = hu_n + k, \quad \forall n \in \mathbb{N}_0. \tag{2.2}$$

So, for all $n \in \mathbb{N}_0$,

$$\begin{cases} u_{n+3} = achu_n + ack + ad + b, \\ v_{n+3} = achv_n + chb + ck + d, \\ w_{n+3} = achw_n + ahd + hb + k. \end{cases}$$

From this, we get, for all $n \in \mathbb{N}_0$, the following linear first order nonhomogeneous difference equations,

$$\begin{cases} u_{3(n+1)+i} = achu_{3n+i} + ack + ad + b, \\ v_{3(n+1)+i} = achv_{3n+i} + chb + ck + d, \\ w_{3(n+1)+i} = achw_{3n+i} + ahd + hb + k, \end{cases} \quad for \;\; i = 0, 1, 2.$$

Then, from Lemma (1.2) we get for all $n \in \mathbb{N}_0$ and for $i = 0, 1, 2$,

$$\begin{aligned} u_{3n+i} = & \begin{cases} u_i + (ack + ad + b)n, & ach = 1, \\ (ach)^n u_i + (\frac{1-(ach)^n}{1-ach})(ack + ad + b), & otherwise, \end{cases} \\ = & \begin{cases} u_i + (ack + ad + b)n, & ach = 1, \\ \frac{ack+ad+b+(ach)^n(u_i(1-ach)-ack-ad-b)}{1-ach}, & otherwise, \end{cases} ; \end{aligned} \tag{2.3}$$

$$\begin{aligned} v_{3n+i} = & \begin{cases} v_i + (chb + ck + d)n, & ach = 1, \\ (ach)^n v_i + (\frac{1-(ach)^n}{1-ach})(chb + ck + d), & otherwise, \end{cases} \\ = & \begin{cases} v_i + (ack + ad + b)n, & ach = 1, \\ \frac{chb+ck+d+(ach)^n(v_i(1-ach)-chb-ck-d)}{1-ach}, & otherwise, \end{cases} ; \end{aligned} \tag{2.4}$$

$$\begin{aligned} w_{3n+i} = & \begin{cases} w_i + (ahd + hb + k)n, & ach = 1, \\ (ach)^n w_i + (\frac{1-(ach)^n}{1-ach})(ahd + hb + k), & otherwise, \end{cases} \\ = & \begin{cases} w_i + (ahd + hb + k)n, & ach = 1, \\ \frac{ahd+hb+k+(ach)^n(w_i(1-ach)-ahd-hb-k)}{1-ach}, & otherwise, \end{cases} . \end{aligned} \tag{2.5}$$

Now, by rearranging equations in (2.1), we get for all $n \in \mathbb{N}_0$,

$$\begin{cases} x_{n+1} = \frac{u_n^p}{u_{n+1}} x_{n-1}^{p^2}, \\ y_{n+1} = \frac{v_n^q}{v_{n+1}} y_{n-1}^{q^2}, \\ z_{n+1} = \frac{w_n^r}{w_{n+1}} z_{n-1}^{r^2}, \end{cases} \tag{2.6}$$

from which it follows that

$$\begin{cases} x_{2n} = x_0^{p^{2n}} \prod_{j=0}^{n-1} \frac{u_{2j+1}^{p^{(2(n-j-1)+1)}}}{u_{2j+2}^{p^{2(n-j-1)}}} = x_0^{p^{2n}} \prod_{j=0}^{n-1} (\frac{u_{2j+1}^{p}}{u_{2j+2}})^{p^{2(n-j-1)}}, \\ x_{2n-1} = x_{-1}^{p^{2n}} \prod_{j=0}^{n-1} \frac{u_{2j}^{p^{(2(n-j-1)+1)}}}{u_{2j+1}^{p^{2(n-j-1)}}} = x_{-1}^{p^{2n}} \prod_{j=0}^{n-1} (\frac{u_{2j}^{p}}{u_{2j+1}})^{p^{2(n-j-1)}}, \end{cases} \tag{2.7}$$

$$\begin{cases} y_{2n} = y_0^{q^{2n}} \prod_{j=0}^{n-1} \frac{v_{2j+1}^{q^{(2(n-j-1)+1)}}}{v_{2j+2}^{q^{2(n-j-1)}}} = y_0^{q^{2n}} \prod_{j=0}^{n-1} (\frac{v_{2j+1}^{p}}{v_{2j+2}})^{q^{2(n-j-1)}}, \\ y_{2n-1} = y_{-1}^{q^{2n}} \prod_{j=0}^{n-1} \frac{v_{2j}^{q^{(2(n-j-1)+1)}}}{v_{2j+1}^{q^{2(n-j-1)}}} = y_{-1}^{q^{2n}} \prod_{j=0}^{n-1} (\frac{v_{2j}^{p}}{v_{2j+1}})^{q^{2(n-j-1)}}, \end{cases} \tag{2.8}$$

$$\begin{cases} z_{2n} = z_0^{r^{2n}} \prod_{j=0}^{n-1} \frac{w_{2j+1}^{r^{(2(n-j-1)+1)}}}{w_{2j+2}^{r^{2(n-j-1)}}} = z_0^{r^{2n}} \prod_{j=0}^{n-1} (\frac{w_{2j+1}^{r}}{w_{2j+2}})^{r^{2(n-j-1)}}, \\ z_{2n-1} = z_{-1}^{r^{2n}} \prod_{j=0}^{n-1} \frac{w_{2j}^{p^{(2(n-j-1)+1)}}}{w_{2j+1}^{r^{2(n-j-1)}}} = z_{-1}^{r^{2n}} \prod_{j=0}^{n-1} (\frac{w_{2j}^{r}}{w_{2j+1}})^{r^{2(n-j-1)}}. \end{cases} \tag{2.9}$$

Then for all $n \in \mathbb{N}_0$ and $i \in \{0, 1\}$, we have

$$x_{2n-i} = x_{-i}^{p2n} \prod_{j=0}^{n-1} (\frac{u_{2j+1-i}^{p}}{u_{2j+2-i}})^{p^{2(n-j-1)}}, \tag{2.10}$$

$$y_{2n-i} = y_{-i}^{q^{2n}} \prod_{j=0}^{n-1} (\frac{v_{2j+1-i}^{q}}{v_{2j+2-i}})^{q^{2(n-j-1)}}, \tag{2.11}$$

$$z_{2n-i} = z_{-i}^{r^{2n}} \prod_{j=0}^{n-1} (\frac{w_{2j+1-i}^{p}}{w_{2j+2-i}})^{r^{2(n-j-1)}}. \tag{2.12}$$

Now, we will distinguish the following situations:

i) If $n = 3m$, $m \in \mathbb{N}_0$, then, from (2.7), (2.8) and (2.9), we have

$$\begin{cases} x_{2(3m)} = x_0^{p^{6m}} \prod_{j=0}^{m-1} (\frac{u_{3(2j)+1}^{p}}{u_{3(2j)+2}})^{p^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} (\frac{u_{3(2j+1)}^{p}}{u_{3(2j+1)+1}})^{p^{2(3(m-j)-2)}} \prod_{j=0}^{m-1} (\frac{u_{3(2j+1)+2}^{p}}{u_{3(2j+2)}})^{p^{6(m-j-1)}}, \\ x_{2(3m)-1} = x_{-1}^{p^{6m}} \prod_{j=0}^{m-1} (\frac{u_{3(2j)}^{p}}{u_{3(2j)+1}})^{p^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} (\frac{u_{3(2j)+2}^{p}}{u_{3(2j+1)}})^{p^{2(3(m-j)-2)}} \prod_{j=0}^{m-1} (\frac{u_{3(2j+1)+1}^{p}}{u_{3(2j+1)+2}})^{p^{6(m-j-1)}}, \end{cases} \tag{2.13}$$

$$\begin{cases} y_{2(3m)} = y_0^{q^{6m}} \prod_{j=0}^{m-1} (\frac{v_{3(2j)+1}^{q}}{v_{3(2j)+2}})^{q^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} (\frac{v_{3(2j+1)}^{q}}{v_{3(2j+1)+1}})^{q^{2(3(m-j)-2)}} \prod_{j=0}^{m-1} (\frac{v_{3(2j+1)+2}^{q}}{v_{3(2j+2)}})^{q^{6(m-j-1)}}, \\ y_{2(3m)-1} = y_{-1}^{q^{6m}} \prod_{j=0}^{m-1} (\frac{v_{3(2j)}^{q}}{v_{3(2j)+1}})^{q^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} (\frac{v_{3(2j)+2}^{q}}{v_{3(2j+1)}})^{q^{2(3(m-j)-2)}} \prod_{j=0}^{m-1} (\frac{v_{3(2j+1)+1}^{q}}{v_{3(2j+1)+2}})^{q^{6(m-j-1)}}, \end{cases} \tag{2.14}$$

$$\begin{cases} z_{2(3m)} = z_0^{r^{6m}} \prod_{j=0}^{m-1} (\frac{w_{3(2j)+1}^{r}}{w_{3(2j)+2}})^{r^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} (\frac{w_{3(2j+1)}^{r}}{w_{3(2j+1)+1}})^{r^{2(3(m-j)-2)}} \prod_{j=0}^{m-1} (\frac{w_{3(2j+1)+2}^{r}}{w_{3(2j+2)}})^{r^{6(m-j-1)}}, \\ z_{2(3m)-1} = z_{-1}^{r^{6m}} \prod_{j=0}^{m-1} (\frac{w_{3(2j)}^{r}}{w_{3(2j)+1}})^{r^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} (\frac{w_{3(2j)+2}^{r}}{w_{3(2j+1)}})^{r^{2(3(m-j)-2)}} \prod_{j=0}^{m-1} (\frac{w_{3(2j+1)+1}^{r}}{w_{3(2j+1)+2}})^{r^{6(m-j-1)}}. \end{cases} \tag{2.15}$$

ii) If $n = 3m + 1$, $m \in \mathbb{N}_0$, then, from (2.7), (2.8) and (2.9), we get

$$\begin{cases} x_{2(3m+1)} = x_0^{p^{6m+2}} \prod_{j=0}^{m} (\frac{u_{3(2j)+1}^{p}}{u_{3(2j)+2}})^{p^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{u_{3(2j+1)}^{p}}{u_{3(2j+1)+1}})^{p^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} (\frac{u_{3(2j+1)+2}^{p}}{u_{3(2j+2)}})^{p^{2(3(m-j)-2)}}, \\ x_{2(3m)+1} = x_{-1}^{p^{6m+2}} \prod_{j=0}^{m} (\frac{u_{3(2j)}^{p}}{u_{3(2j)+1}})^{p^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{u_{3(2j)+2}^{p}}{u_{3(2j+1)}})^{p^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} (\frac{u_{3(2j+1)+1}^{p}}{u_{3(2j+1)+2}})^{p^{2(3(m-j)-2)}}, \end{cases} \tag{2.16}$$

$$\begin{cases} y_{2(3m+1)} = y_0^{q^{6m+2}} \prod_{j=0}^{m} (\frac{v^q_{3(2j)+1}}{v_{3(2j)+2}})^{q^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{v^q_{3(2j+1)}}{v_{3(2j+1)+1}})^{q^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} (\frac{v^q_{3(2j+1)+2}}{v_{3(2j+2)}})^{q^{2(3(m-j)-2)}}, \\ y_{2(3m)+1} = y_{-1}^{q^{6m+2}} \prod_{j=0}^{m} (\frac{v^q_{3(2j)}}{v_{3(2j)+1}})^{q^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{v^q_{3(2j)+2}}{v_{3(2j+1)}})^{q^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} (\frac{v^q_{3(2j+1)+1}}{v_{3(2j+1)+2}})^{q^{2(3(m-j)-2)}}, \end{cases} \tag{2.17}$$

$$\begin{cases} z_{2(3m+1)} = z_0^{r^{6m+2}} \prod_{j=0}^{m} (\frac{w^r_{3(2j)+1}}{w_{3(2j)+2}})^{r^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{w^r_{3(2j+1)}}{w_{3(2j+1)+1}})^{r^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} (\frac{w^r_{3(2j+1)+2}}{w_{3(2j+2)}})^{r^{2(3(m-j)-2)}}, \\ z_{2(3m)+1} = z_{-1}^{r^{6m+2}} \prod_{j=0}^{m} (\frac{w^r_{3(2j)}}{w_{3(2j)+1}})^{r^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{w^r_{3(2j)+2}}{w_{3(2j+1)}})^{r^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} (\frac{w^r_{3(2j+1)+1}}{w_{3(2j+1)+2}})^{r^{2(3(m-j)-2)}}. \end{cases} \tag{2.18}$$

iii) If $n = 3m + 2$, $m \in \mathbb{N}_0$, then, from (2.7), (2.8) and (2.9), we obtain

$$\begin{cases} x_{2(3m+2)} = x_0^{p^{6m+4}} \prod_{j=0}^{m} (\frac{u^p_{3(2j)+1}}{u_{3(2j)+2}})^{p^{2(3(m-j)+1)}} \prod_{j=0}^{m} (\frac{u^p_{3(2j+1)}}{u_{3(2j+1)+1}})^{p^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{u^p_{3(2j+1)+2}}{u_{3(2j+2)}})^{p^{2(3(m-j)-1)}}, \\ x_{2(3m)+3} = x_{-1}^{p^{6m+4}} \prod_{j=0}^{m} (\frac{u^p_{3(2j)}}{u_{3(2j)+1}})^{p^{2(3(m-j)+1)}} \prod_{j=0}^{m} (\frac{u^p_{3(2j)+2}}{u_{3(2j+1)}})^{p^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{u^p_{3(2j+1)+1}}{u_{3(2j+1)+2}})^{p^{2(3(m-j)-1)}}, \end{cases} \tag{2.19}$$

$$\begin{cases} y_{2(3m+2)} = y_0^{q^{6m+4}} \prod_{j=0}^{m} (\frac{v^q_{3(2j)+1}}{v_{3(2j)+2}})^{q^{2(3(m-j)+1)}} \prod_{j=0}^{m} (\frac{v^q_{3(2j+1)}}{v_{3(2j+1)+1}})^{q^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{v^q_{3(2j+1)+2}}{v_{3(2j+2)}})^{q^{2(3(m-j)-1)}}, \\ y_{2(3m)+3} = y_{-1}^{q^{6m+4}} \prod_{j=0}^{m} (\frac{v^q_{3(2j)}}{v_{3(2j)+1}})^{q^{2(3(m-j)+1)}} \prod_{j=0}^{m} (\frac{v^q_{3(2j)+2}}{v_{3(2j+1)}})^{q^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{v^q_{3(2j+1)+1}}{v_{3(2j+1)+2}})^{q^{2(3(m-j)-1)}}, \end{cases} \tag{2.20}$$

$$\begin{cases} z_{2(3m+2)} = z_0^{r^{6m+4}} \prod_{j=0}^{m} (\frac{w^r_{3(2j)+1}}{w_{3(2j)+2}})^{r^{2(3(m-j)+1)}} \prod_{j=0}^{m} (\frac{w^r_{3(2j+1)}}{w_{3(2j+1)+1}})^{r^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{w^r_{3(2j+1)+2}}{w_{3(2j+2)}})^{r^{2(3(m-j)-1)}}, \\ z_{2(3m)+3} = z_{-1}^{r^{6m+4}} \prod_{j=0}^{m} (\frac{w^r_{3(2j)}}{w_{3(2j)+1}})^{r^{2(3(m-j)+1)}} \prod_{j=0}^{m} (\frac{w^r_{3(2j)+2}}{w_{3(2j+1)}})^{r^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{w^r_{3(2j+1)+1}}{w_{3(2j+1)+2}})^{r^{2(3(m-j)-1)}}. \end{cases} \tag{2.21}$$

Again we will consider two cases. Using (2.3), (2.4) and (2.5), we have

First Case $ach \neq 1$.

i) If $n = 3m$, $m \in \mathbb{N}_0$, then, from (2.13), (2.14) and (2.15), we have

$$\begin{cases} x_{2(3m)} = x_0^{p^{6m}} \prod_{j=0}^{m-1} (\frac{(ack+ad+b+(ach)^{2j}(u_1(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j}(u_2(1-ach)-ack-ad-b))(1-ach)^{p-1}})^{p^{2(3(m-j)-1)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(ack+ad+b+(ach)^{2j+1}(u_0(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j+1}(u_1(1-ach)-ack-ad-b))(1-ach)^{p-1}})^{p^{2(3(m-j)-2)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(ack+ad+b+(ach)^{2j+1}(u_2(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j+2}(u_0(1-ach)-ack-ad-b))(1-ach)^{p-1}})^{p^{6(m-j-1)}}, \\ x_{2(3m)-1} = x_{-1}^{p^{6m}} \prod_{j=0}^{m-1} (\frac{(ack+ad+b+(ach)^{2j}(u_0(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j}(u_1(1-ach)-ack-ad-b))(1-ach)^{p-1}})^{p^{2(3(m-j)-1)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(ack+ad+b+(ach)^{2j}(u_2(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j+1}(u_0(1-ach)-ack-ad-b))(1-ach)^{p-1}})^{p^{2(3(m-j)-2)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(ack+ad+b+(ach)^{2j+1}(u_1(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j+1}(u_2(1-ach)-ack-ad-b))(1-ach)^{p-1}})^{p^{6(m-j-1)}}. \end{cases} \tag{2.22}$$

$$\begin{cases} y_{2(3m)}=y_0^{q^{6m}}\prod\limits_{j=0}^{m-1}\Big(\frac{(chb+ck+d+(ach)^{2j}(v_1(1-ach)-chb-ck-d))^q}{(chb+ck+d+(ach)^{2j}(v_2(1-ach)-chb-ck-d))(1-ach)^{q-1}}\Big)^{q^{2(3(m-j)-1)}}\\ \qquad\prod\limits_{j=0}^{m-1}\Big(\frac{(chb+ck+d+(ach)^{2j+1}(v_0(1-ach)-chb-ck-d))^q}{(chb+ck+d+(ach)^{2j+1}(v_1(1-ach)-chb-ck-d))(1-ach)^{q-1}}\Big)^{q^{2(3(m-j)-2)}}\\ \qquad\prod\limits_{j=0}^{m-1}\Big(\frac{(chb+ck+d+(ach)^{2j+1}(v_2(1-ach)-chb-ck-d))^p}{(chb+ck+d+(ach)^{2j+2}(v_0(1-ach)-chb-ck-d))(1-ach)^{q-1}}\Big)^{q^{6(m-j-1)}},\\ y_{2(3m)-1}=y_{-1}^{q^{6m}}\prod\limits_{j=0}^{m-1}\Big(\frac{(chb+ck+d+(ach)^{2j}(v_0(1-ach)-chb-ck-d))^q}{(chb+ck+d+(ach)^{2j}(v_1(1-ach)-chb-ck-d))(1-ach)^{q-1}}\Big)^{q^{2(3(m-j)-1)}}\\ \qquad\prod\limits_{j=0}^{m-1}\Big(\frac{(chb+ck+d+(ach)^{2j}(v_2(1-ach)-chb-ck-d))^q}{(chb+ck+d+(ach)^{2j+1}(v_0(1-ach)-chb-ck-d))(1-ach)^{q-1}}\Big)^{q^{2(3(m-j)-2)}}\\ \qquad\prod\limits_{j=0}^{m-1}\Big(\frac{(chb+ck+d+(ach)^{2j+1}(v_1(1-ach)-chb-ck-d))^q}{(chb+ck+d+(ach)^{2j+1}(v_2(1-ach)-chb-ck-d))(1-ach)^{q-1}}\Big)^{q^{6(m-j-1)}}. \end{cases} \tag{2.23}$$

$$\begin{cases} z_{2(3m)}=z_0^{r^{6m}}\prod\limits_{j=0}^{m-1}\Big(\frac{(ahd+hb+k+(ach)^{2j}(w_1(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j}(w_2(1-ach)-ahd-hb-k))(1-ach)^{r-1}}\Big)^{r^{2(3(m-j)-1)}}\\ \qquad\prod\limits_{j=0}^{m-1}\Big(\frac{(ahd+hb+k+(ach)^{2j+1}(w_0(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j+1}(w_1(1-ach)-ahd-hb-k))(1-ach)^{r-1}}\Big)^{r^{2(3(m-j)-2)}}\\ \qquad\prod\limits_{j=0}^{m-1}\Big(\frac{(ahd+hb+k+(ach)^{2j+1}(w_2(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j+2}(w_0(1-ach)-ahd-hb-k))(1-ach)^{r-1}}\Big)^{r^{6(m-j-1)}},\\ z_{2(3m)-1}=z_{-1}^{r^{6m}}\prod\limits_{j=0}^{m-1}\Big(\frac{(ahd+hb+k+(ach)^{2j}(w_0(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j}(w_1(1-ach)-ahd-hb-k))(1-ach)^{r-1}}\Big)^{r^{2(3(m-j)-1)}}\\ \qquad\prod\limits_{j=0}^{m-1}\Big(\frac{(ahd+hb+k+(ach)^{2j}(w_2(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j+1}(w_0(1-ach)-ahd-hb-k))(1-ach)^{r-1}}\Big)^{r^{2(3(m-j)-2)}}\\ \qquad\prod\limits_{j=0}^{m-1}\Big(\frac{(ahd+hb+k+(ach)^{2j+1}(w_1(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j+1}(w_2(1-ach)-ahd-hb-k))(1-ach)^{r-1}}\Big)^{r^{6(m-j-1)}}. \end{cases} \tag{2.24}$$

$ii)$ If $n=3m+1$, $m\in\mathbb{N}_0$, then, from (2.16), (2.17) and (2.18), we have

$$\begin{cases} x_{2(3m+1)}=x_0^{p^{6m+2}}\prod\limits_{j=0}^{m}\Big(\frac{(ack+ad+b+(ach)^{2j}(u_1(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j}(u_2(1-ach)-ack-ad-b))(1-ach)^{p-1}}\Big)^{p^{6(m-j)}}\\ \qquad\prod\limits_{j=0}^{m-1}\Big(\frac{(ack+ad+b+(ach)^{2j+1}(u_0(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j+1}(u_1(1-ach)-ack-ad-b))(1-ach)^{p-1}}\Big)^{p^{2(3(m-j)-1)}}\\ \qquad\prod\limits_{j=0}^{m-1}\Big(\frac{(ack+ad+b+(ach)^{2j+1}(u_2(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j+2}(u_0(1-ach)-ack-ad-b))(1-ach)^{p-1}}\Big)^{p^{2(3(m-j)-2)}},\\ x_{2(3m)+1}=x_{-1}^{p^{6m+2}}\prod\limits_{j=0}^{m}\Big(\frac{(ack+ad+b+(ach)^{2j}(u_0(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j}(u_1(1-ach)-ack-ad-b))(1-ach)^{p-1}}\Big)^{p^{6(m-j)}}\\ \qquad\prod\limits_{j=0}^{m-1}\Big(\frac{(ack+ad+b+(ach)^{2j}(u_2(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j+1}(u_0(1-ach)-ack-ad-b))(1-ach)^{p-1}}\Big)^{p^{2(3(m-j)-1)}}\\ \qquad\prod\limits_{j=0}^{m-1}\Big(\frac{(ack+ad+b+(ach)^{2j+1}(u_1(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j+1}(u_2(1-ach)-ack-ad-b))(1-ach)^{p-1}}\Big)^{p^{2(3(m-j)-2)}}. \end{cases} \tag{2.25}$$

$$\begin{cases} y_{2(3m+1)} = y_0^{q^{6m+2}} \prod_{j=0}^{m} \left(\frac{(chb+ck+d+(ach)^{2j}(v_1(1-ach)-chb-ck-d))^q}{(chb+ck+d+(ach)^{2j}(v_2(1-ach)-chb-ck-d))(1-ach)^{q-1}}\right)^{q^{6(m-j)}} \\ \qquad \prod_{j=0}^{m-1} \left(\frac{(chb+ck+d+(ach)^{2j+1}(v_0(1-ach)-chb-ck-d))^q}{(chb+ck+d+(ach)^{2j+1}(v_1(1-ach)-chb-ck-d))(1-ach)^{q-1}}\right)^{q^{2(3(m-j)-1)}} \\ \qquad \prod_{j=0}^{m-1} \left(\frac{(chb+ck+d+(ach)^{2j+1}(v_2(1-ach)-chb-ck-d))^p}{(chb+ck+d+(ach)^{2j+2}(v_0(1-ach)-chb-ck-d))(1-ach)^{q-1}}\right)^{q^{2(3(m-j)-2)}}, \\ y_{2(3m)+1} = y_{-1}^{q^{6m+2}} \prod_{j=0}^{m} \left(\frac{(chb+ck+d+(ach)^{2j}(v_0(1-ach)-chb-ck-d))^q}{(chb+ck+d+(ach)^{2j}(v_1(1-ach)-chb-ck-d))(1-ach)^{q-1}}\right)^{q^{6(m-j)}} \\ \qquad \prod_{j=0}^{m-1} \left(\frac{(chb+ck+d+(ach)^{2j}(v_2(1-ach)-chb-ck-d))^q}{(chb+ck+d+(ach)^{2j+1}(v_0(1-ach)-chb-ck-d))(1-ach)^{q-1}}\right)^{q^{2(3(m-j)-1)}} \\ \qquad \prod_{j=0}^{m-1} \left(\frac{(chb+ck+d+(ach)^{2j+1}(v_1(1-ach)-chb-ck-d))^q}{(chb+ck+d+(ach)^{2j+1}(v_2(1-ach)-chb-ck-d))(1-ach)^{q-1}}\right)^{q^{2(3(m-j)-2)}}. \end{cases} \tag{2.26}$$

$$\begin{cases} z_{2(3m+1)} = z_0^{r^{6m+2}} \prod_{j=0}^{m} \left(\frac{(ahd+hb+k+(ach)^{2j}(w_1(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j}(w_2(1-ach)-ahd-hb-k))(1-ach)^{r-1}}\right)^{r^{6(m-j))}} \\ \qquad \prod_{j=0}^{m-1} \left(\frac{(ahd+hb+k+(ach)^{2j+1}(w_0(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j+1}(w_1(1-ach)-ahd-hb-k))(1-ach)^{r-1}}\right)^{r^{2(3(m-j)-1)}} \\ \qquad \prod_{j=0}^{m-1} \left(\frac{(ahd+hb+k+(ach)^{2j+1}(w_2(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j+2}(w_0(1-ach)-ahd-hb-k))(1-ach)^{r-1}}\right)^{r^{2(3(m-j)-2)}}, \\ z_{2(3m)+1} = z_{-1}^{r^{6m+2}} \prod_{j=0}^{m} \left(\frac{(ahd+hb+k+(ach)^{2j}(w_0(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j}(w_1(1-ach)-ahd-hb-k))(1-ach)^{r-1}}\right)^{r^{6(m-j)}} \\ \qquad \prod_{j=0}^{m-1} \left(\frac{(ahd+hb+k+(ach)^{2j}(w_2(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j+1}(w_0(1-ach)-ahd-hb-k))(1-ach)^{r-1}}\right)^{r^{2(3(m-j)-1)}} \\ \qquad \prod_{j=0}^{m-1} \left(\frac{(ahd+hb+k+(ach)^{2j+1}(w_1(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j+1}(w_2(1-ach)-ahb-hb-k))(1-ach)^{r-1}}\right)^{r^{2(3(m-j)-2)}}. \end{cases} \tag{2.27}$$

iii) If $n = 3m+2$, $m \in \mathbb{N}_0$, then, from (2.19), (2.20) and (2.21), we get

$$\begin{cases} x_{2(3m+2)} = x_0^{p^{6m+4}} \prod_{j=0}^{m} \left(\frac{(ack+ad+b+(ach)^{2j}(u_1(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j}(u_2(1-ach)-ack-ad-b))(1-ach)^{p-1}}\right)^{p^{2(3(m-j)+1)}} \\ \qquad \prod_{j=0}^{m} \left(\frac{(ack+ad+b+(ach)^{2j+1}(u_0(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j+1}(u_1(1-ach)-ack-ad-b))(1-ach)^{p-1}}\right)^{p^{6(m-j)}} \\ \qquad \prod_{j=0}^{m-1} \left(\frac{(ack+ad+b+(ach)^{2j+1}(u_2(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j+2}(u_0(1-ach)-ack-ad-b))(1-ach)^{p-1}}\right)^{p^{2(3(m-j)-1)}}, \\ x_{2(3m)+3} = x_{-1}^{p^{6m+4}} \prod_{j=0}^{m-1} \left(\frac{(ack+ad+b+(ach)^{2j}(u_0(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j}(u_1(1-ach)-ack-ad-b))(1-ach)^{p-1}}\right)^{p^{2(3(m-j)+1)}} \\ \qquad \prod_{j=0}^{m-1} \left(\frac{(ack+ad+b+(ach)^{2j}(u_2(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j+1}(u_0(1-ach)-ack-ad-b))(1-ach)^{p-1}}\right)^{p^{6(m-j)}} \\ \qquad \prod_{j=0}^{m-1} \left(\frac{(ack+ad+b+(ach)^{2j+1}(u_1(1-ach)-ack-ad-b))^p}{(ack+ad+b+(ach)^{2j+1}(u_2(1-ach)-ack-ad-b))(1-ach)^{p-1}}\right)^{p^{2(3(m-j)-1)}}. \end{cases} \tag{2.28}$$

$$\begin{cases} y_{2(3m+2)} = y_0^{q^{6m+4}} \prod_{j=0}^{m} \left(\frac{(chb+ck+d+(ach)^{2j}(v_1(1-ach)-chb-ck-d))^q}{(chb+ck+d+(ach)^{2j}(v_2(1-ach)-chb-ck-d))(1-ach)^{q-1}}\right)^{q^{2(3(m-j)+1)}} \\ \prod_{j=0}^{m} \left(\frac{(chb+ck+d+(ach)^{2j+1}(v_0(1-ach)-chb-ck-d))^q}{(chb+ck+d+(ach)^{2j+1}(v_1(1-ach)-chb-ckb-d))(1-ach)^{q-1}}\right)^{q^{6(m-j)}} \\ \prod_{j=0}^{m-1} \left(\frac{(chb+ck+d+(ach)^{2j+1}(v_2(1-ach)-chb-ck-d))^p}{(chb+ck+d+(ach)^{2j+2}(v_0(1-ach)-chb-ck-d))(1-ach)^{q-1}}\right)^{q^{2(3(m-j)-1)}}, \\ y_{2(3m)+3} = y_{-1}^{q^{6m+4}} \prod_{j=0}^{m-1} \left(\frac{(chb+ck+d+(ach)^{2j}(v_0(1-ach)-chb-ck-d))^q}{(chb+ck+d+(ach)^{2j}(v_1(1-ach)-chb-ck-d))(1-ach)^{q-1}}\right)^{q^{2(3(m-j)+1)}} \\ \prod_{j=0}^{m} \left(\frac{(chb+ck+d+(ach)^{2j}(v_2(1-ach)-chb-ck-d))^q}{(chb+ck+d+(ach)^{2j+1}(v_0(1-ach)-chb-ck-d))(1-ach)^{q-1}}\right)^{q^{6(m-j)}} \\ \prod_{j=0}^{m} \left(\frac{(chb+ck+d+(ach)^{2j+1}(v_1(1-ach)-chb-ck-d))^q}{(chb+ck+d+(ach)^{2j+1}(v_2(1-ach)-chb-ck-d))(1-ach)^{q-1}}\right)^{q^{2(3(m-j)-1)}}. \end{cases} \tag{2.29}$$

$$\begin{cases} z_{2(3m+2)} = z_0^{r^{6m+4}} \prod_{j=0}^{m} \left(\frac{(ahd+hb+k+(ach)^{2j}(w_1(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j}(w_2(1-ach)-ahd-hb-k))(1-ach)^{r-1}}\right)^{r^{2(3(m-j)+1)}} \\ \prod_{j=0}^{m} \left(\frac{(ahd+hb+k+(ach)^{2j+1}(w_0(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j+1}(w_1(1-ach)-ahd-hb-k))(1-ach)^{r-1}}\right)^{r^{6(m-j)}} \\ \prod_{j=0}^{m-1} \left(\frac{(ahd+hb+k+(ach)^{2j+1}(w_2(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j+2}(w_0(1-ach)-ahd-hb-k))(1-ach)^{r-1}}\right)^{r^{2(3(m-j)-1)}}, \\ z_{2(3m)+3} = z_{-1}^{r^{6m+4}} \prod_{j=0}^{m} \left(\frac{(ahd+hb+k+(ach)^{2j}(w_0(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j}(w_1(1-ach)-ahd-hb-k))(1-ach)^{r-1}}\right)^{r^{2(3(m-j)+1)}} \\ \prod_{j=0}^{m} \left(\frac{(ahd+hb+k+(ach)^{2j}(w_2(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j+1}(w_0(1-ach)-ahd-hb-k))(1-ach)^{r-1}}\right)^{r^{6(m-j)}} \\ \prod_{j=0}^{m-1} \left(\frac{(ahd+hb+k+(ach)^{2j+1}(w_1(1-ach)-ahd-hb-k))^r}{(ahd+hb+k+(ach)^{2j+1}(w_2(1-ach)-ahd-hb-k))(1-ach)^{r-1}}\right)^{r^{2(3(m-j)-1)}}. \end{cases} \tag{2.30}$$

Second case $ach = 1$. We have

i) If $n = 3m$, $m \in \mathbb{N}_0$, then, from (2.13), (2.14) and (2.15), we have

$$\begin{cases} x_{2(3m)} = x_0^{p^{6m}} \prod_{j=0}^{m-1} \left(\frac{(u_1+(ack+ad+b)(2j))^p}{u_2+(ack+ad+b)(2j)}\right)^{p^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} \left(\frac{(u_0+(ack+ad+b)(2j+1))^p}{u_1+(ack+ad+b)(2j+1)}\right)^{p^{2(3(m-j)-2)}} \\ \prod_{j=0}^{m-1} \left(\frac{(u_2+(ack+ad+b)(2j+1))^p}{u_0+(ack+ad+b)(2j+2)}\right)^{p^{6(m-j-1)}}, \\ x_{2(3m)-1} = x_{-1}^{p^{6m}} \prod_{j=0}^{m-1} \left(\frac{(u_0+(ack+ad+b)(2j))^p}{u_1+(ack+ad+b)(2j)}\right)^{p^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} \left(\frac{(u_2+(ack+ad+b)(2j))^p}{u_0+(ack+ad+b)(2j+1)}\right)^{p^{2(3(m-j)-2)}} \\ \prod_{j=0}^{m-1} \left(\frac{(u_1+(ack+ad+b)(2j+1))^p}{u_2+(ack+ad+b)(2j+1)}\right)^{p^{6(m-j-1)}}. \end{cases} \tag{2.31}$$

$$\begin{cases} y_{2(3m)} = y_0^{q^{6m}} \prod_{j=0}^{m-1} \left(\frac{(v_1+(chb+ck+d)(2j))^q}{v_2+(chb+ck+d)(2j)}\right)^{q^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} \left(\frac{(v_0+(chb+ck+d)(2j+1))^q}{v_1+(chb+ck+d)(2j+1)}\right)^{q^{2(3(m-j)-2)}} \\ \prod_{j=0}^{m-1} \left(\frac{(v_2+(chb+ck+d)(2j+1))^q}{v_0+(chb+ck+d)(2j+2)}\right)^{q^{6(m-j-1)}}, \\ y_{2(3m)-1} = x_{-1}^{q^{6m}} \prod_{j=0}^{m-1} \left(\frac{(v_0+(chb+ck+d)(2j))^q}{v_1+(chb+ck+d)(2j)}\right)^{q^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} \left(\frac{(v_2+(chb+ck+d)(2j))^q}{v_0+(chb+ck+d)(2j+1)}\right)^{q^{2(3(m-j)-2)}} \\ \prod_{j=0}^{m-1} \left(\frac{(v_1+(chb+ck+d)(2j+1))^q}{v_2+(chb+ck+d)(2j+1)}\right)^{q^{6(m-j-1)}}. \end{cases} \tag{2.32}$$

$$\begin{cases} z_{2(3m)} = z_0^{r^{6m}} \prod_{j=0}^{m-1} (\frac{(w_1+(ahd+hb+k)(2j))^r}{w_2+(ahd+hb+k)(2j)})^{r^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} (\frac{(w_0+(ahd+hb+k)(2j+1))^r}{w_1+(ahd+hb+k)(2j+1)})^{r^{2(3(m-j)-2)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(w_2+(ahd+hb+k)(2j+1))^r}{w_0+(ahd+hb+k)(2j+2)})^{r^{6(m-j-1)}}, \\ z_{2(3m)-1} = z_{-1}^{r^{6m}} \prod_{j=0}^{m-1} (\frac{(w_0+(ahd+hb+k)(2j))^r}{w_1+(ahd+hb+k)(2j)})^{r^{2(3(m-j)-1)}} \prod_{j=0}^{m-1} (\frac{(w_2+(ahd+hb+k)(2j))^p}{w_0+(ahd+hb+k)(2j+1)})^{r^{2(3(m-j)-2)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(w_1+(ahd+hb+k)(2j+1))^r}{w_2+(ahd+hb+k)(2j+1)})^{r^{6(m-j-1)}}. \end{cases} \tag{2.33}$$

ii) If $n = 3m+1$, $m \in \mathbb{N}_0$, then, from (2.16), (2.17) and (2.18), we obtain

$$\begin{cases} x_{2(3m+1)} = x_0^{p^{6m+2}} \prod_{j=0}^{m} (\frac{(u_1+(ack+ad+b)(2j))^p}{u_2+(ack+ad+b)(2j)})^{p^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{(u_0+(ack+ad+b)(2j+1))^p}{u_1+(ack+ad+b)(2j+1)})^{p^{2(3(m-j)-1)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(u_2+(ack+ad+b)(2j+1))^p}{u_0+(ack+ad+b)(2j+2)})^{p^{2(3(m-j)-2)}}, \\ x_{2(3m)+1} = x_{-1}^{p^{6m+2}} \prod_{j=0}^{m} (\frac{(u_0+(ack+ad+b)(2j))^p}{u_1+(ack+ad+b)(2j)})^{p^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{(u_2+(ack+ad+b)(2j))^p}{u_0+(ack+ad+b)(2j+1)})^{p^{2(3(m-j)-1)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(u_1+(ack+ad+b)(2j+1))^p}{u_2+(ack+ad+b)(2j+1)})^{p^{2(3(m-j)-2)}}. \end{cases} \tag{2.34}$$

$$\begin{cases} y_{2(3m+1)} = y_0^{q^{6m+2}} \prod_{j=0}^{m} (\frac{(v_1+(chb+ck+d)(2j))^q}{v_2+(chb+ck+d)(2j)})^{q^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{(v_0+(chb+ck+d)(2j+1))^q}{v_1+(chb+ck+d)(2j+1)})^{q^{2(3(m-j)-1)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(v_2+(chb+ck+d)(2j+1))^q}{v_0+(chb+ck+d)(2j+2)})^{q^{2(3(m-j)-2)}}, \\ y_{2(3m)+1} = y_{-1}^{q^{6m+2}} \prod_{j=0}^{m} (\frac{(v_0+(chb+ck+d)(2j))^q}{v_1+(chb+ck+d)(2j)})^{q^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{(v_2+(chb+ck+d)(2j))^q}{v_0+(chb+ck+d)(2j+1)})^{q^{2(3(m-j)-1)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(v_1+(chb+ck+d)(2j+1))^q}{v_2+(chb+ck+d)(2j+1)})^{q^{2(3(m-j)-2)}}. \end{cases} \tag{2.35}$$

$$\begin{cases} z_{2(3m+1)} = z_0^{r^{6m+2}} \prod_{j=0}^{m} (\frac{(w_1+(ahd+hb+k)(2j))^r}{w_2+(ahd+hb+k)(2j)})^{r^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{(w_0+(ahd+hb+k)(2j+1))^r}{w_1+(ahd+hb+k)(2j+1)})^{r^{2(3(m-j)-1)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(w_2+(ahd+hb+k)(2j+1))^r}{w_0+(ahd+hb+k)(2j+2)})^{r^{2(3(m-j)-2)}}, \\ z_{2(3m)+1} = z_{-1}^{r^{6m+2}} \prod_{j=0}^{m} (\frac{(w_0+(ahd+hb+k)(2j))^r}{w_1+(ahd+hb+k)(2j)})^{r^{6(m-j)}} \prod_{j=0}^{m-1} (\frac{(w_2+(ahd+hb+k)(2j))^r}{w_0+(ahd+hb+k)(2j+1)})^{r^{2(3(m-j)-1)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(w_1+(ahd+hb+k)(2j+1))^r}{w_2+(ahd+hb+k)(2j+1)})^{r^{2(3(m-j)-2)}}. \end{cases} \tag{2.36}$$

iii) If $n = 3m+2$, $m \in \mathbb{N}_0$, then, from (2.19), (2.20) and (2.21), we get

$$\begin{cases} x_{2(3m+2)} = x_0^{p^{6m+4}} \prod_{j=0}^{m} (\frac{(u_1+(ack+ad+b)(2j))^p}{u_2+(ack+ad+b)(2j)})^{p^{2(3(m-j)+1)}} \prod_{j=0}^{m} (\frac{(u_0+(ack+ad+b)(2j+1))^p}{u_1+(ack+ad+b)(2j+1)})^{p^{6(m-j)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(u_2+(ack+ad+b)(2j+1))^p}{u_0+(ack+ad+b)(2j+2)})^{p^{2(3(m-j)-1)}}, \\ x_{2(3m)+3} = x_{-1}^{p^{6m+4}} \prod_{j=0}^{m} (\frac{(u_0+(ack+ad+b)(2j))^p}{u_1+(ack+ad+b)(2j)})^{p^{2(3(m-j)+1)}} \prod_{j=0}^{m} (\frac{(u_2+(ack+ad+b)(2j))^p}{u_0+(ack+ad+b)(2j+1)})^{p^{6(m-j)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(u_1+(ack+ad+b)(2j+1))^p}{u_2+(ack+ad+b)(2j+1)})^{p^{2(3(m-j)-1)}}. \end{cases} \tag{2.37}$$

$$\begin{cases} y_{2(3m+2)} = y_0^{q^{6m+4}} \prod_{j=0}^{m} (\frac{(v_1+(chb+ck+d)(2j))^q}{v_2+(chb+ck+d)(2j)})^{q^{2(3(m-j)+1)}} \prod_{j=0}^{m} (\frac{(v_0+(chb+ck+d)(2j+1))^q}{v_1+(chb+ck+d)(2j+1)})^{q^{6(m-j)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(v_2+(chb+ck+d)(2j+1))^q}{v_0+(chb+ck+d)(2j+2)})^{q^{2(3(m-j)-1)}}, \\ y_{2(3m)+3} = y_{-1}^{q^{6m+4}} \prod_{j=0}^{m} (\frac{(v_0+(chb+ck+d)(2j))^q}{v_1+(chb+ck+d)(2j)})^{q^{2(3(m-j)+1)}} \prod_{j=0}^{m} (\frac{(v_2+(chb+ck+d)(2j))^q}{v_0+(chb+ck+d)(2j+1)})^{q^{6(m-j)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(v_1+(chb+ck+d)(2j+1))^q}{v_2+(chb+ck+d)(2j+1)})^{q^{2(3(m-j)-1)}}. \end{cases} \tag{2.38}$$

$$\begin{cases} z_{2(3m+2)} = z_0^{r^{6m+4}} \prod_{j=0}^{m} (\frac{(w_1+(ahd+hb+k)(2j))^r}{w_2+(ahd+hb+k)(2j)})^{r^{2(3(m-j)+1)}} \prod_{j=0}^{m} (\frac{(w_0+(ahd+hb+k)(2j+1))^r}{w_1+(ahd+hb+k)(2j+1)})^{r^{6(m-j)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(w_2+(ahd+hb+k)(2j+1))^r}{w_0+(ahd+hb+k)(2j+2)})^{r^{2(3(m-j)-1)}}, \\ z_{2(3m)+3} = z_{-1}^{r^{6m+4}} \prod_{j=0}^{m} (\frac{(w_0+(ahd+hb+k)(2j))^r}{w_1+(ahd+hb+k)(2j)})^{r^{2(3(m-j)+1)}} \prod_{j=0}^{m} (\frac{(w_2+(ahd+hb+k)(2j))^r}{w_0+(ahd+hb+k)(2j+1)})^{r^{6(m-j)}} \\ \qquad \prod_{j=0}^{m-1} (\frac{(w_1+(ahd+hb+k)(2j+1))^r}{w_2+(ahd+hb+k)(2j+1)})^{r^{2(3(m-j)-1)}}. \end{cases} \tag{2.39}$$

In summary, we have the following theorem.

Theorem 2.1. *Consider System (1.3). Then, the following statements hold:*

(*a*) *If* $n = 3m, \quad m \in \mathbb{N}_0$, *every solution of System (1.3) is given by (2.22), (2.23), (2.24), (2.31), (2.32) and (2.33).*

(*b*) *If* $n = 3m+1, \quad m \in \mathbb{N}_0$, *every solution of System (1.3) is given by (2.25), (2.26), (2.27), (2.34), (2.35) and (2.36).*

(*c*) *If* $n = 3m+2, \quad m \in \mathbb{N}_0$, *every solution of System (1.3) is given by (2.28), (2.29), (2.30), (2.37), (2.38) and (2.39).*

3. Asymptotics behavior of the solutions of System (1.3) when $p = q = r = 0$ and $p = q = r = 1$

In this section, we focus our attention on two special cases of System (1.3). We examine the boundendness, periodicity and asymptotic behavior of solutions of System (1.3) when $p = q = r = 0$ and $p = q = r = 1$.

3.1. Case $p = q = r = 0$.

If $p = q = r = 0$, System (1.3) takes the form

$$\begin{cases} x_{n+1} = \frac{y_n}{a+by_n}, \\ y_{n+1} = \frac{z_n}{c+dz_n}, \qquad n \in \mathbb{N}_0. \\ z_{n+1} = \frac{x_n}{h+kx_n}, \end{cases} \tag{3.1}$$

As in the previous section every solution $(x_n, y_n, z_n)_{n\in\mathbb{N}_0}$ of System (3.1) is assumed to be well defined, which means that

$$(a + by_n)(c + dz_n)(h + kx_n) \neq 0, \ n \in \mathbb{N}_0.$$

As the initial values are such that $x_0y_0z_0 \neq 0$, it is not hard to see that

$$x_ny_nz_n \neq 0, \ n \in \mathbb{N}_0.$$

By using the following changes of variables

$$u_n = \frac{1}{x_n}, \quad v_n = 1 \qquad — \tag{3.2}$$

then, for $n \in \mathbb{N}_0$ and $i = 0, 1, 2$, every solution of System (3.1) is given by

$$x_{3n+i} = \begin{cases} \frac{1}{u_i+(ack+ad+b)n}, & ach = 1, \\ \frac{1}{(ach)^n u_i + (\frac{1-(ach)^n}{1-ach})(ack+ad+b)}, & otherwise, \end{cases}$$

$$= \begin{cases} \frac{1}{u_i+(ack+ad+b)n}, & ach = 1, \\ \frac{1-ach}{ack+ad+b+(ach)^n(u_i(1-ach)-ack-ad-b)}, & otherwise, \end{cases}; \tag{3.3}$$

$$y_{3n+i} = \begin{cases} \frac{1}{v_i+(chb+ck+d)n}, & ach = 1, \\ \frac{1}{(ach)^n v_i + (\frac{1-(ach)^n}{1-ach})(chb+ck+d)}, & otherwise, \end{cases}$$

$$= \begin{cases} \frac{1}{v_i+(chb+ck+d)n}, & ach = 1, \\ \frac{1-ach}{chb+ck+d+(ach)^n(v_i(1-ach)-chb-ck-d)}, & otherwise, \end{cases}; \tag{3.4}$$

$$z_{3n+i} = \begin{cases} \frac{1}{w_i+(ahd+hb+k)n}, & ach = 1, \\ \frac{1}{(ach)^n w_i + (\frac{1-(ach)^n}{1-ach})(ahd+hb+k)}, & otherwise, \end{cases}$$

$$= \begin{cases} \frac{1}{w_i+(ahd+hb+k)n}, & ach = 1, \\ \frac{1-ach}{ahd+hb+k+(ach)^n(w_i(1-ach)-ahd-hb-k)}, & otherwise, \end{cases}. \tag{3.5}$$

Theorem 3.1. *Let $(x_n, y_n, z_n)_{n\geq 0}$ be a solution of System (3.1). Then the following statements are true.*

(a) *If $ach = 1$, $ack + ad + b = 0$, $chb + ck + d = 0$, and $ahd + hb + k = 0$, then for $n \in \mathbb{N}_0$ and $i = 0, 1, 2$, we get $x_{3n+i} = x_i$, $y_{3n+i} = y_i$ and $z_{3n+i} = z_i$. That is the solution $(x_n, y_n, z_n)_{n\geq 0}$ is periodic of period 3.*

(b) *If $ach = 1$, $(ack + ad + b)(chb + ck + d)(ahd + hb + k) \neq 0$, then $(x_n, y_n, z_n) \to (0, 0, 0)$, as $n \to \infty$.*

(c) *If $(ack + ad + b)(chb + ck + d)(ahd + hb + k) \neq 0$ and $\mid ach \mid < 1$, then $x_n \to \frac{1-ach}{ack+ad+b}$, $y_n \to \frac{1-ach}{chb+ck+d}$ and $z_n \to \frac{1-ach}{ahd+hb+k}$, as $n \to \infty$.*

(d) *If $ach \neq 1$, $u_0 = u_1 = u_2 = \frac{ack+ad+b}{1-ach}$, then the sequence $(x_n)_{n\geq 0}$ is constant.*

(e) *If $ach \neq 1$, $v_0 = v_1 = v_2 = \frac{chb+ck+d}{1-ach}$, then the sequence $(y_n)_{n\geq 0}$ is constant.*

(f) *If $ach \neq 1$, $w_0 = w_1 = w_2 = \frac{ahd+hb+k}{1-ach}$, then the sequence $(z_n)_{n\geq 0}$ is constant.*

(g) *If $u_i \neq \frac{ack+ad+b}{1-ach}$, $i = 0, 1, 2$ and $\mid ach \mid > 1$ then $x_n \to 0$, as $n \to \infty$.*

(h) *If $v_i \neq \frac{chb+ck+d}{1-ach}$, $i = 0, 1, 2$ and $\mid ach \mid > 1$ then $y_n \to 0$, as $n \to \infty$.*

(i) *If $w_i \neq \frac{ahd+hb+k}{1-ach}$, $i = 0, 1, 2$ and $\mid ach \mid > 1$ then $z_n \to 0$, as $n \to \infty$.*

Proof.

From (3.3), (3.4) and (3.5), the proof is easily seen. □

3.2. Case $p = q = r = 1$

If $p = q = r = 1$ then System (1.3) will be

$$\begin{cases} x_{n+1} = \frac{y_n y_{n-1}}{x_n(a+by_n y_{n-1})}, \\ y_{n+1} = \frac{z_n z_{n-1}}{y_n(c+dz_n z_{n-1})}, \qquad n \in \mathbb{N}_0. \\ z_{n+1} = \frac{x_n x_{n-1}}{z_n(h+kx_n x_{n-1})}, \end{cases} \tag{3.6}$$

Remark 3.2. Before we deal with this system we want to mention the following.
- System (3.6) was treated by A. Cete under the supervision of Y. Yazlik in [4] and separately by M. C. Bekara Mostefa and N. Touafek.
- It is worth noting that in dealing with System (3.6), we are mainly inspired by the paper of Stevic et al. [16], this is why we use and follows practically the same notations and technics.

From the general case, we have the following formulas of the well-defined solutions of System (3.6), depending on $ach \neq 1$ or $ach = 1$. Let $(x_n, y_n, z_n)_{n\geq -1}$ be a well-defined solution of System (3.6), we have

1) Case $ach \neq 1$.

i) If $n = 3m$, $m \in \mathbb{N}_0$. Then, from (2.22), (2.23) and (2.24), we have

$$\begin{cases} x_{2(3m)} = x_0 \prod_{j=0}^{m-1} \frac{ack+ad+b+(ach)^{2j}(u_1(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2j}(u_2(1-ach)-ack-ad-b)} \\ \qquad \prod_{j=0}^{m-1} \frac{ack+ad+b+(ach)^{2j+1}(u_0(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2j+1}(u_1(1-ach)-ack-ad-b)} \\ \qquad \prod_{j=0}^{m-1} \frac{ack+ad+b+(ach)^{2j+1}(u_2(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2j+2}(u_0(1-ach)-ack-ad-b)}, \\ x_{2(3m)-1} = x_{-1} \prod_{j=0}^{m-1} \frac{ack+ad+b+(ach)^{2j}(u_0(1-ach)-ack-ad-b))}{ack+ad+b+(ach)^{2j}(u_1(1-ach)-ack-ad-b)} \\ \qquad \prod_{j=0}^{m-1} \frac{ack+ad+b+(ach)^{2j}(u_2(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2j+1}(u_0(1-ach)-ack-ad-b)} \\ \qquad \prod_{j=0}^{m-1} \frac{ack+ad+b+(ach)^{2j+1}(u_1(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2j+1}(u_2(1-ach)-ack-ad-b)}. \end{cases} \tag{3.7}$$

$$\begin{cases} y_{2(3m)} = y_0 \prod_{j=0}^{m-1} \frac{chb+ck+d+(ach)^{2j}(v_1(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2j}(v_2(1-ach)-chb-ck-d)} \\ \qquad \prod_{j=0}^{m-1} \frac{chb+ck+d+(ach)^{2j+1}(v_0(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2j+1}(v_1(1-ach)-chb-ck-d)} \\ \qquad \prod_{j=0}^{m-1} \frac{(chb+ck+d+(ach)^{2j+1}(v_2(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2j+2}(v_0(1-ach)-chb-ck-d)}, \\ y_{2(3m)-1} = y_{-1} \prod_{j=0}^{m-1} \frac{chb+ck+d+(ach)^{2j}(v_0(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2j}(v_1(1-ach)-chb-ck-d)} \\ \qquad \prod_{j=0}^{m-1} \frac{chb+ck+d+(ach)^{2j}(v_2(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2j+1}(v_0(1-ach)-chb-ck-d)} \\ \qquad \prod_{j=0}^{m-1} \frac{chb+ck+d+(ach)^{2j+1}(v_1(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2j+1}(v_2(1-ach)-chb-ck-d)}. \end{cases} \tag{3.8}$$

$$\begin{cases} z_{2(3m)} = z_0 \prod_{j=0}^{m-1} \frac{ahd+hb+k+(ach)^{2j}(w_1(1-ach)-ahd-hb-k))}{ahd+hb+k+(ach)^{2j}(w_2(1-ach)-ahd-hb-k)} \\ \qquad \prod_{j=0}^{m-1} \frac{ahd+hb+k+(ach)^{2j+1}(w_0(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2j+1}(w_1(1-ach)-ahd-hb-k)} \\ \qquad \prod_{j=0}^{m-1} \frac{ahd+hb+k+(ach)^{2j+1}(w_2(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2j+2}(w_0(1-ach)-ahd-hb-k)}, \\ z_{2(3m)-1} = z_{-1} \prod_{j=0}^{m-1} \frac{ahd+hb+k+(ach)^{2j}(w_0(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2j}(w_1(1-ach)-ahd-hb-k)} \\ \qquad \prod_{j=0}^{m-1} \frac{ahd+hb+k+(ach)^{2j}(w_2(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2j+1}(w_0(1-ach)-ahd-hb-k)} \\ \qquad \prod_{j=0}^{m-1} \frac{ahd+hb+k+(ach)^{2j+1}(w_1(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2j+1}(w_2(1-ach)-ahd-hb-k)}. \end{cases} \tag{3.9}$$

ii) If $n = 3m+1$, $m \in \mathbb{N}_0$, then, from (2.25), (2.26) and (2.27), we have

$$\begin{cases} x_{2(3m+1)} = x_0 \prod_{j=0}^{m} \frac{ack+ad+b+(ach)^{2j}(u_1(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2j}(u_2(1-ach)-ack-ad-b)} \\ \qquad \prod_{j=0}^{m-1} \frac{ack+ad+b+(ach)^{2j+1}(u_0(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2j+1}(u_1(1-ach)-ack-ad-b)} \\ \qquad \prod_{j=0}^{m-1} \frac{ack+ad+b+(ach)^{2j+1}(u_2(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2j+2}(u_0(1-ach)-ack-ad-b)}, \\ x_{2(3m)+1} = x_{-1} \prod_{j=0}^{m} \frac{ack+ad+b+(ach)^{2j}(u_0(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2j}(u_1(1-ach)-ack-ad-b)} \\ \qquad \prod_{j=0}^{m-1} \frac{ack+ad+b+(ach)^{2j}(u_2(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2j+1}(u_0(1-ach)-ack-ad-b)} \\ \qquad \prod_{j=0}^{m-1} \frac{ack+ad+b+(ach)^{2j+1}(u_1(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2j+1}(u_2(1-ach)-ack-ad-b)}. \end{cases} \tag{3.10}$$

$$\begin{cases} y_{2(3m+1)} = y_0 \prod_{j=0}^{m} \frac{chb+ck+d+(ach)^{2j}(v_1(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2j}(v_2(1-ach)-chb-ck-d)} \\ \qquad \prod_{j=0}^{m-1} \frac{chb+ck+d+(ach)^{2j+1}(v_0(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2j+1}(v_1(1-ach)-chb-ck-d)} \\ \qquad \prod_{j=0}^{m-1} \frac{chb+ck+d+(ach)^{2j+1}(v_2(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2j+2}(v_0(1-ach)-chb-ck-d)}, \\ y_{2(3m)+1} = y_{-1} \prod_{j=0}^{m} \frac{chb+ck+d+(ach)^{2j}(v_0(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2j}(v_1(1-ach)-chb-ck-d)} \\ \qquad \prod_{j=0}^{m-1} \frac{chb+ck+d+(ach)^{2j}(v_2(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2j+1}(v_0(1-ach)-chb-ck-d)} \\ \qquad \prod_{j=0}^{m-1} \frac{chb+ck+d+(ach)^{2j+1}(v_1(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2j+1}(v_2(1-ach)-chb-ck-d)}. \end{cases} \tag{3.11}$$

$$\begin{cases} z_{2(3m+1)} = z_0 \prod_{j=0}^{m} \frac{ahd+hb+k+(ach)^{2j}(w_1(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2j}(w_2(1-ach)-ahd-hb-k)} \\ \qquad \prod_{j=0}^{m-1} \frac{ahd+hb+k+(ach)^{2j+1}(w_0(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2j+1}(w_1(1-ach)-ahd-hb-k)} \\ \qquad \prod_{j=0}^{m-1} \frac{ahd+hb+k+(ach)^{2j+1}(w_2(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2j+2}(w_0(1-ach)-ahd-hb-k)}, \\ z_{2(3m)+1} = z_{-1} \prod_{j=0}^{m} \frac{ahd+hb+k+(ach)^{2j}(w_0(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2j}(w_1(1-ach)-ahd-hb-k)} \\ \qquad \prod_{j=0}^{m-1} \frac{ahd+hb+k+(ach)^{2j}(w_2(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2j+1}(w_0(1-ach)-ahd-hb-k)} \\ \qquad \prod_{j=0}^{m-1} (\frac{(ahd+hb+k+(ach)^{2j+1}(w_1(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2j+1}(w_2(1-ach)-ahb-hb-k)}. \end{cases} \tag{3.12}$$

iii) If $n = 3m + 2$, $m \in \mathbb{N}_0$, then, from (2.28), (2.29) and (2.30), we get

$$\begin{cases} x_{2(3m+2)} = x_0 \prod_{j=0}^{m} \frac{ack+ad+b+(ach)^{2j}(u_1(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2j}(u_2(1-ach)-ack-ad-b)} \\ \qquad \prod_{j=0}^{m} \frac{ack+ad+b+(ach)^{2j+1}(u_0(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2j+1}(u_1(1-ach)-ack-ad-b)} \\ \qquad \prod_{j=0}^{m-1} \frac{ack+ad+b+(ach)^{2j+1}(u_2(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2j+2}(u_0(1-ach)-ack-ad-b)}, \\ x_{2(3m)+3} = x_{-1} \prod_{j=0}^{m} \frac{ack+ad+b+(ach)^{2j}(u_0(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2j}(u_1(1-ach)-ack-ad-b)} \\ \qquad \prod_{j=0}^{m} \frac{ack+ad+b+(ach)^{2j}(u_2(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2j+1}(u_0(1-ach)-ack-ad-b)} \\ \qquad \prod_{j=0}^{m-1} \frac{ack+ad+b+(ach)^{2j+1}(u_1(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2j+1}(u_2(1-ach)-ack-ad-b)}. \end{cases} \tag{3.13}$$

$$\begin{cases} y_{2(3m+2)} = y_0 \prod_{j=0}^{m} \frac{chb+ck+d+(ach)^{2j}(v_1(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2j}(v_2(1-ach)-chb-ck-d)} \\ \qquad \prod_{j=0}^{m} \frac{(chb+ck+d+(ach)^{2j+1}(v_0(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2j+1}(v_1(1-ach)-chb-ckb-d)} \\ \qquad \prod_{j=0}^{m-1} \frac{chb+ck+d+(ach)^{2j+1}(v_2(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2j+2}(v_0(1-ach)-chb-ck-d)}, \\ y_{2(3m)+3} = y_{-1} \prod_{j=0}^{m} \frac{chb+ck+d+(ach)^{2j}(v_0(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2j}(v_1(1-ach)-chb-ck-d)} \\ \qquad \prod_{j=0}^{m} \frac{chb+ck+d+(ach)^{2j}(v_2(1-ach)-chb-ck-d)}{(chb+ck+d+(ach)^{2j+1}(v_0(1-ach)-chb-ck-d)} \\ \qquad \prod_{j=0}^{m-1} \frac{chb+ck+d+(ach)^{2j+1}(v_1(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2j+1}(v_2(1-ach)-chb-ck-d)}. \end{cases} \tag{3.14}$$

$$\left\{\begin{array}{l} z_{2(3m+2)} = z_0 \prod_{j=0}^{m} \frac{ahd+hb+k+(ach)^{2j}(w_1(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2j}(w_2(1-ach)-ahd-hb-k)} \\ \qquad \prod_{j=0}^{m} \frac{ahd+hb+k+(ach)^{2j+1}(w_0(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2j+1}(w_1(1-ach)-ahd-hb-k)} \\ \qquad \prod_{j=0}^{m-1} \frac{ahd+hb+k+(ach)^{2j+1}(w_2(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2j+2}(w_0(1-ach)-ahd-hb-k)}, \\ z_{2(3m)+3} = z_{-1} \prod_{j=0}^{m} \frac{ahd+hb+k+(ach)^{2j}(w_0(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2j}(w_1(1-ach)-ahd-hb-k)} \\ \qquad \prod_{j=0}^{m} \frac{ahd+hb+k+(ach)^{2j}(w_2(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2j+1}(w_0(1-ach)-ahd-hb-k)} \\ \qquad \prod_{j=0}^{m-1} \frac{ahd+hb+k+(ach)^{2j+1}(w_1(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2j+1}(w_2(1-ach)-ahd-hb-k)}. \end{array}\right. \tag{3.15}$$

2) Case $ach = 1$. We have

i) If $n = 3m$, $m \in \mathbb{N}_0$, then, from (2.31), (2.32) and (2.33), we obtain

$$\left\{\begin{array}{l} x_{2(3m)} = x_0 \prod_{j=0}^{m-1} \frac{u_1+(ack+ad+b)(2j)}{u_2+(ack+ad+b)(2j)} \prod_{j=0}^{m-1} \frac{u_0+(ack+ad+b)(2j+1)}{u_1+(ack+ad+b)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{u_2+(ack+ad+b)(2j+1)}{u_0+(ack+ad+b)(2j+2)}, \\ x_{2(3m)-1} = x_{-1} \prod_{j=0}^{m-1} \frac{u_0+(ack+ad+b)(2j)}{u_1+(ack+ad+b)(2j)} \prod_{j=0}^{m-1} \frac{u_2+(ack+ad+b)(2j)}{u_0+(ack+ad+b)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{u_1+(ack+ad+b)(2j+1)}{u_2+(ack+ad+b)(2j+1)}. \end{array}\right. \tag{3.16}$$

$$\left\{\begin{array}{l} y_{2(3m)} = y_0 \prod_{j=0}^{m-1} \frac{v_1+(chb+ck+d)(2j)}{v_2+(chb+ck+d)(2j)} \prod_{j=0}^{m-1} \frac{v_0+(chb+ck+d)(2j+1)}{v_1+(chb+ck+d)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{v_2+(chb+ck+d)(2j+1)}{v_0+(chb+ck+d)(2j+2)}, \\ y_{2(3m)-1} = y_{-1} \prod_{j=0}^{m-1} \frac{v_0+(chb+ck+d)(2j)}{v_1+(chb+ck+d)(2j)} \prod_{j=0}^{m-1} \frac{v_2+(chb+ck+d)(2j)}{v_0+(chb+ck+d)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{v_1+(chb+ck+d)(2j+1)}{v_2+(chb+ck+d)(2j+1)}. \end{array}\right. \tag{3.17}$$

$$\left\{\begin{array}{l} z_{2(3m)} = z_0 \prod_{j=0}^{m-1} \frac{w_1+(ahd+hb+k)(2j)}{w_2+(ahd+hb+k)(2j)} \prod_{j=0}^{m-1} \frac{w_0+(ahd+hb+k)(2j+1)}{w_1+(ahd+hb+k)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{w_2+(ahd+hb+k)(2j+1)}{w_0+(ahd+hb+k)(2j+2)}, \\ z_{2(3m)-1} = z_{-1} \prod_{j=0}^{m-1} \frac{w_0+(ahd+hb+k)(2j)}{w_1+(ahd+hb+k)(2j)} \prod_{j=0}^{m-1} \frac{w_2+(ahd+hb+k)(2j)}{w_0+(ahd+hb+k)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{w_1+(ahd+hb+k)(2j+1)}{w_2+(ahd+hb+k)(2j+1)}. \end{array}\right. \tag{3.18}$$

ii) If $n = 3m + 1$, $m \in \mathbb{N}_0$, then, from (2.34), (2.35) and (2.36), we have

$$\begin{cases} x_{2(3m+1)} = x_0 \prod_{j=0}^{m} \frac{u_1+(ack+ad+b)(2j)}{u_2+(ack+ad+b)(2j)} \prod_{j=0}^{m-1} \frac{u_0+(ack+ad+b)(2j+1)}{u_1+(ack+ad+b)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{u_2+(ack+ad+b)(2j+1)}{u_0+(ack+ad+b)(2j+2)}, \\ x_{2(3m)+1} = x_{-1} \prod_{j=0}^{m} \frac{u_0+(ack+ad+b)(2j)}{u_1+(ack+ad+b)(2j)} \prod_{j=0}^{m-1} \frac{u_2+(ack+ad+b)(2j)}{u_0+(ack+ad+b)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{u_1+(ack+ad+b)(2j+1)}{u_2+(ack+ad+b)(2j+1)}. \end{cases} \tag{3.19}$$

$$\begin{cases} y_{2(3m+1)} = y_0 \prod_{j=0}^{m} \frac{v_1+(chb+ck+d)(2j)}{v_2+(chb+ck+d)(2j)} \prod_{j=0}^{m-1} \frac{v_0+(chb+ck+d)(2j+1)}{v_1+(chb+ck+d)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{v_2+(chb+ck+d)(2j+1)}{v_0+(chb+ck+d)(2j+2)}, \\ y_{2(3m)+1} = y_{-1} \prod_{j=0}^{m} \frac{v_0+(chb+ck+d)(2j)}{v_1+(chb+ck+d)(2j)} \prod_{j=0}^{m-1} \frac{v_2+(chb+ck+d)(2j)}{v_0+(chb+ck+d)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{v_1+(chb+ck+d)(2j+1)}{v_2+(chb+ck+d)(2j+1)}. \end{cases} \tag{3.20}$$

$$\begin{cases} z_{2(3m+1)} = z_0 \prod_{j=0}^{m} \frac{w_1+(ahd+hb+k)(2j)}{w_2+(ahd+hb+k)(2j)} \prod_{j=0}^{m-1} \frac{w_0+(ahd+hb+k)(2j+1)}{w_1+(ahd+hb+k)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{w_2+(ahd+hb+k)(2j+1))}{w_0+(ahd+hb+k)(2j+2)}, \\ z_{2(3m)+1} = z_{-1} \prod_{j=0}^{m} \frac{w_0+(ahd+hb+k)(2j)}{w_1+(ahd+hb+k)(2j)} \prod_{j=0}^{m-1} \frac{w_2+(ahd+hb+k)(2j)}{w_0+(ahd+hb+k)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{w_1+(ahd+hb+k)(2j+1)}{w_2+(ahd+hb+k)(2j+1)}. \end{cases} \tag{3.21}$$

iii) If $n = 3m+2$, $m \in \mathbb{N}_0$, then, from (2.37), (2.38) and (2.39), we get

$$\begin{cases} x_{2(3m+2)} = x_0 \prod_{j=0}^{m} \frac{u_1+(ack+ad+b)(2j)}{u_2+(ack+ad+b)(2j)} \prod_{j=0}^{m} \frac{u_0+(ack+ad+b)(2j+1)}{u_1+(ack+ad+b)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{u_2+(ack+ad+b)(2j+1)}{u_0+(ack+ad+b)(2j+2)}, \\ x_{2(3m)+3} = x_{-1} \prod_{j=0}^{m} \frac{u_0+(ack+ad+b)(2j)}{u_1+(ack+ad+b)(2j)} \prod_{j=0}^{m} \frac{u_2+(ack+ad+b)(2j)}{u_0+(ack+ad+b)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{u_1+(ack+ad+b)(2j+1)}{u_2+(ack+ad+b)(2j+1)}. \end{cases} \tag{3.22}$$

$$\begin{cases} y_{2(3m+2)} = y_0 \prod_{j=0}^{m} \frac{v_1+(chb+ck+d)(2j)}{v_2+(chb+ck+d)(2j)} \prod_{j=0}^{m} \frac{v_0+(chb+ck+d)(2j+1)}{v_1+(chb+ck+d)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{v_2+(chb+ck+d)(2j+1)}{v_0+(chb+ck+d)(2j+2)}, \\ y_{2(3m)+3} = y_{-1} \prod_{j=0}^{m} \frac{v_0+(chb+ck+d)(2j)}{v_1+(chb+ck+d)(2j)} \prod_{j=0}^{m} \frac{v_2+(chb+ck+d)(2j)}{v_0+(chb+ck+d)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{v_1+(chb+ck+d)(2j+1)}{v_2+(chb+ck+d)(2j+1)}. \end{cases} \tag{3.23}$$

$$\begin{cases} z_{2(3m+2)} = z_0 \prod_{j=0}^{m} \frac{w_1+(ahd+hb+k)(2j)}{w_2+(ahd+hb+k)(2j)} \prod_{j=0}^{m} \frac{w_0+(ahd+hb+k)(2j+1)}{w_1+(ahd+hb+k)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{w_2+(ahd+hb+k)(2j+1)}{w_0+(ahd+hb+k)(2j+2)}, \\ z_{2(3m)+3} = z_{-1} \prod_{j=0}^{m} \frac{w_0+(ahd+hb+k)(2j)}{w_1+(ahd+hb+k)(2j)} \prod_{j=0}^{m} \frac{w_2+(ahd+hb+k)(2j)}{w_0+(ahd+hb+k)(2j+1)} \\ \qquad \prod_{j=0}^{m-1} \frac{w_1+(ahd+hb+k)(2j+1)}{w_2+(ahd+hb+k)(2j+1)}. \end{cases} \tag{3.24}$$

In the following theorems, we are interested in the long-term behavior of well-defined solutions of System (3.6).

Theorem 3.3. *Let $(x_n, y_n, z_n)_{n\geq -1}$ be a well-defined solution of System (3.6). Assume that $|ach| \neq 1$, then the following statements are true.*

a) If $ack + ad + b \neq 0$, $chb + ck + d \neq 0$, $ahd + hb + k \neq 0$ and $| ach |< 1$, then (x_n, y_n, z_n) converges to a periodic solution.

b) If $u_0 = u_1 = u_2 = \frac{ack+ad+b}{1-ach}$, then the sequences $(x_{6m+j})_{m\geq 0}$, for $j \in \{-1, 0, 1, 2, 3, 4\}$, are constant.

c) If $v_0 = v_1 = v_2 = \frac{chb+ck+d}{1-ach}$, then the sequences $(y_{6m+j})_{m\geq 0}$, for $j \in \{-1, 0, 1, 2, 3, 4\}$, are constant.

d) If $w_0 = w_1 = w_2 = \frac{ahd+hb+k}{1-ach}$, then the sequences $(z_{6m+j})_{m\geq 0}$, for $j \in \{-1, 0, 1, 2, 3, 4\}$, are constant.

e) If $ack + ad + b = 0$ and $| ach |< 1$, then $| x_n |\to \infty$, as $n \to \infty$.

f) If $chb + ck + d = 0$ and $| ach |< 1$, then $| y_n |\to \infty$, as $n \to \infty$.

g) If $ahd + hb + k = 0$ and $| ach |< 1$, then $| z_n |\to \infty$, as $n \to \infty$.

h) If $ack + ad + b = 0$ and $| ach |> 1$, then $x_n \to 0$, as $n \to \infty$.

j) If $chb + ck + d = 0$ and $| ach |> 1$, then $y_n \to 0$, as $n \to \infty$.

k) If $ahd + hb + k = 0$ and $| ach |> 1$, then $z_n \to 0$, as $n \to \infty$.

l) If $| ach |> 1$, $u_0 = u_1 = \frac{ack+ad+b}{1-ach} \neq u_2$, then $x_n \to 0$, as $n \to \infty$.

m) If $| ach |> 1$, $u_0 = u_2 = \frac{ack+ad+b}{1-ach} \neq u_1$, then $x_n \to 0$, as $n \to \infty$.

n) If $| ach |> 1$, $u_1 = u_2 = \frac{ack+ad+b}{1-ach} \neq u_0$, then $x_n \to 0$, as $n \to \infty$.

o) If $| ach |> 1$, $v_0 = v_1 = \frac{chb+ck+d}{1-ach} \neq v_2$, then $y_n \to 0$, as $n \to \infty$.

p) If $| ach |> 1$, $v_0 = v_2 = \frac{chb+ck+d}{1-ach} \neq v_1$, then $y_n \to 0$, as $n \to \infty$.

q) If $| ach |> 1$, $v_1 = v_2 = \frac{chb+ck+d}{1-ach} \neq v_0$, then $y_n \to 0$, as $n \to \infty$.

r) If $| ach |> 1$, $w_0 = w_1 = \frac{ahd+hb+k}{1-ach} \neq w_2$, then $z_n \to 0$, as $n \to \infty$.

s) If $| ach |> 1$, $w_0 = w_2 = \frac{ahd+hb+k}{1-ach} \neq w_1$, then $z_n \to 0$, as $n \to \infty$.

t) If $| ach |> 1$, $w_1 = w_2 = \frac{ahd+hb+k}{1-ach} \neq w_0$, then $z_n \to 0$, as $n \to \infty$.

u) If $| ach |> 1$ and $u_i \neq \frac{ack+ad+b}{1-ach}$, for every $i \in \{0, 1, 2\}$, then $x_n \to 0$, as $n \to \infty$.

v) If $| ach |> 1$ and $v_i \neq \frac{ack+ad+b}{1-ach}$, for every $i \in \{0, 1, 2\}$, then $y_n \to 0$, as $n \to \infty$.

y) If $\mid ach \mid > 1$ *and* $w_i \neq \frac{ack+ad+b}{1-ach}$*, for every* $i \in \{0,1,2\}$*, then* $z_n \to 0$*, as* $n \to \infty$*.*

Proof. Let

$$\begin{aligned}
p_m^{(0)} &= \frac{ack+ad+b+(ach)^{2m+1}(u_0(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2m+1}(u_1(1-ach)-ack-ad-b)},\\
p_m^{(1)} &= \frac{ack+ad+b+(ach)^{2m}(u_1(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2m}(u_2(1-ach)-ack-ad-b)},\\
p_m^{(2)} &= \frac{ack+ad+b+(ach)^{2m+1}(u_2(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2m+2}(u_0(1-ach)-ack-ad-b)},
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
q_m^{(0)} &= \frac{chb+ck+d+(ach)^{2m+1}(v_0(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2m+1}(v_1(1-ach)-chb-ck-d)},\\
q_m^{(1)} &= \frac{chb+ck+d+(ach)^{2m}(v_1(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2m}(v_2(1-ach)-chb-ck-d)},\\
q_m^{(2)} &= \frac{chb+ck+d+(ach)^{2m+1}(v_2(1-ach)-chb-ck-d)}{chb+ck+d+(ach)^{2m+2}(v_0(1-ach)-chb-ck-d)},
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
r_m^{(0)} &= \frac{ahd+hb+k+(ach)^{2m+1}(w_0(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2m+1}(w_1(1-ach)-ahd-hb-k)},\\
r_m^{(1)} &= \frac{ahd+hb+k+(ach)^{2m}(w_1(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2m}(w_2(1-ach)-ahd-hb-k)},\\
r_m^{(2)} &= \frac{ahd+hb+k+(ach)^{2m+1}(w_2(1-ach)-ahd-hb-k)}{ahd+hb+k+(ach)^{2m+2}(w_0(1-ach)-ahd-hb-k)}.
\end{aligned} \tag{3.27}$$

Using the result $(1+x)^{-1} = 1 - x + \mathcal{O}\left(x^2\right), \quad as\ x \to 0$, we have

$$\begin{aligned}
p_m^{(0)} &= \frac{1+(ach)^{2m+1}(u_0(1-ach)-ack-ad-b)(ack+ad+b)^{-1}}{1+(ach)^{2m+1}(u_1(1-ach)-ack-ad-b)(ack+ad+b)^{-1}},\\
&= 1+(u_0-u_1)(1-ach)(ack+ad+b)^{-1}(ach)^{2m+1}+\mathcal{O}\left((ach)^{2m}\right),
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
p_m^{(1)} &= \frac{1+(ach)^{2m}(u_1(1-ach)-ack-ad-b)(ack+ad+b)^{-1}}{1+(ach)^{2m}(u_2(1-ach)-ack-ad-b)(ack+ad+b)^{-1}},\\
&= 1+(u_1-u_2)(1-ach)(ack+ad+b)^{-1}(ach)^{2m}+\mathcal{O}\left((ach)^{2m}\right),
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
p_m^{(2)} &= \frac{1+(ach)^{2m+1}(u_2(1-ach)-ack-ad-b)(ack+ad+b)^{-1}}{1+(ach)^{2m+2}(u_0(1-ach)-ack-ad-b)(ack+ad+b)^{-1}},\\
&= 1+(u_2-achu_0-ack-ad-b)(1-ach)(ack+ad+b)^{-1}(ach)^{2m+1}+\mathcal{O}\left((ach)^{2m}\right),
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
q_m^{(0)} &= \frac{1+(ach)^{2m+1}(v_0(1-ach)-chb-ck-d)(chb+ck+d)^{-1}}{1+(ach)^{2m+1}(v_1(1-ach)-chb-ck-d)(chb+ck+d)^{-1}},\\
&= 1+(v_0-v_1)(1-ach)(chb+ck+d)^{-1}(ach)^{2m+1}+\mathcal{O}\left((ach)^{2m}\right),
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
q_m^{(1)} &= \frac{1+(ach)^{2m}(v_1(1-ach)-chb-ck-d)(chb+ck+d)^{-1}}{1+(ach)^{2m}(v_2(1-ach)-chb-ck-d)(chb+ck+d)^{-1}},\\
&= 1+(v_1-v_2)(1-ach)(chb+ck+d)^{-1}(ach)^{2m}+\mathcal{O}\left((ach)^{2m}\right),
\end{aligned} \tag{3.32}$$

$$q_m^{(2)} = \frac{1+(ach)^{2m+1}(v_2(1-ach)-chb-ck-d)(chb+ck+d)^{-1}}{1+(ach)^{2m+2}(v_0(1-ach)-chb-ck-d)(chb+ck+d)^{-1}},$$
$$= 1+(v_2-achv_0-chb-ck-d)(1-ach)(chb+ck+d)^{-1}(ach)^{2m+1}+\mathcal{O}\left((ach)^{2m}\right), \quad (3.33)$$
$$r_m^{(0)} = \frac{1+(ach)^{2m+1}(w_0(1-ach)-ahd-hb-k)(ahd+hb+k)^{-1}}{1+(ach)^{2m+1}(w_1(1-ach)-ahd-hb-k)(ahd+hb+k)^{-1}},$$
$$= 1+(w_0-w_1)(1-ach)(ahd+hb+k)^{-1}(ach)^{2m+1}+\mathcal{O}\left((ach)^{2m}\right), \quad (3.34)$$
$$r_m^{(1)} = \frac{1+(ach)^{2m}(w_1(1-ach)-ahd-hb-k)(ahd+hb+k)^{-1}}{1+(ach)^{2m}(w_2(1-ach)-ahd-hb-k)(ahd+hb+k)^{-1}},$$
$$= 1+(w_1-w_2)(1-ach)(ahd+hb+k)^{-1}(ach)^{2m}+\mathcal{O}\left((ach)^{2m}\right), \quad (3.35)$$
$$r_m^{(2)} = \frac{1+(ach)^{2m+1}(w_2(1-ach)-ahd-hb-k)(ahd+hb+k)^{-1}}{1+(ach)^{2m+2}(w_0(1-ach)-ahd-hb-k)(ahd+hb+k)^{-1}},$$
$$= 1+(w_2-achw_0-ahd-hb-k)(1-ach)(ahd+hb+k)^{-1}(ach)^{2m+1}+\mathcal{O}\left((ach)^{2m}\right), \quad (3.36)$$

for sufficiently large m.

(a) From (3.28)-(3.36), (3.7)-(3.15) and using condition $\mid ach \mid < 1$ and a well-know criterion for convergence of products, the result easily follows.

(b) By employing the condition $u_0 = u_1 = u_2 = \frac{ack+ad+b}{1-ach}$ in (3.7), (3.10) and (3.13), the result can be easily obtained.

(c) By employing the condition $v_0 = v_1 = v_2 = \frac{chb+ck+d}{1-ach}$ in (3.8), (3.11) and (3.14), the result can be easily seen.

(d) By employing the condition $w_0 = w_1 = w_2 = \frac{ahd+hb+k}{1-ach}$ in (3.9), (3.12) and (3.15), the result easily follows.

(e)-(k) In here we will only prove (e) and (h) since the proofs of the others can be done in a similar way. By using the condition $ack+ad+b=0$ in (3.25), we have

$$p_m^{(0)}p_m^{(1)}p_m^{(2)} = \frac{1}{ach}, \quad (3.37)$$

from which along with (3.7), (3.10), (3.13) and using the assumptions $|ach| < 1$ in the case (e) and $|ach| > 1$ in the case (h), the results immediately follow.

(l) By using the condition $u_0 = u_1 = \frac{ack+ad+b}{1-ach} \neq u_2$ in (3.25), we obtain

$$p_m^{(0)}p_m^{(1)}p_m^{(2)} = \frac{ack+ad+b+(ach)^{2m+1}(u_2(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2m}(u_2(1-ach)-ack-ad-b)}. \quad (3.38)$$

Letting $m \to \infty$ in (3.38) and using the assumption $|ach| > 1$, we obtain $|p_m^{(0)}p_m^{(1)}p_m^{(2)}| \to \infty$, from which along with (3.7), (3.10) and (3.13), the result can be easily seen.

(m) By using the condition $u_0 = u_2 = \frac{ack+ad+b}{1-ach} \neq u_1$ in (3.25), we get

$$p_m^{(0)}p_m^{(1)}p_m^{(2)} = \frac{ack+ad+b+(ach)^{2m}(u_1(1-ach)-ack-ad-b)}{ack+ad+b+(ach)^{2m+1}(u_1(1-ach)-ack-ad-b)}. \quad (3.39)$$

Letting $m \to \infty$ in (3.39) and using the assumption $|ach| > 1$, we have $p_m^{(0)}p_m^{(1)}p_m^{(2)} \to 0$, from which along with (3.7), (3.10) and (3.13), the result immediately follows.

(n) By using the condition $u_1 = u_2 = \frac{ack+ad+b}{1-ach} \neq u_0$ in (3.25), we get

$$p_m^{(0)}p_m^{(1)}p_m^{(2)} = \frac{ack + ad + b + (ach)^{2m}(u_1(1 - ach) - ack - ad - b)}{ack + ad + b + (ach)^{2m+1}(u_1(1 - ach) - ack - ad - b)}. \tag{3.40}$$

Letting $m \to \infty$ in (3.40) and using the assumption $|ach| > 1$, we obtain $p_m^{(0)}p_m^{(1)}p_m^{(2)} \to 0$, from which along with (3.7), (3.10) and (3.13), the result can be easily seen.

(o)-(t) Since the proofs of (o)-(q) and (r)-(t) are similar to (l)-(n), we will omit them in here.

(u)-(y) From (3.25)-(3.27), we have

$$\lim_{m\to\infty} p_m^{(0)}p_m^{(1)}p_m^{(2)} = \lim_{m\to\infty} q_m^{(0)}q_m^{(1)}q_m^{(2)} = \lim_{m\to\infty} r_m^{(0)}r_m^{(1)}r_m^{(2)} = \frac{1}{ach} \tag{3.41}$$

from which along with (3.7)-(3.15) and using the assumption $|ach| > 1$, the results immediately follow.

□

Theorem 3.4. *Assume that $(x_n, y_n, z_n)_{n\geq -1}$ is a well-defined solution of System (3.6). Assume that $ach = 1$, then the following statements are true.*

a) If $ack + ad + b = 0$, then the sequence $(x_n)_{n\geq -1}$ is six-periodic.

b) If $chb + ck + d = 0$, then the sequence $(y_n)_{n\geq -1}$ is six-periodic.

c) If $ahb + hb + k = 0$, then the sequence $(z_n)_{n\geq -1}$ is six-periodic.

d) If $ack + ad + b \neq 0$, then $x_m \to 0$, as $m \to \infty$.

e) If $chb + ck + d \neq 0$, then $y_m \to 0$, as $m \to \infty$.

f) If $ahb + hb + k \neq 0$, then $z_m \to 0$, as $m \to \infty$.

Proof.

Let

$$\begin{aligned}
R_m^{(0)} &= \tfrac{u_0+(ack+ad+b)(2m+1)}{u_1+(ack+ad+b)(2m+1)}, & R_m^{(1)} &= \tfrac{u_1+(ack+ad+b)(2m)}{u_2+(ack+ad+b)(2m)}, & R_m^{(2)} &= \tfrac{u_2+(ack+ad+b)(2m+1)}{u_0+(ack+ad+b)(2m+2)},\\
S_m^{(0)} &= \tfrac{v_0+(chb+ck+d)(2m+1)}{v_1+(chb+ck+d)(2m+1)}, & S_m^{(1)} &= \tfrac{v_1+(chb+ck+d)(2m)}{v_2+(chb+ck+d)(2m)}, & S_m^{(2)} &= \tfrac{v_2+(chb+ck+d)(2m+1)}{v_0+(chb+ck+d)(2m+2)},\\
T_m^{(0)} &= \tfrac{w_0+(ahd+hb+k)(2m+1)}{w_1+(ahd+hb+k)(2m+1)}, & T_m^{(1)} &= \tfrac{w_1+(ahd+hb+k)(2m)}{w_2+(ahd+hb+k)(2m)}, & T_m^{(2)} &= \tfrac{w_2+(ahd+hb+k)(2m+1)}{w_0+(ahd+hb+k)(2m+2)},
\end{aligned} \tag{3.42}$$

for $m \in \mathbb{N}_0$.

(a)-(d): From (3.16)-(3.24), these statements easily follow.

(e)-(f): Employing the Taylor expansion for $(1 + x)^{-1}$, we have for sufficiently large m

$$\begin{aligned}
R_m^{(0)}R_m^{(1)}R_m^{(2)} &= \left(\frac{u_0 + (ack + ad + b)(2m + 1)}{u_1 + (ack + ad + b)(2m + 1)}\right)\left(\frac{u_1 + (ack + ad + b)(2m)}{u_2 + (ack + ad + b)(2m)}\right)\left(\frac{u_2 + (ack + ad + b)(2m + 1)}{u_0 + (ack + ad + b)(2m + 2)}\right)\\
&= \left(1 + \frac{u_0 - u_1}{(ack + ad + b)2m} + \mathcal{O}(\frac{1}{m^2})\right)\left(1 + \frac{u_1 - u_2}{(ack + ad + b)2m} + \mathcal{O}(\frac{1}{m^2})\right)\\
&\times \left(1 + \frac{u_2 - u_0 - ack - ad - b}{(ack + ad + b)2m} + \mathcal{O}(\frac{1}{m^2})\right)\\
&= 1 - \frac{1}{2m} + \mathcal{O}(\frac{1}{m^2}),
\end{aligned} \tag{3.43}$$

$$\begin{aligned} S_m^{(0)}S_m^{(1)}S_m^{(2)} &= \left(\frac{v_0+(chb+ck+d)(2m+1)}{v_1+(chb+ck+d)(2m+1)}\right)\left(\frac{v_1+(chb+ck+d)(2m)}{v_2+(chb+ck+d)(2m)}\right)\left(\frac{v_2+(chb+ck+d)(2m+1)}{v_0+(chb+ck+d)(2m+2)}\right)\\ &= \left(1+\frac{v_0-v_1}{(chb+ck+d)2m}+\mathcal{O}(\frac{1}{m^2})\right)\left(1+\frac{v_1-v_2}{(chb+ck+d)2m}+\mathcal{O}(\frac{1}{m^2})\right)\\ &\times\left(1+\frac{v_2-v_0-chb-ck-d}{(chb+ck+d)2m}+\mathcal{O}(\frac{1}{m^2})\right)\\ &= 1-\frac{1}{2m}+\mathcal{O}(\frac{1}{m^2}), \end{aligned} \tag{3.44}$$

$$\begin{aligned} T_m^{(0)}T_m^{(1)}T_m^{(2)} &= \left(\frac{w_0+(ahd+hb+k)(2m+1)}{w_1+(ahd+hb+k)(2m+1)}\right)\left(\frac{w_1+(ahd+hb+k)(2m)}{w_2+(ahd+hb+k)(2m)}\right)\left(\frac{w_2+(ahd+hb+k)(2m+1)}{w_0+(ahd+hb+k)(2m+2)}\right)\\ &= \left(1+\frac{w_0-w_1}{(ahd+hb+k)2m}+\mathcal{O}(\frac{1}{m^2})\right)\left(1+\frac{w_1-w_2}{(ahd+hb+k)2m}+\mathcal{O}(\frac{1}{m^2})\right)\\ &\times\left(1+\frac{w_2-w_0-ahd-hb-k}{(ahd+hb+k)2m}+\mathcal{O}(\frac{1}{m^2})\right)\\ &= 1-\frac{1}{2m}+\mathcal{O}(\frac{1}{m^2}). \end{aligned} \tag{3.45}$$

From (3.43)-(3.45) and (3.16)-(3.23), we can write the following equality

$$\prod_{j=1}^{m}\left(1-\frac{1}{2m}+\mathcal{O}(\frac{1}{m^2})\right)=e^{\sum_{j=1}^{m}\ln\left(1-\frac{1}{2j}+\mathcal{O}(\frac{1}{j^2})\right)}=e^{-\frac{1}{2}\sum_{j=1}^{m}\left(\frac{1}{j}+\mathcal{O}(\frac{1}{j^2})\right)}, \tag{3.46}$$

from which along with $\lim_{m\to\infty}\sum_{j=1}^{m}\frac{1}{j}=\infty$ and $\sum_{j=1}^{\infty}|\mathcal{O}(\frac{1}{j^2}|<\infty$, these statements can be easily seen.

□

Theorem 3.5. *Let $(x_n, y_n, z_n)_{n\geq -1}$ be a well-defined solution of System (3.6). Assume that $ach=-1$, then the following statements are true.*

a) If $|L_1|<1$, then $|x_{6m+2j}|\to\infty$ and $x_{6m+2j-1}\to 0$, for $j\in\{0,1,2\}$, as $m\to\infty$.

b) If $|L_1|>1$, then $x_{6m+2j}\to 0$ and $|x_{6m+2j-1}|\to\infty$, for $j\in\{0,1,2\}$, as $m\to\infty$.

c) If $L_1=1$, then the sequence $(x_n)_{n\geq -1}$ is six-periodic.

d) If $L_1=-1$, then the sequence $(x_n)_{n\geq -1}$ is twelve-periodic.

e) If $|L_2|<1$, then $|y_{6m+2j}|\to\infty$ and $y_{6m+2j-1}\to 0$, for $j\in\{0,1,2\}$, as $m\to\infty$.

f) If $|L_2|>1$, then $y_{6m+2j}\to 0$ and $|y_{6m+2j-1}|\to\infty$, for $j\in\{0,1,2\}$, as $m\to\infty$.

g) If $L_2=1$, then the sequence $(y_n)_{n\geq -1}$ is six-periodic.

h) If $L_2=-1$, then the sequence $(y_n)_{n\geq -1}$ is twelve-periodic.

i) If $|L_3|<1$, then $|z_{6m+2j}|\to\infty$ and $z_{6m+2j-1}\to 0$, for $j\in\{0,1,2\}$, as $m\to\infty$.

j) If $|L_3|>1$, then $z_{6m+2j}\to 0$ and $|z_{6m+2j-1}|\to\infty$, for $j\in\{0,1,2\}$, as $m\to\infty$.

k) If $L_3=1$, then the sequence $(z_n)_{n\geq -1}$ is six-periodic.

l) If $L_3 = -1$, then the sequence $(z_n)_{n\geq -1}$ is twelve-periodic.

where

$$L_1 := \frac{u_0u_2\left(ack+ad+b-u_1\right)}{u_1\left(ack+ad+b-u_0\right)\left(ack+ad+b-u_2\right)},$$
$$L_2 := \frac{v_0v_2\left(chb+ck+d-v_1\right)}{v_1\left(chb+ck+d-v_0\right)\left(chb+ck+d-v_2\right)},$$
$$L_3 := \frac{w_0w_2\left(ahb+hb+k-w_1\right)}{w_1\left(ahb+hb+k-w_0\right)\left(ahb+hb+k-w_2\right)}.$$

Proof. By taking $ach = -1$ in (3.7)-(3.15), we obtain

$$\begin{aligned} &x_{6m-1} = x_{-1}L_1^m, && x_{6m} = x_0\tfrac{1}{L_1^m}, && x_{6m+1} = x_{-1}\tfrac{u_0}{u_1}L_1^m,\\ &x_{6m+2} = x_0\tfrac{u_1}{u_2}\tfrac{1}{L_1^m}, && x_{6m+3} = x_{-1}\tfrac{u_0u_2}{u_1(ack+ad+b-u_0)}L_1^m, && x_{6m+4} = x_0\tfrac{u_1(ack+ad+b-u_0)}{u_2(ack+ad+b-u_1)}\tfrac{1}{L_1^m}, \end{aligned} \tag{3.47}$$

$$\begin{aligned} &y_{6m-1} = y_{-1}L_2^m, && y_{6m} = y_0\tfrac{1}{L_2^m}, && y_{6m+1} = y_{-1}\tfrac{v_0}{v_1}L_2^m,\\ &y_{6m+2} = y_0\tfrac{v_1}{v_2}\tfrac{1}{L_2^m}, && y_{6m+3} = y_{-1}\tfrac{v_0v_2}{v_1(chb+ck+d-v_0)}L_2^m, && y_{6m+4} = y_0\tfrac{v_1(chb+ck+d-v_0)}{v_2(chb+ck+d-v_1)}\tfrac{1}{L_2^m}, \end{aligned} \tag{3.48}$$

$$\begin{aligned} &z_{6m-1} = z_{-1}L_3^m, && z_{6m} = z_0\tfrac{1}{L_3^m}, && z_{6m+1} = z_{-1}\tfrac{w_0}{w_1}L_3^m,\\ &z_{6m+2} = z_0\tfrac{w_1}{w_2}\tfrac{1}{L_3^m}, && z_{6m+3} = z_{-1}\tfrac{w_0w_2}{w_1(ahb+hb+k-w_0)}L_3^m, && z_{6m+4} = z_0\tfrac{w_1(ahb+hb+k-w_0)}{w_2(ahb+hb+k-w_1)}\tfrac{1}{L_3^m}, \end{aligned} \tag{3.49}$$

for $m \in \mathbb{N}_0$. From (3.47)-(3.49), all of the statements can be easliy seen. □

Lemma 3.6.

If $ach \neq 1$, $ack + ad + b \neq 0$, $chb + ck + d \neq 0$, $ahd + hb + k \neq 0$. Then system (3.6) has two-periodic solutions.

Proof.

The equilibrium solution to system (2.2) is

$$\begin{cases} u_n = u^* = \frac{ack+ad+b}{1-ach},\\ v_n = v^* = \frac{chb+ck+d}{1-ach}, \quad n \in \mathbb{N}_0.\\ w_n = w^* = \frac{ahd+hb+k}{1-ach}. \end{cases} \tag{3.50}$$

From (2.1), (3.50) and $p = q = r = 1$, it follows that

$$x_{n+1} = \frac{1-ach}{(ack+ad+b)x_n} = x_{n-1}, \quad n \in \mathbb{N}_0, \tag{3.51}$$

$$y_{n+1} = \frac{1-ach}{(chb+ck+d)y_n} = y_{n-1}, \quad n \in \mathbb{N}_0, \tag{3.52}$$

and

$$z_{n+1} = \frac{1-ach}{(ahd+hb+k)z_n} = z_{n-1}, \quad n \in \mathbb{N}_0, \tag{3.53}$$

which is desired. □

Acknowledgements:

The authors, thanks the two referees for their comments and suggestions.

References

[1] Y. Akrour, M. Kara, N. Touafek, Y. Yazlik, *Solutions formulas for some general systems of difference equations*, Miskolc Math. Notes **22(2)** (2022), 529–555. 1

[2] Y. Akrour, M. Kara, N. Touafek, Y. Yazlik, *On a three dimensional higher order system of difference equations*, Dyn. Contin. Discrete Impuls. Syst., Ser. B, Appl. Algorithms **29(2)** (2022), 77–108. 1

[3] E.O. Alzahrani, M.M. El-Dessoky, E.M. Elsayed, Y. Kuang, *Solutions and properties of some degenerate systems of difference equations*, J. Comput. Anal. Appl. **18(2)** (2015), 321–333. 1

[4] A. Çete, *The solutions to three-dimensional symmetric system of difference equations with second-order and the asymptotic behavior of their solutions*, Master Thesis, Graduate School of Naturel and Applied Sciences, Nevsehir Haci Bektas Veli University, (2023). 3.2

[5] E.M. Elsayed, *Solution for systems of difference equations of rational form of order two*, Comp. Appl. Math. **33(3)** (2014),751–765.

[6] E.M. Elsayed, T.F. Ibrahim, *Periodicity and solutions for some systems of nonlinear rational difference equations*, Hacet. J. Math. Stat. **44(6)** (2015), 1361–1390.

[7] M. Kara, Y. Yazlik, *On a solvable system of rational difference equations of higher order*, Turk. J. Math. **46(2)** (2022), 587–611.

[8] M. Kara, Y. Yazlik, *Solvable three-dimensional system of higher-order nonlinear difference equations*, Filomat **36(10)** (2022), 3449–3469.

[9] M. Kara, Y. Yazlik, *On the solutions of three-dimensional system of difference equations via recursive relations of order two and applications*, J. Appl. Anal. Comput. **12(2)** (2022), 736–753.

[10] M. Kara, Y. Yazlik, *Solvability of a nonlinear three-dimensional system of difference equations with constant coefficients*, Math. Slovaca **71(5)** (2021), 1133–1148.

[11] N. Haddad, N. Touafek, J.F.T. Rabago, *Well-defined solutions of a system of difference equations*, J. Appl. Math. Comput. **56(1-2)** (2018), 439–458.

[12] N. Haddad, N. Touafek, J.F.T. Rabago, *Solution form of a higher-order system of difference equations and dynamical behavior of its special case*, Math. Methods Appl. Sci. **40(10)** (2017), 3599–3607. 1

[13] Y. Halim, A. Khelifa, M. Berkal, A. Bouchair, *On a solvable system of p difference equations of higher order*, Period. Math. Hung. **85** (2022), 109–127.

[14] H. Hamioud, I. Dekkar, N. Touafek, N., *Solvability of a third order system of nonlinear difference equations via a generalized Fibonacci sequence*, Miskolc Math. Notes (2023), to appear. 1

[15] T.F. Ibrahim, N. Touafek, *On a third order rational difference equation with variable coeffitients*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms **20(2)** (2013), 251–264, (2013).

[16] S. Stevic, B. Iricanin, Z. Smarda, *On a close to symmetric system of difference equations of second order*, Adv. Difference Equ. **Article ID 264** (2015), 17 pages. 1, 1, 3.2

[17] N. Touafek, E.M. Elsayed, *On a third order rational systems of difference equations*, An. Stiint. Univ. Al. I. Cuza IaSi, Ser. Noua, Mat. **61(2)** (2015), 367–380. 1

[18] N. Touafek, *On some fractional systems of difference equations*, Iran. J Math Sci Inform. **9(1)** (2014), 73–86.