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Translation Factorable (TF) Surfaces in Three-Dimensional Isotropic Space: Unveiling Future Prospects for Control Systems

Brahim Medjahdi[a], Abdelakder Belhenniche[b,*], Hanifi Zoubir[c]

[a]*Ecole Normale Supérieure D'Oran, Ahmed Ammour B.P 1063 Saim Mohamed Oran 31003, Algérie.*

[b]*SYSTEC - Research Center for Systems and Technologies, FEUP - Faculty of Engineering, Porto University, Rua Dr. Roberto Frias sn, 4200-465 Porto, Portugal.*

[c]*École Nationale Polytechnique d'Oran Maurice Audin, B.P 1523 El M'naouer Oran 31000, Algeria.*

Abstract

This article introduces the concept of translation-factorable surfaces in the isotropic space $\mathbb{I}^3$ and presents classification theorems for these surfaces based on their isotropic mean and isotropic Gaussian curvatures, considering both zero and nonzero values. Furthermore, an additional investigation is conducted to classify translation-factorable (TF) surfaces in $\mathbb{I}^3$ under the condition $H^2 = K$.

Keywords: Isotropic space, translation-factorable surface, mean curvatures, Gaussian curvatures.
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1. Introduction

An important problem in classical differential geometry is finding the mean curvature H and Gaussian curvature K of surfaces in three-dimensional Euclidean space $\mathbb{E}^3$ and other spaces.
In particular, for the immersed graph z into $\mathbb{E}^3$, such a problem is reduced to solve the Monge-Ampère equation given by ([18], [26]). Specifically, when considering an immersed graph z in $\mathbb{E}^3$, this problem can be reduced to solving the Monge-Ampère equation, as shown in references [18] and [26]:

$$\det\left(\frac{\partial z}{\partial x \partial y}\right) = K(1 + |\nabla z|^2)^2, \tag{1.1}$$

*Corresponding author
Email addresses: `medjahdib41@gmail.com` (Brahim Medjahdi), `belhenniche@fe.up.pt` (Abdelakder Belhenniche), `zoubirhanifi@yahoo.fr` (Hanifi Zoubir)

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and the equation of mean curvature type in divergence form

$$div\left(\frac{\nabla z}{\sqrt{1+|\nabla z|^2}}\right) = H, \tag{1.2}$$

where ∇ denotes the gradient of $\mathbb{E}^2$ ([11], [22], [21]).

One class of surfaces in $\mathbb{E}^3$ that is of particular interest is that of translation surfaces, which can be locally parametrized as:

$$r(s,t) = (s,t,u(s)+v(t)), \tag{1.3}$$

where u and v are smooth functions. Geometers have investigated these surfaces from various perspectives.

H. Liu gives a classification of translation surfaces with constant mean curvature or constant Gaussian curvature in both $\mathbb{E}^3$ and three-dimensional Minkowski space $\mathbb{E}^3_1$ in [10]. Similarly, in [23] Z.M. Sipus classifies translation surfaces of constant curvature created by two planar curves in isotropic space $\mathbb{I}^3$. In addition, in[27] , L. Verstraelen, J. Wahare, and Y. Yaprak examine minimal translation surfaces in n-dimensional Euclidean space. Furthermore, in [20], K. Seo provides a categorization of translation hypersurfaces with constant mean curvature or constant Kronecker curvature in space forms.

On the other hand, a surface in $\mathbb{E}^3$ that is the graph of the function $z(s,t) = u(s).v(t)$ is said to be factorable or homothetical surface. These surfaces have been extensively studied in the Euclidian space and semi-Euclidean space. In [2], M.E Aydin classifies such surfaces with constant Gaussian and mean curvature in $\mathbb{I}^3$ and he provide a non- existence result related with the surfaces satisfying $H/K = const$. Y.Yu and H.Liu [28] study the factorable minimal surfaces in $\mathbb{E}^3$and $\mathbb{E}^3_1$ and give soms classifacations theorems.

Next, we introduce an extension of surfaces in $\mathbb{I}^3$ based on the definitions mentioned earlier. We refer to these surfaces as translation-factorable (TF) surfaces.

Definition 1.1. A surface M^2 in the three dimensinal space $\mathbb{I}^3$ is T.F - surface if it can be parametrized locally either by patch:

$$r(s,t) = (s,t,A(u(s)+v(t))+Bu(s)v(t)) \tag{1.4}$$

or

$$r(s,t) = (s,A(u(s)+v(t))+Bu(s)v(t),t), \tag{1.5}$$

where u and v are smooth functions and A, B non zero reals.

In the present study, we consider a translation-factorable (T.F)- surfaces in isotropic space $\mathbb{I}^3$, which can be parametrized locally as

$$r(s,t) = (s,t,u(s)+v(t)+u(s)v(t), \tag{1.6}$$

where u and v are smooth functions. We characterize such surfaces in terms of their isotropic mean and isotropic Gaussian functions.

Medjahdi et al in [16] classify the Graph Translation Surface in the Lorentz-Heisenberg 3-space with constant curvatures.

Recently, in [8] S.A. Difi and A. Hakem carried out a classification of translation-factorable (TF) surfaces in both three-dimensional Euclidean space and Lorentz-Minkowski space. This classification was performed under the condition that the coordinate functions of the surface, denoted by r_i, satisfy the equation:

$$\triangle r_i = \lambda_i r_i. \tag{1.7}$$

where $\triangle$ denotes the Laplace operator and the coordinate functions $\lambda_i \in \mathbb{R}$ and r_i describe the surface.

2. Preliminaries

The concept of differential geometry in relation to isotopic spaces was first introduced by K. Strubacker [24], H. Sachs [19], and several other researchers [15],[23].

$\mathbb{I}^3$ denotes an isotropic space, and its properties can be observed in [19].

The term used to describe such affine transformations is isotropic congruence transformations, commonly referred to as i-motions.

On the contrary, the isotropic distance, referred to as the i-distance, between two points $A\left(x_1, x_2, x_3\right)$ and $B\left(y_1, y_2, y_3\right)$ is defined as follows

$$\|A-B\|_i=\sqrt{\left(y_1-x_1\right)^2+\left(y_2-x_2\right)^2}, \tag{2.1}$$

the i-motions is degenerate along the lines in z-direction and such lines are called isotropic lines.

Let $v_1=\left(x_1, x_2, x_3\right)$ and $v_2=\left(y_1, y_2, y_3\right)$ be vectors in $\mathbb{I}^3$. The isotropic inner product of v_1 and v_2 is defined by

$$\langle v_1, v_2\rangle_i=\begin{cases} x_3 y_3 & \text{if} \quad x_i=y_i=0 \\ x_1 y_1+x_2 y_2 & \text{if} \quad \text{otherwise}\end{cases} \tag{2.2}$$

An isotropic vector in $\mathbb{I}^3$ is classified as a vector of the form $v=(0,0,x)$, otherwise, it is considered non-isotropic. A surface M^2 is immersed in $\mathbb{I}^3$ and is referred to as admissible if it lacks isotropic tangent planes. For our purposes, we only consider admissible regular surfaces. Let M^2 be a regular admissible graph surface in $\mathbb{I}^3$, that is parameterized locally by

$$r\left(u, v\right)=\left(u, v, z\left(u, v\right)\right). \tag{2.3}$$

The components E, F and G of the first fundamental form I for the surface M^2 can be determined by utilizing the metric derived from $\mathbb{I}^3$. This gives us

$$E=\left\langle r_u, r_u\right\rangle_i, F=\left\langle r_u, r_v\right\rangle_i, G=\left\langle r_v, r_v\right\rangle_i. \tag{2.4}$$

The surface M^2 exhibits complete isotropy in its unit normal vector. Additionally, the components of the second fundamental form II are

$$L=\frac{\det(r_{uu}, r_u, r_v)}{\sqrt{EG-F^2}}, M=\frac{\det(r_{uv}, r_u, r_v)}{\sqrt{EG-F^2}}, N=\frac{\det(r_{vv}, r_u, r_v)}{\sqrt{EG-F^2}}. \tag{2.5}$$

The mean curvature K and the Gaussian curvature H in the isotropic context are in the following manner

$$K=\frac{LN-M^2}{EG-F^2}, \qquad 2H=\frac{EN-2FM+GL}{EG-F^2}. \tag{2.6}$$

3. Translation-factorable (TF) surfaces in isotropic space $\mathbb{I}^3$

Let M^2 a TF- surface in $\mathbb{I}^3$ parametrized locally by a patch

$$r(s,t)=(s,t,z(s,t)=u(s)+v(t)+u(s)v(t)). \tag{3.1}$$

where u and v smooth functions.

Then the coefficients of the first fundamental form of M^2 are calculated by induced metric from $\mathbb{I}^3$as follow

$$E=1,\ F=0,\ G=1, \tag{3.2}$$

and the coefficients of the second fundamental form of M^2 are

$$L = (1+v)u'',\ M = u'v',\ N = (1+u)v''. \tag{3.3}$$

Where $u' = \frac{du}{ds}$ and $v' = \frac{dv}{dt}$.

Therefore, it follows from (2.6), the definition of the isotropic mean curvature H and the isotropic Gaussian curvature K of M^2 is as follows:

$$2H = (1+v)u'' + (1+u)v'', \tag{3.4}$$

and

$$K = (1+u)(1+v)u''v'' - u'^2v'^2. \tag{3.5}$$

The surface M^2 is considered isotropic minimal (or isotropic flat) when the values of H (or K) are zero. Similarly, it is referred to as having constant isotropic mean (or isotropic Gaussian) curvature when H (or K) remains constant throughout the whole surface.

4. TF-surfaces with zero curvatures in $\mathbb{I}^3$

In this section, our objective is to provide a description of the TF-surfaces where H equals zero and K equals zero.

4.1. Isotropic minimal TF-surfaces in $\mathbb{I}^3$

Let M^2 be a TF- surface in $\mathbb{I}^3$ representid as the graph of $z(s,t) = u(s) + v(t) + u(s)v(t)$.
If M^2 is isotropic minimal then from (3.4), we can derive the following equation

$$(1+v)u'' + (1+u)v'' = 0. \tag{4.1}$$

Consequently, based on (4.1), we can conclude that

$$\frac{u''}{1+u} = -\frac{v''}{1+v}.$$

As both sides of this equation contain a function dependent on s and another function dependent on t, there exists a real number $\lambda_1 \in \mathbb{R}$ for which:

$$\frac{u''}{1+u} = \lambda_1 = -\frac{v''}{1+v}. \tag{4.2}$$

If $\lambda_1 = 0$ in (4.2), then u an v becomes linear functions, denotd as $u(s) = a_1 s + a_2$ and $v(t) = a_3 t + a_4$, where a_1, a_2, a_3 and a_4 are constants.

Now let us assume that $\lambda_1 \neq 0$, that is u and v are nonlinear functions. Then the equation in (4.2) can be rewritten as

$$u'' - \lambda_1(1+u) = 0, \tag{4.3}$$

and

$$v'' + \lambda_1(1+v) = 0. \tag{4.4}$$

If $\lambda_1 > 0$, by solving (4.3) and (4.4), we derive

$$u(s) = -1 + b_1 e^{\sqrt{\lambda_1}s} + b_2 e^{-\sqrt{\lambda_1}s}, \tag{4.5}$$

and

$$v(t) = -1 + b_3 cos(\sqrt{\lambda_1}t) + b_4 \sin(\sqrt{\lambda_1}t),$$

where b_1, b_2, b_3 and b_4 are constants.

If $\lambda_1 < 0$, we obtain

$$u(s) = -1 + c_1 cos(\sqrt{-\lambda_1}s) + c_2 \sin(\sqrt{-\lambda_1}s), \tag{4.6}$$

and

$$v(t) = -1 + c_3 e^{\sqrt{-\lambda_1}t} + c_4 e^{-\sqrt{-\lambda_1}t}, \tag{4.7}$$

where c_1, c_2 , c_3 and c_4 are constants.

Therefore we have proved the following :

Theorem 4.1. *Let M^2 be a isotropic minimal TF-surface which is the graph of $z(s,t) = u(s)+v(t)+u(s)v(t)$ in $\mathbb{I}^3$. Then we have either*

1. $u(s) = a_1 s + a_2$ and $v(t) = a_3 t + a_4$, $\quad a_i \in \mathbb{R}$.
2. $u(s) = -1 + b_1 e^{\sqrt{\lambda_1}s} + b_2 e^{-\sqrt{\lambda_1}s}$ and $v(t) = -1 + b_3 cos(\sqrt{\lambda_1}t) + b_4 \sin(\sqrt{\lambda_1}t)$, $b_i \in \mathbb{R}$ and $\lambda_1 > 0$.
3. $u(s) = -1 + c_1 cos(\sqrt{-\lambda_1}s) + c_2 \sin(\sqrt{-\lambda_1}s)$ and $v(t) = -1 + c_3 e^{\sqrt{-\lambda_1}t} + c_4 e^{-\sqrt{-\lambda_1}t}$, $c_i \in \mathbb{R}$ and $\lambda_1 < 0$.

Exercise 4.2. Let us consider TF-surfaces with constant isotropic mean curvatures H_0 given by

1. $z(s,t) = s + t + st$, $t, s \in [-1, 1]$.
2. $z(s,t) = -2 + e^s + e^{-s} + cos(t) + \sin(t) + (-1 + e^s + e^{-s})(-1 + cos(t) + \sin(t))$, $t \in [-\pi, \pi]$, $s \in [-1, 1]$.

These surfaces can be drawn respectively as in figures 1 and 2

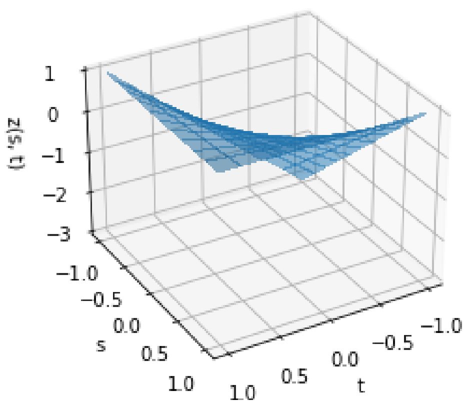

Figure 1: Isotropic minimal TF-surface$z(s,t) = s + t + st$

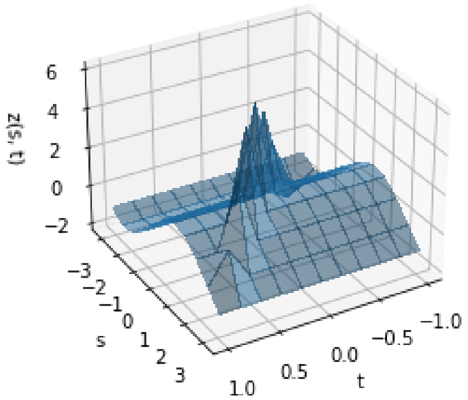

Figure 2: Isotropic minimal TF-surface $z(s,t) = -2 + e^s + e^{-s} + cos(t) + sin(t) + \left(-1 + e^s + e^{-s}\right)(-1 + cos(t) + sin(t))$

4.2. Isotropic flat TF-surfaces in $\mathbb{I}^3$

Let M^2 be a graph surface immersed in $\mathbb{I}^3$ corresponding to a real-valued smooth function $z(s,t) = u(s) + v(t) + u(s)v(t)$. Assume that M^2 is a isotropic flat TF-surface.
Then from (3.5) we have

$$(1+u)(1+v)u''v'' - u'^2v'^2 = 0. \tag{4.8}$$

For solving this equation, we need to address the following cases:

Case 1 If $u' = 0$. we have $u(s) = d_1$. Thus $z(s,t) = (1+d_1)v(t) + d_1$. Analogously, if $v' = 0$, we have $v(t) = d_2$, then $z(s,t) = (1+d_2)u(s) + d_2$.

Case 2 If $u'u'' \neq 0$. By symmetry on the arguments, we also suppose $v'v'' \neq 0$. Then (4.8) can be rewritten as

$$\frac{(1+u)u''}{u'^2} = \frac{v'^2}{(1+v)v''}. \tag{4.9}$$

Therefore, we can infer the existence of a real number $\lambda_2 \in \mathbb{R}$, $\lambda_2 \neq 0$, such that

$$\frac{(1+u)u''}{u'^2} = \lambda_2 \text{ and } \frac{(1+v)v''}{v'^2} = \frac{1}{\lambda_2}. \tag{4.10}$$

In order to solve (4.10) we distinguish two situations :

Case 2.1 If $\lambda_2 = 1$, solving (4.10) yields

$$u(s) = d_3 e^{d_4 s} - 1 \text{ and } v(t) = d_5 e^{d_6 t} - 1, \tag{4.11}$$

where d_3, d_4, d_5 and d_6 are constants.

Case 2.2 If $\lambda_2 \neq 1$, by solving (4.10), we obtain

$$u(s) = \left[(1-\lambda_2)(d_7 s + d_8)\right]^{\frac{1}{1-\lambda_2}} - 1 \text{ and } v(t) = \left[\left(\frac{\lambda_2 - 1}{\lambda_2}\right)(d_9 t + d_{10})\right]^{\frac{\lambda_2}{\lambda_2 - 1}} - 1, \tag{4.12}$$

where d_7, d_8, d_9 and d_{10} are constants.

Consequently, we have successfully demonstrated the following:

Theorem 4.3. *Let M^2 be a isotropic flat TF-surface which is the graph $z(s,t) = u(s) + v(t) + u(s)v(t)$ in $\mathbb{I}^3$. Then this surface will be one of the following:*

1. $z(s,t) = a_1 v(t) + a_2$ or $z(s,t) = a_3 u(s) + a_4,\ \ a_i \in \mathbb{R}$,
2. $u(s) = b_1 e^{b_2 s} - 1$ and $v(t) = b_3 e^{b_4 t} - 1,\ \ b_i \in \mathbb{R}$,
3. $u(s) = [(1-\lambda)(c_1 s + c_2)]^{\frac{1}{1-\lambda}} - 1$ and $v(t) = \left[\frac{\lambda-1}{\lambda}(c_3 t + c_4)\right]^{\frac{\lambda}{\lambda-1}} - 1,\ \ c_i \in \mathbb{R}$ and $\lambda \neq 1$.

Exercise 4.4. Let us consider isotropic flat TF-surface given by

1. $z(s,t) = e^s + e^t - 2 + (e^s - 1)\left(e^t - 1\right)\ t, s \in [-1,1]$
2. $z(s,t) = (-s)^{-1} + \left(\frac{1}{2}t\right)^2 - 2 + \left((-s)^{-1} - 1\right)\left(\left(\frac{1}{2}t\right)^2 - 1\right)\ t, s \in [-1,1]$

These surfaces can be drawn respectively as in figures 3 and 4

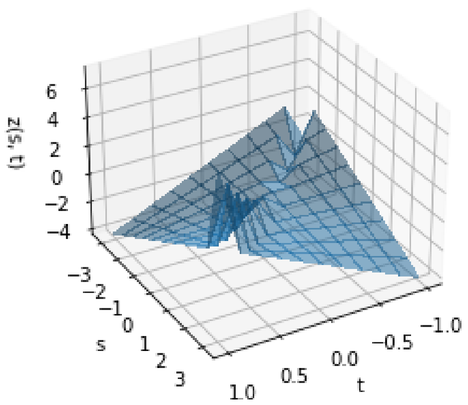

Figure 3: Isotropic flat TF-surface $z(s,t) = e^s + e^t - 2 + (e^s - 1)\left(e^t - 1\right)$

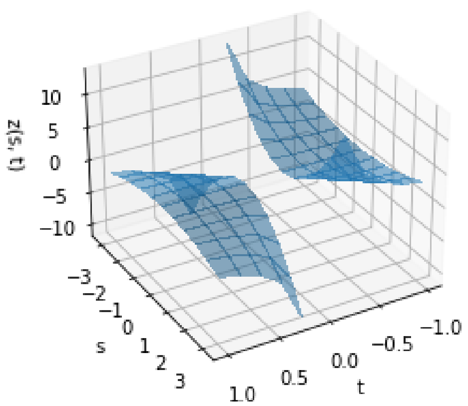

Figure 4: Isotropic flat TF-surface $z(s,t) = (-s)^{-1} + \left(\frac{1}{2}t\right)^2 - 2 + \left((-s)^{-1} - 1\right)\left(\left(\frac{1}{2}t\right)^2 - 1\right)$

5. TF-surfaces with non zero curvatures in $\mathbb{I}^3$

In this section we describe the TF-surfaces when $H \neq 0$ and $K \neq 0$.

5.1. TF-surfaces with constant isotropic mean curvature

Let M^2 be a TF-surface in $\mathbb{I}^3$, i.e., the graph of $z(s,t) = u(s) + v(t) + u(s)v(t)$. Assume that M^2 has constant isotropic mean curvature H_0. Then, it follows from (3.4) that

$$2H_0 = (1+v)\,u'' + (1+u)\,v''. \tag{5.1}$$

For solving this equation, we distinguish the following cases :

Case 1 $u = const = a_1 \neq 0$. (5.1) immediately implies

$$v(t) = \frac{H_0}{1+a_1}t^2 + a_2 t + a_3, \quad a_2, a_3 \in \mathbb{R}. \tag{5.2}$$

Case 2 $v = b_1 \neq 0$, we get

$$u(s) = \frac{H_0}{1+b_1}s^2 + b_2 s + b_3, \quad b_2, b_3 \in \mathbb{R} \tag{5.3}$$

Now suppose that u and v non constants functions. Then (5.1) can be rewritten as

$$\frac{2H_0}{(1+u)\,(1+v)} = \frac{u"}{1+u} + \frac{v"}{1+v}. \tag{5.4}$$

Upon computing partial derivatives of the equation (5.4) concerning s and t, we derive

$$H_0\left(\frac{1}{1+v}\right)'\left(\frac{1}{1+u}\right)' = 0. \tag{5.5}$$

This equation results in a contradiction since both u and v are non-constant functions. The above analysis enables us to state the following theorem:

Theorem 5.1. *Consider M^2 as a TF-surface having a constant isotropic mean curvature H_0 which is the graph $z(s,t) = u(s) + v(t) + u(s)v(t)$ in $\mathbb{I}^3$. Then we have either*

1. $z(s,t) = H_0 s^2 + c_1 s + c_2$,
2. $z(s,t) = H_0 t^2 + c_3 t + c_4$.

Where $c_i \in \mathbb{R}$ such that $i = 1, 2, 3, 4$.

Exercise 5.2. Let us consider TF-surfaces with constant isotropic mean curvature given by

$$z(s,t) = s^2 + s, \quad s \in [-1, 1]$$

.

These surfaces can be drawn as in figure 5

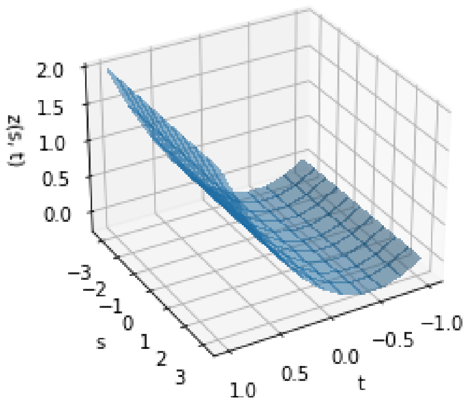

Figure 5: TF-surfaces with constant isotropic mean curvature

5.2. TF-surfaces with constant isotropic Gaussian curvature K_0

Suppose M^2 is a TF-surface in $\mathbb{I}^3$ with constant isotropic Gaussian curvature. From equation (3.5), it can be deduced that

$$K_0 = (1+u)\,(1+v)\,u''{v''}^{2}v'. \tag{5.6}$$

In order to solve equation (5.6), we distinguish three cases.

Case 1 $u' = c_1 \neq 0, c_1 \in \mathbb{R}$. Then (5.6) leads to

$$v' = \frac{\sqrt{-K_0}}{c_1}, K_0 < 0. \tag{5.7}$$

Consequently, v is also a linear function, denoted by

$$v(t) = \frac{\sqrt{-K_0}}{c_1}t + c_2, \quad c_2 \in \mathbb{R}. \tag{5.8}$$

Case 2 $v' = c_3 \neq 0, c_3 \in \mathbb{R}$, we deduce

$$u(s) = \frac{\sqrt{-K_0}}{c_3}s + c_4\ , \quad c_4\ \in \mathbb{R} \tag{5.9}$$

Case 3 Now suppose that u is nonlinear function. From the symmetry v is also nonlinear function . By dividing (5.6) with the product $(1+v)\,u'^2 v''$ we get

$$\frac{K_0}{(1+v)\,u'^2 v''} = \frac{(1+u)\,u''}{u'^2} + \frac{v'^2}{(1+v)\,v''}. \tag{5.10}$$

Upon computing partial derivatives of equation (5.10) with respect to both s and t, we arrive at the following expressions:

$$K_0\left(\frac{1}{u'^2}\right)'\left(\frac{1}{(1+v)\,v''}\right)' = 0 \tag{5.11}$$

From (5.11), if $\left(\frac{1}{u'^2}\right)'' = 0$, i.e., u'^2 nonzero constant which is not possible because u is nonlinear function.

If $\left(\frac{1}{(1+v)v''}\right)' = 0$, this means that $(1+v)\,v''$ is nonzero constant c_1. Considering it into (5.10) gives

$$\frac{K_0}{c_1 u'^2} - \frac{(1+u)\,u''}{u'^2} = \frac{v'^2}{c_1}. \tag{5.12}$$

Given that v is a nonlinear function, the right-hand side of equation (5.12) depends on t. However, the left-hand side of (5.12) is either a constant or a function of s,and both possibilities are not feasible.
We can formally state the following theorem:

Theorem 5.3. *Let M^2 be TF-surface with constant isotropic Gaussian curvature $K_0 (K_0 < 0)$ which is the graph $z(s,t) = u(s) + v(t) + u(s)v(t)$ in $\mathbb{I}^3$.Then we have either*

1. $u(s) = c_1 s + c_2$ and $v(t) = \frac{\sqrt{-K_0}}{c_1} t + c_3,\ c_1, c_2, c_3 \in \mathbb{R}$.
2. $u(s) = \frac{\sqrt{-K_0}}{b_1} s + b_2$ and $v(t) = b_1 t + b_3,\ \ b_1, b_2, b_3 \in \mathbb{R}$.

6. TF-surfaces in $\mathbb{I}^3$ satisfying $K = H^2$.

Consider M^2 as a surface in the Euclidean three-space $\mathbb{R}^3$. The Euler inequality for M^2 encompasses both the Gaussian and mean curvature and can be expressed as follows

$$K \leqslant H^2. \tag{6.1}$$

The equality sign of (6.1) holds on M^2 if and only if it is totally umbilical. For more generalizations, see ([4], [9], [13], [14]).

Now, we aim to classify the translation- factorable (TF) surfaces in $\mathbb{I}^3$ that satisfy $H^2 = K$. By considering (3.4) and (3.5), we have

$$\left[\frac{(1+v)\,u'' + (1+u)\,v''}{2}\right]^2 = (1+u)\,(1+v)\,u''v'' - u'^2 v'^2, \tag{6.2}$$

The previous equation may be rewritten as

$$\left[(1+v)\,u'' - (1+u)\,v''\right]^2 + 4u'^2 v'^2 = 0, \tag{6.3}$$

which implies

$$(1+v)\,u'' - (1+u)\,v'' = 0 \ \text{ and } \ u'v' = 0, \tag{6.4}$$

it follow from (6.4) that either $u = const = c_0$ and $v(t) = c_1 t + c_2$ or $v = const = b_0$ and $u(s) = b_1 s + b_2$.

Therfore, we have the following:

Theorem 6.1. *The TF-surface which is the graph $z(s,t) = u(s) + v(t) + u(s)v(t)$ in $\mathbb{I}^3$satisfying $K = H^2$ is either*

1. $z(s,t) = c_1 s + c_2$, c_1 , $c_2 \in \mathbb{R}$,
2. $z(s,t) = b_1 t + b_2,\ b_1, b_2 \in \mathbb{R}$.

Exercise 6.2. Let us consider TF-surfaces in $\mathbb{I}^3$ with $K = H^2$ given by

$$z(s,t) = s + 1, \quad s \in [-1, 1]$$

These surfaces can be drawn as in figure 6

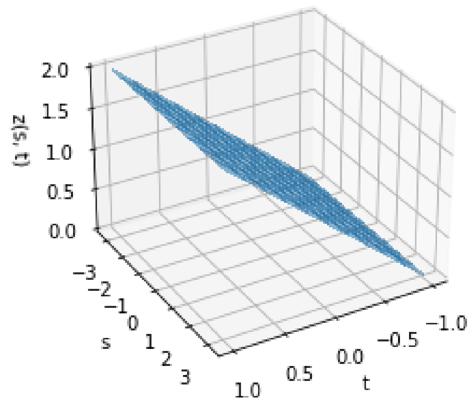

Figure 6: TF-surfaces in $\mathbb{I}^3$ with $K = H^2$

7. Conclusion

In this article, we have presented an overview of factorable-translation surfaces in isotropic space that possess constant mean and Gaussian curvatures. Additionally, we have provided a classification of surfaces that satisfy the condition $K = H^2$. Furthermore, our results pave the way for future research in this field. Exploring the geometric characteristics and differential equations associated with these surfaces, as well as analyzing their behavior under other restrictions or in alternative spacetime environments, are potential future research paths. We can improve our grasp of factorable-translation surfaces and their unique mathematical characteristics by going deeper into these issues.

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References

[1] K. Arslan, B. Bayram, B. Bulca and G. Ozturk, On Translation Surfaces in 4-dimensional Euclidean Space, Acta et Comm. Univ. Tartuensis Math., Vol:20, No.2 (2016), 123-133.

[2] M.E., Aydin, A generalization of translation surfaces with constant curvature in the isotropic space, J. Geom., V. 107 (2016), 603-615. 1

[3] B-Y, Chen, Total mean curvature and submanifolds of finite type.World Scientific, Singapore, (1984).

[4] B-Y, Chen, Mean curvature and chape operator of isometric immersions in real-space forms. Glasg. Math. J. 38 (1996), 87-97. 6

[5] M, Choi, On the Gauss map of surfaces of revolution in a 3-dimensional Minkowski space, Tsukuba J. Math., V. 19, (1995), 351-367.

[6] S, Decu, L, Verstraelen, A note on the isotropical geometry of production surfaces, Kragujevac J. Math., V. 37, (2013), 217-220 .

[7] F, Dillen, J, Pas, L, Vertraelen, On the Gauss map of surfaces of revolution, Bull. Inst. Math. Acad. Sinica, V. 18, (1990), 239-246 .

[8] S.A. Difi , A. Hakem and H. Zoubir,. Translation-Factorable (TF) surfaces in the 3-dimensional Euclidean and Lorentz space satisfying $\triangle r_i = \lambda_i r_i$. Electronic Journal of Mathematical Analysis and Applications. V. 6(2), (2018), 227-236. 1

[9] Z. H., Hou and F. Ji. Helicoidal surfaces with $H^2 = K$ in Minkowski 3-space; J. Math. Anal. Appl. V, 325, (2007) 101-113. 6

[10] H. Liu,. Translation surfaces with constant mean curvature in three dimensional spaces,J. Geom. V. 64 (1999), 141-149. 1

[11] R. Lopez,. Separtion of variables in equations of mean-curvature type, Proc. Roy. Soc. Edinburgh Sect. V146 , no. 5, (2016),1017-1035. 1

[12] R, Lopez, M, Moruz,. Translation and homothetical surfaces in Euclidean space with constant curvature, J. Korean Math. Soc, V. 52, (2015),523-535 .

[13] A. Mihai,. Geometric inequalities for purely real submanifolds in complex space forms, Results Math, V55 (2009), 457-468. 6

[14] A.Mihai,. on the generalized Wintgen inequality for Lagrangian submanifolds in complex space forms, Nonlinear Analysis. V 95. (2014), 714-720 .

[15] B. Medjahdi and H. Zoubir,. Affine Factorable Surfaces in the Three-Dimensional Simply Isotropic Space IAENG International Journal of Applied Mathematics, Vol, 51:2, IJAM (2021).

[16] B.Medjahdi and H.Zoubir,. Graph Translation Surface in the Lorentz-Heisenberg 3-space with constant curvatures, çankaya University Journal of Science and Engineering. CUJSE V19(01)(2022), 013-021.

[17] H. Meng, H. Liu, Factorables surfaces in three Minkowski space, Bulle Korean-Math. Soc V. 46 (2009), 155-169. 6
2

[18] A. D. Polyanin. W. E. Schiesser, and A. I. Zhurov, Partial differentialbequation, Scholarpedia, 3 (2008), no 10, 4605, revision 121514. 1

[19] H, Sachs, Isotrope Geometrie des Raumes, Vieweg Verlag, Braunschweig, (1990).

[20] K. Seo,Translation Hypersurfaces with constant curvatures inspace form,Osaka J .Math. V. 50, (2013), 631-641. 1

[21] L. Simon. equations of mean curvature type in 2 independent variables, Pacific j. Math V.69 , no 1, (1977), 245-268. 2

[22] L. Simon. The minimal surface equation, Geometry, Encyclopaedia Math.Sci. Springer, Berlin,V90, (1997),239-272. 1

[23] Z. M. Sipus: Translation surfaces of constant curvatures in a simply isotropic space,period.math.Hung V. 68, (2014), 160-175. 1

[24] K. Strubecker, differenti geometrie des isotropic Raumes III, Flachentheoie, Math, Zeitsch, V. 48, (1942), 369-342. 1

[25] T, Takahashi, Minimal immersions of Riemannian manifolds, J.Math.Soc. Japan, V.18, (1966), 380-385. 1, 2

[26] V. Ushakov, The explicit general solution of trivial Monge-Ampère equation, Comment. Math. Helv. V. 75 , no. 1, (2000), 125-133. 2

[27] L. Verstraelen, J.Walrave and S. Yaprak, the minimalt ranslation surfaces in Euclidian space, Soochow J. Math V. 20 (1994) 77-82.

[28] Y.Yu and H. Liu, The factorables minimals surfaces,Proceedings of the Eleventh International worshop on Diff.Geom. V.11 (2007) 33-39. 1