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Characterizing Irregularity in Planar Graph Structures

Abdul Aleem Mughal, Raja Noshad Jamil, Abaid ur Rehman Virk **

[a]University of Management and Technology, Lahore, Pakistan.

[b]University of Management and Technology, Lahore, Pakistan.

Abstract

Face irregularity strength under $\rho-$labeling ξ with class $(\alpha_1, \beta_1, \gamma_1)$ of plane graphs is a labeling from the set of graph elements into the set of integers, that is, $\xi : \{V \cup E \cup F\} \rightarrow \{1, 2, 3, .., \rho\}$, such that the face weights are distinct at any stage in the graph labeling, that is, $W_{\xi(\alpha_1,\beta_1,\gamma_1)}(f) \neq W_{\xi(\alpha_1,\beta_1,\gamma_1)}(g)$, for any two faces f and g of the graph G. The face irregular strength of a plane graph G is the least possible integer ρ such that G admits face irregular $\rho-$labeling. In this research, authors have examined the exact tight lower bounds for the face irregular strength of generalized plane graphs under $\rho-$labeling of class $(\alpha_1, \beta_1, \gamma_1)$ for vertex $(1, 0, 0)$, edge $(0, 1, 0)$, face $(0, 0, 1)$, vertex-face $(1, 0, 1)$, edge-face $(0, 1, 1)$ and entire $(1, 1, 1)$. Results are verified by examples.

Keywords: Cartesian product, Face labeling, Face irregularity strength, Face weights.

1. Introduction and Preliminaries

In this article, all graphs are finite, simple, planar and undirected. They can be represented by $G = (V, E, F)$, where V is the set of vertices, E is the set of edges and F is the set of faces. Graph labeling has its origin long back ago but in 1967, Alexander Rosa did valuable work on graph labeling [1]. Since then, many researchers put their interest in vertex labeling and edge labeling [4, 6, 10, 5]. Vertex labeling of a graph means to assign some positive integer to all the vertices of a graph under some conditions. Similarly, edge and face labeling of a graph have the same strategy. In this article, the labeling under discussion is $\rho-$labeling. The parameter ξ is a mapping from the set of graph elements $\{V \cup E \cup F\}$ into the set of integers $\{1, 2, 3, .., \rho\}$, where ρ is the largest label in a graph labeling on which the face weights are distinct [15]. Recent research on $\rho-$labeling of graphs can be seen in [3, 5, 7, 18]. The letters $\alpha_1, \beta_1, \gamma_1 \in \{0, 1\}$ have

*Corresponding author

Email addresses: f2016265007@umt.edu.pk (Abdul Aleem Mughal), noshad.jamil@yahoo.com (Raja Noshad Jamil), abaidrehman@umt.edu.pk (Abaid ur Rehman Virk *)

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association with graph vertices, edges and faces respectively according to [21]. The weight of a face f under the ρ-labeling ξ of class $(\alpha_1, \beta_1, \gamma_1)$, is

$$W_{\xi_{(\alpha_1,\beta_1,\gamma_1)}}(f) = \alpha_1 \sum_{v\sim f} \xi(v) + \beta_1 \sum_{e\sim f} \xi(e) + \gamma_1\, \xi(f) \tag{1.1}$$

In this research, authors have investigated face irregularity strength of grid graphs. A grid graph is represented by $G_s^t = P_{s+1}\Box P_{t+1}$ such that $2 \leq t < s$, where s and t are the number of row-faces and column-faces of the graph respectively. To understand the basics of irregularity strength in grid graphs, one can study [14, 3, 19, 22]. Face irregularity strength and grid graphs will be abbreviated by FIS and GG respectively. This research paper is based on the calculation of face irregularity strength under $\rho-$labeling ξ by using vertex $\rho-$labeling $(1, 0, 0)$, edge $\rho-$labeling $(0, 1, 0)$, face $\rho-$labeling $(0, 0, 1)$, vertex-face $\rho-$labeling $(1, 0, 1)$, edge-face $\rho-$labeling $(0, 1, 1)$ and entire $\rho-$labeling $(1, 1, 1)$ for grid graphs. In 2020, Baca et al. investigated exact value for the FIS of ladder graphs [13]. Noshad and Aleem proved the following Theorem 1.1 to investigate the exact value for total FIS of grid graphs [21] and investigated the case $(1, 1, 0)$.

Theorem 1.1. *Suppose that $G = (V, E, F)$ is a plane graph with d_i where d_i is the number of $i-$sided faces with $i \geq 3$. Let $(\alpha_1, \beta_1, \gamma_1) \in \{0, 1\}$, $[a = min\{i\}$, for $d_i \neq 0]$, $[b = max\{i\}$, for $d_i \neq 0]$, $d_b = 1$ and $[c = max\{i, i < b\}$, for $d_i \neq 0]$. Then FIS with class $(\alpha_1, \beta_1, \gamma_1)$ of G is*

$$fs_{(\alpha_1,\beta_1,\gamma_1)}(G) \geq \left\lceil \frac{(\alpha_1 + \beta_1)\, a + \gamma_1 + |F(G)| - 2}{(\alpha_1 + \beta_1)\, c + \gamma_1} \right\rceil. \tag{1.2}$$

Proof. Proof of the theorem can be seen in [21]. □

2. Main results

Exact value for the lower bound holds if the horizontal differences in face weights are 1 and the vertical differences in face weights are t. From Theorem 1.1, the lower bounds for the FIS of classes $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, $(0, 1, 1)$ and $(1, 1, 1)$ can be calculated easily.

For $i = 1, 2, 3, \ldots, s + 1$ and $j = 1, 2, 3, \cdots, t + 1$, the generalized formulas for vertices and edges around $P_{s+1}\Box P_{t+1}$ are as follows:

Vertices:
$V(G_s^t) = \{v_i^j\}$

Horizontal Edges:
$E(G_s^t) = \{v_i^j v_i^{j+1}\}$

Vertical Edges:
$E(G_s^t) = \{v_i^j v_{i+1}^j\}$

Theorem 2.1. *Let $t, s \in \mathbb{N}$ such that $2 \leq t < s$, then $fs_{(1,0,0)}(G_s^t) = \lceil \frac{mn+3}{4} \rceil$.*

Proof. The generalized labeling is as follows: We define the face irregular ρ-labeling ξ of class $(1, 0, 0)$ of G_s^t in the following way

$$\xi(v_i^j) = \begin{cases} \left\lceil \frac{j}{4} \right\rceil + \left\lfloor \frac{t}{2} \right\rfloor \left\lfloor \frac{i+1}{4} \right\rfloor + \left\lceil \frac{t}{2} \right\rceil \left\lfloor \frac{i}{4} \right\rfloor, \\ \\ \text{if } i = \text{Odd numbers till } s\ ;\ s \equiv 1(2) \text{ or } i = \text{Odd numbers till } s+1\ ;\ s \equiv 0(2). \\ j = \text{Odd numbers till } t\ ;\ t \equiv 1(2) \text{ or } j = \text{Odd numbers till } t+1\ ;\ t \equiv 0(2). \end{cases}$$

$$\xi(v_i^j)=\begin{cases}\left\lceil\frac{j}{4}\right\rceil+\left\lceil\frac{t}{2}\right\rceil\left\lfloor\frac{i+1}{4}\right\rfloor+\left\lfloor\frac{t}{2}\right\rfloor\left\lfloor\frac{i}{4}\right\rfloor, \\ \\ \text{if } i=\text{Odd numbers till } s\ ;\ s\equiv 1(2) \text{ or } i=\text{Odd numbers till } s+1\ ;\ s\equiv 0(2). \\ j=\text{Even numbers till } t+1\ ;\ t\equiv 1(2) \text{ or } j=\text{Even numbers till } t\ ;\ t\equiv 0(2).\end{cases}$$

$$\xi(v_i^j)=\begin{cases}\left\lceil\frac{j+2}{4}\right\rceil+\left\lfloor\frac{t}{2}\right\rfloor\left\lfloor\frac{i}{4}\right\rfloor+\left\lceil\frac{t}{2}\right\rceil\left\lfloor\frac{i-1}{4}\right\rfloor, \\ \\ \text{if } i=\text{Even numbers till } s+1\ ;\ s\equiv 1(2) \text{ or } i=\text{Even numbers till } s\ ;\ s\equiv 0(2). \\ j=\text{Odd numbers till } t\ ;\ t\equiv 1(2) \text{ or } j=\text{Odd numbers till } t+1\ ;\ t\equiv 0(2).\end{cases}$$

$$\xi(v_i^j)=\begin{cases}\left\lceil\frac{j+2}{4}\right\rceil+\left\lceil\frac{t}{2}\right\rceil\left\lfloor\frac{i}{4}\right\rfloor+\left\lfloor\frac{t}{2}\right\rfloor\left\lfloor\frac{i-1}{4}\right\rfloor, \\ \\ \text{if } i=\text{Even numbers till } s+1\ ;\ s\equiv 1(2) \text{ or } i=\text{Even numbers till } s\ ;\ s\equiv 0(2). \\ j=\text{Even numbers till } t+1\ ;\ t\equiv 1(2) \text{ or } j=\text{Even numbers till } t\ ;\ t\equiv 0(2).\end{cases}$$

The weight formula under $\rho-$labeling ξ of class $(1,0,0)$ is defined as

$$W_{\xi(1,0,0)}(f_i^j)=\xi(v_i^j)+\xi(v_i^{j+1})+\xi(v_{i+1}^j)+\xi(v_{i+1}^{j+1}).$$

The differences between the weights of the horizontal faces will be calculated as follows:

For $i=$ Odd numbers till $s\ ;\ s\equiv 1(2)$ or $i=$ Odd numbers till $s+1\ ;\ s\equiv 0(2)$
and
$j=$ Odd numbers till $t\ ;\ t\equiv 1(2)$ or $j=$ Odd numbers till $t+1\ ;\ t\equiv 0(2)$
OR
$j=$ Even numbers till $t+1\ ;\ t\equiv 1(2)$ or $j=$ Even numbers till $t\ ;\ t\equiv 0(2)$.

$$\begin{aligned}W_{\xi(1,0,0)}(f_i^{j+1})-W_{\xi(1,0,0)}(f_i^j)&=\left(\xi(v_i^{j+1})+\xi(v_i^{j+2})+\xi(v_{i+1}^{j+1})+\xi(v_{i+1}^{j+2})\right)\\&-\left(\xi(v_i^j)+\xi(v_i^{j+1})+\xi(v_{i+1}^j)+\xi(v_{i+1}^{j+1})\right)\\&=\left\lceil\frac{j}{4}\right\rceil+\left\lceil\frac{j+1}{4}\right\rceil+\left\lceil\frac{j+2}{4}\right\rceil+\left\lceil\frac{j+3}{4}\right\rceil+2t\left\lfloor\frac{i}{4}\right\rfloor\\&+2t\left\lfloor\frac{i+1}{4}\right\rfloor+1-\left\lceil\frac{j}{4}\right\rceil-\left\lceil\frac{j+1}{4}\right\rceil-\left\lceil\frac{j+2}{4}\right\rceil\\&-\left\lceil\frac{j+3}{4}\right\rceil-2t\left\lfloor\frac{i}{4}\right\rfloor-2t\left\lfloor\frac{i+1}{4}\right\rfloor\\&=1.\end{aligned}$$

For $i =$ Even numbers till $s+1$; $s \equiv 1(2)$ or $i =$ Even numbers till s ; $s \equiv 0(2)$
and
$j =$ Odd numbers till t ; $t \equiv 1(2)$ or $j =$ Odd numbers till $t+1$; $t \equiv 0(2)$
OR
$j =$ Even numbers till $t+1$; $t \equiv 1(2)$ or $j =$ Even numbers till t ; $t \equiv 0(2)$.

$$\begin{aligned} W_{\xi(1,0,0)}(f_i^{j+1}) - W_{\xi(1,0,0)}(f_i^j) &= \Big(\xi(v_i^{j+1}) + \xi(v_i^{j+2}) + \xi(v_{i+1}^{j+1}) + \xi(v_{i+1}^{j+2})\Big) \\ &- \Big(\xi(v_i^j) + \xi(v_i^{j+1}) + \xi(v_{i+1}^j) + \xi(v_{i+1}^{j+1})\Big) \\ &= \left\lceil\frac{j}{4}\right\rceil + \left\lceil\frac{j+1}{4}\right\rceil + \left\lceil\frac{j+2}{4}\right\rceil + \left\lceil\frac{j+3}{4}\right\rceil + t\left\lfloor\frac{i}{4}\right\rfloor + t\left\lfloor\frac{i+2}{4}\right\rfloor \\ &+ t\left\lfloor\frac{i-1}{4}\right\rfloor + t\left\lfloor\frac{i+1}{4}\right\rfloor + 1 - \left\lceil\frac{j}{4}\right\rceil - \left\lceil\frac{j+1}{4}\right\rceil - \left\lceil\frac{j+2}{4}\right\rceil \\ &- \left\lceil\frac{j+3}{4}\right\rceil - t\left\lfloor\frac{i}{4}\right\rfloor - t\left\lfloor\frac{i+2}{4}\right\rfloor - t\left\lfloor\frac{i-1}{4}\right\rfloor - t\left\lfloor\frac{i+1}{4}\right\rfloor \\ &= 1. \end{aligned}$$

So the horizontal differences in face weights are pairwise distinct.

Now let us calculate the vertical faces differences:

For $i =$ Odd numbers till s ; $s \equiv 1(2)$ or $i =$ Odd numbers till $s+1$; $s \equiv 0(2)$
and
$j =$ Odd numbers till t ; $t \equiv 1(2)$ or $j =$ Odd numbers till $t+1$; $t \equiv 0(2)$
OR
$j =$ Even numbers till $t+1$; $t \equiv 1(2)$ or $j =$ Even numbers till t ; $t \equiv 0(2)$.

$$\begin{aligned} W_{\xi(1,0,0)}(f_{i+1}^j) - W_{\xi(1,0,0)}(f_i^j) &= \Big(\xi(v_{i+1}^j) + \xi(v_{i+1}^{j+1}) + \xi(v_{i+2}^j) + \xi(v_{i+2}^{j+1})\Big) \\ &- \Big(\xi(v_i^j) + \xi(v_i^{j+1}) + \xi(v_{i+1}^j) + \xi(v_{i+1}^{j+1})\Big) \\ &= t\left\lfloor\frac{i+3}{4}\right\rfloor + t\left\lfloor\frac{i+2}{4}\right\rfloor - t\left\lfloor\frac{i+1}{4}\right\rfloor - t\left\lfloor\frac{i}{4}\right\rfloor \\ &= t. \end{aligned}$$

For $i =$ Even numbers till $s+1$; $s \equiv 1(2)$ or $i =$ Even numbers till s ; $s \equiv 0(2)$
and
$j =$ Odd numbers till t ; $t \equiv 1(2)$ or $j =$ Odd numbers till $t+1$; $t \equiv 0(2)$
OR
$j =$ Even numbers till $t+1$; $t \equiv 1(2)$ or $j =$ Even numbers till t ; $t \equiv 0(2)$.

$$\begin{aligned}W_{\xi(1,0,0)}(f_{i+1}^j) - W_{\xi(1,0,0)}(f_i^j) &= \Big(\xi(v_{i+1}^j) + \xi(v_{i+1}^{j+1}) + \xi(v_{i+2}^j) + \xi(v_{i+2}^{j+1})\Big) \\ &- \Big(\xi(v_i^j) + \xi(v_i^{j+1}) + \xi(v_{i+1}^j) + \xi(v_{i+1}^{j+1})\Big) \\ &= t\left\lfloor\frac{i+2}{4}\right\rfloor + t\left\lfloor\frac{i+1}{4}\right\rfloor - t\left\lfloor\frac{i}{4}\right\rfloor - t\left\lfloor\frac{i-1}{4}\right\rfloor. \\ &= t.\end{aligned}$$

This shows the differences among the vertically adjacent faces is t. Hence, all the weights are pairwise distinct. □

Theorem 2.2. *Let $t, s \in \mathbb{N}$ such that $2 \le t < s$, then $fs_{(0,1,0)}(P_{s+1}\square P_{t+1}) = \left\lceil \frac{mn+3}{4} \right\rceil$.*

Proof. Theorem can be proved on the similar way as Theorem 2.1. □

Theorem 2.3. *Let $t, s \in \mathbb{N}$ such that $2 \le t < s$, then, $fs_{(0,0,1)}(P_{s+1}\square P_{t+1}) = mn + 1$.*

Proof. Theorem can be proved on the similar way as Theorem 2.1. □

Theorem 2.4. *Let $t, s \in \mathbb{N}$ such that $2 \le t < s$, then for a grid graph $P_{s+1}\square P_{t+1}$, we have*

$$fs_{(1,0,1)}(P_{s+1}\square P_{t+1}) = \left\lceil \frac{mn+4}{5} \right\rceil.$$

Proof. We define vertex-face labeling ξ of class $(1,0,1)$ of G_s^t as follows:

$$\xi(v_i^j) = \begin{cases} 1 + \lfloor \frac{t}{2} \rfloor \lfloor \frac{i+1}{4} \rfloor + \lceil \frac{t}{2} \rceil \lfloor \frac{i-1}{4} \rfloor, & \text{if } i = 1,2,\dots,2\lfloor \frac{2\rho}{t} \rfloor \text{ and } j = \text{Odd numbers till } t+1, \quad t \equiv 0(2) \\ & \text{or } j = \text{Odd numbers till } t, \quad t \equiv 1(2) \\ 1 + \lceil \frac{t}{2} \rceil \lfloor \frac{i+1}{4} \rfloor + \lfloor \frac{t}{2} \rfloor \lfloor \frac{i-1}{4} \rfloor, & \text{if } i = 1,2,\dots,2\lfloor \frac{2\rho}{t} \rfloor \text{ and } j = \text{Even numbers till } t, \quad t \equiv 0(2) \\ & \text{or } j = \text{Even numbers till } t+1,\ t \equiv 1(2) \\ \rho, & \text{if } i = 2\lfloor \frac{2\rho}{t} \rfloor + 1,\dots,s \text{ and } j = \text{Natural numbers till } t+1. \end{cases}$$

$$\xi(f_i^j) = \begin{cases} j, & \text{if } i = 1,2,\dots,2\lfloor \frac{2\rho}{t} \rfloor - 1 \text{ and } j = \text{Natural numbers till } t \\ \Big(i - 2\lfloor \frac{2\rho}{t} \rfloor + 1\Big)\Big(2 + t\lfloor \frac{2\rho}{t} \rfloor - 2\rho\Big) + j, & \text{if } i = 2\lfloor \frac{2\rho}{t} \rfloor, 2\lfloor \frac{2\rho}{t} \rfloor + 1 \text{ and } j = \text{Natural numbers till } t \\ t(i-1) - 4(\rho - 1) + j, & \text{if } i = 2\lfloor \frac{2\rho}{t} \rfloor + 2,\dots,s \text{ and } j = \text{Natural numbers till } t. \end{cases}$$

The weight under labeling ξ of class $(1,0,1)$ can be defined as

$$W_{\xi(1,0,1)}(f_i^j) = \xi(v_i^j) + \xi(v_{i+1}^j) + \xi(v_i^{j+1}) + \xi(v_{i+1}^{j+1}) + \xi(f_i^j).$$

Now let us calculate the horizontally faces differences:

For $i = 1,2,\dots,2\lfloor \frac{2\rho}{t} \rfloor - 1$ and $j =$ Natural numbers till t.

$$\begin{aligned}W_{\xi(1,0,1)}(f_i^{j+1}) - W_{\xi(1,0,1)}(f_i^j) &= \Big(\xi(v_i^{j+1}) + \xi(v_{i+1}^{j+1}) + \xi(v_i^{j+2}) + \xi(v_{i+1}^{j+2}) + \xi(f_i^{j+1})\Big) \\ &- \Big(\xi(v_i^j) + \xi(v_{i+1}^j) + \xi(v_i^{j+1}) + \xi(v_{i+1}^{j+1}) + \xi(f_i^j)\Big) \\ &= (j+1) - j. \\ &= 1.\end{aligned}$$

For $i = 2\lfloor\frac{2\rho}{t}\rfloor, 2\lfloor\frac{2\rho}{t}\rfloor + 1$ and $j =$ Natural numbers till t.

$$\begin{aligned}
W_{\xi(1,0,1)}(f_i^{j+1}) - W_{\xi(1,0,1)}(f_i^j) &= \Big(\xi(v_i^{j+1}) + \xi(v_{i+1}^{j+1}) + \xi(v_i^{j+2}) + \xi(v_{i+1}^{j+2}) + \xi(f_i^{j+1})\Big) \\
&- \Big(\xi(v_i^j) + \xi(v_{i+1}^j) + \xi(v_i^{j+1}) + \xi(v_{i+1}^{j+1}) + \xi(f_i^j)\Big) \\
&= (j+1) - j. \\
&= 1.
\end{aligned}$$

For $i = 2\lfloor\frac{2\rho}{t}\rfloor + 2, \ldots, s$ and $j =$ Natural numbers till t.

$$\begin{aligned}
W_{\xi(1,0,1)}(f_i^{j+1}) - W_{\xi(1,0,1)}(f_i^j) &= \Big(\xi(v_i^{j+1}) + \xi(v_{i+1}^{j+1}) + \xi(v_i^{j+2}) + \xi(v_{i+1}^{j+2}) + \xi(f_i^{j+1})\Big) \\
&- \Big(\xi(v_i^j) + \xi(v_{i+1}^j) + \xi(v_i^{j+1}) + \xi(v_{i+1}^{j+1}) + \xi(f_i^j)\Big) \\
&= (j+1) - j. \\
&= 1.
\end{aligned}$$

Now we measure the differences between the weights for every two vertically adjacent faces:

For $i = 1, 2, \ldots, 2\lfloor\frac{2\rho}{t}\rfloor - 2$ and $j =$ Natural numbers till t.

$$\begin{aligned}
W_{\xi(1,0,1)}(f_{i+1}^j) - W_{\xi(1,0,1)}(f_i^j) &= \Big(\xi(v_{i+1}^j) + \xi(v_{i+1}^{j+1}) + \xi(v_{i+2}^j) + \xi(v_{i+2}^{j+1}) + \xi(f_{i+1}^j)\Big) \\
&- \Big(\xi(v_i^j) + \xi(v_i^{j+1}) + \xi(v_{i+1}^j) + \xi(v_{i+1}^{j+1}) + \xi(f_i^j)\Big) \\
&= \left(2 + \left(\left\lfloor\frac{t}{2}\right\rfloor + \left\lceil\frac{t}{2}\right\rceil\right)\left\lfloor\frac{i+3}{4}\right\rfloor + \left(\left\lceil\frac{t}{2}\right\rceil + \left\lfloor\frac{t}{2}\right\rfloor\right)\left\lfloor\frac{i+1}{4}\right\rfloor\right) \\
&- \left(2 + \left(\left\lfloor\frac{t}{2}\right\rfloor + \left\lceil\frac{t}{2}\right\rceil\right)\left\lfloor\frac{i+1}{4}\right\rfloor + \left(\left\lceil\frac{t}{2}\right\rceil + \left\lfloor\frac{t}{2}\right\rfloor\right)\left\lfloor\frac{i-1}{4}\right\rfloor\right). \\
&= t\left\lfloor\frac{i+3}{4}\right\rfloor + t\left\lfloor\frac{i+1}{4}\right\rfloor - t\left\lfloor\frac{i+1}{4}\right\rfloor - t\left\lfloor\frac{i-1}{4}\right\rfloor. \\
&= t\left(\left\lfloor\frac{i+3}{4}\right\rfloor - \left\lfloor\frac{i-1}{4}\right\rfloor\right). \\
&= t.
\end{aligned}$$

For $i = 2\lfloor\frac{2\rho}{t}\rfloor - 1$ and $j =$ Natural numbers till t.

$$\begin{aligned}
W_{\xi(1,0,1)}(f_{i+1}^j) - W_{\xi(1,0,1)}(f_i^j) &= \Big(\xi(v_{i+1}^j) + \xi(v_{i+1}^{j+1}) + \xi(v_{i+2}^j) + \xi(v_{i+2}^{j+1}) + \xi(f_{i+1}^j)\Big) \\
&- \Big(\xi(v_i^j) + \xi(v_i^{j+1}) + \xi(v_{i+1}^j) + \xi(v_{i+1}^{j+1}) + \xi(f_i^j)\Big). \\
&= 2\rho - \left(2 + t\left\lfloor\frac{i+1}{4}\right\rfloor + t\left\lfloor\frac{i-1}{4}\right\rfloor\right) + \left(2 + t\left\lfloor\frac{2\rho}{t}\right\rfloor - 2\rho + j - j\right). \\
&= 2\rho - \left(2 + t\left\lfloor\frac{\lfloor\frac{2\rho}{t}\rfloor}{2}\right\rfloor + t\left\lfloor\frac{\lfloor\frac{2\rho}{t}\rfloor - 1}{2}\right\rfloor\right) + 2 + t\left\lfloor\frac{2\rho}{t}\right\rfloor - 2\rho. \\
&= t\left\lfloor\frac{2\rho}{t}\right\rfloor - t\left\lfloor\frac{\lfloor\frac{2\rho}{t}\rfloor}{2}\right\rfloor - t\left\lfloor\frac{\lfloor\frac{2\rho}{t}\rfloor - 1}{2}\right\rfloor. \\
&= t\left(\left\lfloor\frac{2\rho}{t}\right\rfloor - \left(\left\lfloor\frac{2\rho}{t}\right\rfloor - 1\right)\right). \\
&= t.
\end{aligned}$$

For $i = 2\lfloor \frac{2\rho}{t} \rfloor$ and $j =$ Natural numbers till t.

$$\begin{aligned}
W_{\xi(1,0,1)}(f_{i+1}^{j}) - W_{\xi(1,0,1)}(f_{i}^{j}) &= \left(\xi(v_{i+1}^{j}) + \xi(v_{i+1}^{j+1}) + \xi(v_{i+2}^{j}) + \xi(v_{i+2}^{j+1}) + \xi(f_{i+1}^{j})\right) \\
&- \left(\xi(v_{i}^{j}) + \xi(v_{i}^{j+1}) + \xi(v_{i+1}^{j}) + \xi(v_{i+1}^{j+1}) + \xi(f_{i}^{j})\right). \\
&= 2\rho - \left(1 + \left\lceil \frac{t}{2} \right\rceil \left\lfloor \frac{2\left\lfloor \frac{2\rho}{t} \right\rfloor + 1}{4} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor \left\lfloor \frac{2\left\lfloor \frac{2\rho}{t} \right\rfloor - 1}{4} \right\rfloor\right) \\
&- \left(1 + \left\lceil \frac{t}{2} \right\rceil \left\lfloor \frac{2\left\lfloor \frac{2\rho}{t} \right\rfloor + 1}{4} \right\rfloor + \left\lfloor \frac{t}{2} \right\rfloor \left\lfloor \frac{2\left\lfloor \frac{2\rho}{t} \right\rfloor - 1}{4} \right\rfloor\right) \\
&+ 2\left(2 + t\left\lfloor \frac{2\rho}{t} \right\rfloor - 2\rho\right) - \left(2 + t\left\lfloor \frac{2\rho}{t} \right\rfloor - 2\rho\right). \\
&= t\left\lfloor \frac{2\rho}{t} \right\rfloor - \left(\left\lfloor \frac{t}{2} \right\rfloor + \left\lceil \frac{t}{2} \right\rceil\right) \left\lfloor \frac{2\left\lfloor \frac{2\rho}{t} \right\rfloor + 1}{4} \right\rfloor - \left(\left\lceil \frac{t}{2} \right\rceil + \left\lfloor \frac{t}{2} \right\rfloor\right) \left\lfloor \frac{2\left\lfloor \frac{2\rho}{t} \right\rfloor - 1}{4} \right\rfloor. \\
&= t\left\lfloor \frac{2\rho}{t} \right\rfloor - t\left(\left\lfloor \frac{2\left\lfloor \frac{2\rho}{t} \right\rfloor + 1}{4} \right\rfloor + \left\lfloor \frac{2\left\lfloor \frac{2\rho}{t} \right\rfloor - 1}{4} \right\rfloor\right). \\
&= t\left\lfloor \frac{2\rho}{t} \right\rfloor - t\left(\left\lfloor \frac{2\rho}{t} \right\rfloor - 1\right). \\
&= t.
\end{aligned}$$

For $i = 2\left\lfloor \frac{2\rho}{t} \right\rfloor + 1$ and $j =$ Natural numbers till t.

$$\begin{aligned}
W_{\xi(1,0,1)}(f_{i+1}^{j}) - W_{\xi(1,0,1)}(f_{i}^{j}) &= \left(\xi(v_{i+1}^{j}) + \xi(v_{i+1}^{j+1}) + \xi(v_{i+2}^{j}) + \xi(v_{i+2}^{j+1}) + \xi(f_{i+1}^{j})\right) \\
&- \left(\xi(v_{i}^{j}) + \xi(v_{i}^{j+1}) + \xi(v_{i+1}^{j}) + \xi(v_{i+1}^{j+1}) + \xi(f_{i}^{j})\right). \\
&= t + 2\left(2 + t\left\lfloor \frac{2\rho}{t} \right\rfloor - 2\rho\right) + j - 2\left(2 + t\left\lfloor \frac{2\rho}{t} \right\rfloor - 2\rho\right) - j. \\
&= t.
\end{aligned}$$

For $i = 2\left\lfloor \frac{2\rho}{t} \right\rfloor + 2, \ldots, s$ and $j =$ Natural numbers till t.

$$\begin{aligned}
W_{\xi(1,0,1)}(f_{i+1}^{j}) - W_{\xi(1,0,1)}(f_{i}^{j}) &= \left(\xi(v_{i+1}^{j}) + \xi(v_{i+1}^{j+1}) + \xi(v_{i+2}^{j}) + \xi(v_{i+2}^{j+1}) + \xi(f_{i+1}^{j})\right) \\
&- \left(\xi(v_{i}^{j}) + \xi(v_{i}^{j+1}) + \xi(v_{i+1}^{j}) + \xi(v_{i+1}^{j+1}) + \xi(f_{i}^{j})\right). \\
&= mi - 4(\rho - 1) + j - (t(i-1) - 4(\rho - 1) + j). \\
&= t.
\end{aligned}$$

Hence all the weights are distinct and the proof is complete. □

Theorem 2.5. *Let $t, s \in \mathbb{N}$ such that $2 \leq t < s$, then for a grid graph G_s^t, we have*

$$fs_{(0,1,1)}(P_{s+1} \Box P_{t+1}) = \left\lceil \frac{4 + mn}{5} \right\rceil.$$

Proof. Theorem can be proved on the similar way as Theorem 2.4. □

Theorem 2.6. *Let s and t be positive integers and $G_s^t = P_{s+1}\square P_{t+1}$ is a collection of grid graphs for different values of s and t where $2 \le t < s$ and $\left\lfloor \frac{t+1}{3} \right\rfloor = t - 2\left\lfloor \frac{t+1}{3} \right\rfloor$. Then the exact value of entire FIS of G_s^t under $\rho-$labeling of type $(1,1,1)$ is*

$$Efs_{(1,1,1)}(G_s^t) = \left\lceil \frac{mn+8}{9} \right\rceil.$$

Proof. To prove the exact value for the entire FIS under $\rho-$labeling ξ of type $(1,1,1)$ of grid graphs which satisfy the condition $\left\lfloor \frac{t+1}{3} \right\rfloor = t - 2\left\lfloor \frac{t+1}{3} \right\rfloor$ where $2 \le t < s$, the weights differences of horizontally and vertically adjacent faces should be 1 and t respectively.

Generalized results for vertex of grid graphs under $\rho-$labeling ξ of type $(1,1,1)$ can be elaborated as follows:

$$\xi\left(v_i^j\right) = \begin{cases} 1 + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor & ; i = 1,2,3,\ldots,2\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil \ ; \ j = 1,2,\ldots,t+1 \\ \rho & ; i = 2\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil + 1,\ldots,s+1 \ ; \ j = 1,2,\ldots,t+1. \end{cases}$$

Generalized results for face of grid graphs under $\rho-$labeling ξ of type $(1,1,1)$ can be elaborated as follows:

$$\xi\left(f_i^j\right) = \begin{cases} j \\ ; i = 1,2,\cdots,2\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil + 1 + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil}{\left\lceil \frac{t}{2} \right\rceil} \right\rfloor \ ; \ j = 1,2,\cdots,t \\ j + \left(9 + t\left(2\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil}{\left\lceil \frac{t}{2} \right\rceil} \right\rfloor + 1\right) - 8\rho - 1\right) \\ ; i = 2\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil}{\left\lceil \frac{t}{2} \right\rceil} \right\rfloor + 2 \ ; \ j = \text{Natural numbers till } t \\ j + 9 + t\left(2\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil}{\left\lceil \frac{t}{2} \right\rceil} \right\rfloor + 1\right) - 8\rho - 1 \\ +t\left(i - 2\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil - \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil}{\left\lceil \frac{t}{2} \right\rceil} \right\rfloor - 2\right) \\ ; i = 2\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil}{\left\lceil \frac{t}{2} \right\rceil} \right\rfloor + 3,\cdots,s \ ; \ j = \text{Natural numbers till } t. \end{cases}$$

Generalized results for horizontal edge of grid graphs under $\rho-$labeling ξ of type $(1,1,1)$ can be elaborated as follows:

$$\xi\left(v_i^j v_i^{j+1}\right) = \begin{cases} 1 + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor & ; i = 1,2,3,\ldots,2\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil \ ; \ j = \text{Natural numbers till } t \\ \rho & ; i = 2\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil + 1,\ldots,s \ ; \ j = \text{Natural numbers till } t. \end{cases}$$

Generalized results for vertical edge of grid graphs under $\rho-$labeling ξ of type $(1,1,1)$ can be elaborated as follows:

$$\xi\left(v_i^j v_{i+1}^j\right)=\begin{cases} 1 & ;\\ i=1,2,\ldots,2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil-1\ ;\ j=\text{Natural numbers till } t+1 & \\ 1+\left\lfloor\frac{1}{2}\left(t\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+3\left\lfloor\frac{t+1}{3}\right\rfloor-t-3\rho+3\right)\right\rfloor\left\lceil\frac{1}{2}\left(i-2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+1\right)\right\rceil & \\ +\left\lceil\frac{1}{2}\left(t\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+3\left\lfloor\frac{t+1}{3}\right\rfloor-t-3\rho+3\right)\right\rceil\left\lceil\frac{1}{2}\left(i-2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil\right)\right\rceil & ;\\ i=2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil,2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+1\ ; & \\ j=\text{Odd numbers till } t\ ;\ t\equiv 1\,(2)\ \ OR\ j=\text{Odd numbers till } t+1\ ;\ t\equiv 0\,(2) & \\ 1+\left\lceil\frac{1}{2}\left(t\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+3\left\lfloor\frac{t+1}{3}\right\rfloor-t-3\rho+3\right)\right\rceil\left\lceil\frac{1}{2}\left(i-2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+1\right)\right\rceil & \\ +\left\lfloor\frac{1}{2}\left(t\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+3\left\lfloor\frac{t+1}{3}\right\rfloor-t-3\rho+3\right)\right\rfloor\left\lceil\frac{1}{2}\left(i-2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil\right)\right\rceil & ;\\ i=2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil,2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+1\ ; & \\ j=\text{Even numbers till } t+1\ ;\ t\equiv 1\,(2)\ \ OR\ j=\text{Even numbers till } t\ ;\ t\equiv 0\,(2)\,. & \end{cases}$$

$$\xi\left(v_i^j v_{i+1}^j\right)=\begin{cases} 1+\left\lfloor\frac{t}{2}\right\rfloor\left\lceil\frac{1}{2}\left(i-2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil-1\right)\right\rceil+\left\lceil\frac{t}{2}\right\rceil\left\lceil\frac{1}{2}\left(i-2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil-2\right)\right\rceil & \\ +t\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+3\left\lfloor\frac{t+1}{3}\right\rfloor-t-3\rho+3 & ;\\ i=2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+2,\ldots,2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+2+\left\lfloor\frac{4\rho-t-t\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil}{\lceil\frac{t}{2}\rceil}\right\rfloor\ ; & \\ j=\text{Odd numbers till } t\ ;\ t\equiv 1\,(2)\ \ OR\ j=\text{Odd numbers till } t+1\ ;\ t\equiv 0\,(2) & \\ 1+\left\lceil\frac{t}{2}\right\rceil\left\lceil\frac{1}{2}\left(i-2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil-1\right)\right\rceil+\left\lfloor\frac{t}{2}\right\rfloor\left\lceil\frac{1}{2}\left(i-2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil-2\right)\right\rceil & \\ +t\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+3\left\lfloor\frac{t+1}{3}\right\rfloor-t-3\rho+3 & ;\\ i=2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+2,\ldots,2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+2+\left\lfloor\frac{4\rho-t-t\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil}{\lceil\frac{t}{2}\rceil}\right\rfloor\ ; & \\ j=\text{Even numbers till } t+1\ ;\ t\equiv 1\,(2)\ \ OR\ j=\text{Even numbers till } t\ ;\ t\equiv 0\,(2) & \\ \rho & ;\\ i=2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+\left\lfloor\frac{4\rho-t-t\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil}{\lceil\frac{t}{2}\rceil}\right\rfloor+2,\cdots,s\ ;\ j=\text{Natural numbers till } t. & \end{cases}$$

Let us first calculate the horizontal differences in weights among different intervals of i and j.

For $i = 1, 2, 3, \ldots, 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil - 2$; $j = 1, 2, \ldots, t$.

$$W_{\xi(1,1,1)}\left(f_i^{j+1}\right) - W_{\xi(1,1,1)}\left(f_i^j\right) = \xi\left(v_i^{j+1}\right) + \xi\left(v_i^{j+2}\right) + \xi\left(v_{i+1}^{j+1}\right) + \xi\left(v_{i+1}^{j+2}\right) + \xi\left(v_i^{j+1}v_i^{j+2}\right)$$
$$+ \xi\left(v_i^{j+1}v_{i+1}^{j+1}\right) + \xi\left(v_{i+1}^{j+1}v_{i+1}^{j+2}\right) + \xi\left(v_i^{j+2}v_{i+1}^{j+2}\right) - \xi\left(v_i^j\right) - \xi\left(v_i^{j+1}\right) - \xi\left(v_{i+1}^j\right) - \xi\left(v_{i+1}^{j+1}\right)$$
$$- \xi\left(v_i^j v_i^{j+1}\right) - \xi\left(v_i^j v_{i+1}^j\right) - \xi\left(v_{i+1}^j v_{i+1}^{j+1}\right) - \xi\left(v_i^{j+1}v_{i+1}^{j+1}\right) + \xi\left(f_i^{j+1}\right) - \xi\left(f_i^j\right).$$
$$= 1 + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + 1 + \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + 1 + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + 1 + \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor$$
$$+ 1 + j + 1 - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1$$
$$- \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - j.$$
$$= 1.$$

For $i = 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil - 1$; $j = 1, 2, \ldots, t$.

$$W_{\xi(1,1,1)}\left(f_i^{j+1}\right) - W_{\xi(1,1,1)}\left(f_i^j\right) = \xi\left(v_i^{j+1}\right) + \xi\left(v_i^{j+2}\right) + \xi\left(v_{i+1}^{j+1}\right) + \xi\left(v_{i+1}^{j+2}\right) + \xi\left(v_i^{j+1}v_i^{j+2}\right)$$
$$+ \xi\left(v_i^{j+1}v_{i+1}^{j+1}\right) + \xi\left(v_{i+1}^{j+1}v_{i+1}^{j+2}\right) + \xi\left(v_i^{j+2}v_{i+1}^{j+2}\right) - \xi\left(v_i^j\right) - \xi\left(v_i^{j+1}\right) - \xi\left(v_{i+1}^j\right) - \xi\left(v_{i+1}^{j+1}\right)$$
$$- \xi\left(v_i^j v_i^{j+1}\right) - \xi\left(v_i^j v_{i+1}^j\right) - \xi\left(v_{i+1}^j v_{i+1}^{j+1}\right) - \xi\left(v_i^{j+1}v_{i+1}^{j+1}\right) + \xi\left(f_i^{j+1}\right) - \xi\left(f_i^j\right).$$
$$= 1 + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + 1 + \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + 1 + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + 1 + \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor$$
$$+ 1 + j + 1 - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1$$
$$- \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - j.$$
$$= 1.$$

For $i = 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil$; $j = 1, 2, \ldots, t.$

$$W_{\xi(1,1,1)}\left(f_i^{j+1}\right) - W_{\xi(1,1,1)}\left(f_i^{j}\right) = \xi\left(v_i^{j+1}\right) + \xi\left(v_i^{j+2}\right) + \xi\left(v_{i+1}^{j+1}\right) + \xi\left(v_{i+1}^{j+2}\right) + \xi\left(v_i^{j+1}v_i^{j+2}\right)$$
$$+\xi\left(v_i^{j+1}v_{i+1}^{j+1}\right) + \xi\left(v_{i+1}^{j+1}v_{i+1}^{j+2}\right) + \xi\left(v_i^{j+2}v_{i+1}^{j+2}\right) - \xi\left(v_i^{j}\right) - \xi\left(v_i^{j+1}\right) - \xi\left(v_{i+1}^{j}\right) - \xi\left(v_{i+1}^{j+1}\right)$$
$$-\xi\left(v_i^{j}v_i^{j+1}\right) - \xi\left(v_i^{j}v_{i+1}^{j}\right) - \xi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right) - \xi\left(v_i^{j+1}v_{i+1}^{j+1}\right) + \xi\left(f_i^{j+1}\right) - \xi\left(f_i^{j}\right).$$
$$= 1 + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + \rho + 1 + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + \rho + 1$$
$$+\left\lfloor \frac{1}{2}\left(t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3\right)\right\rfloor \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 1\right)\right\rceil$$
$$+\left\lceil \frac{1}{2}\left(t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3\right)\right\rceil \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil\right)\right\rceil$$
$$+j + 1 - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - \rho - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - \rho - 1$$
$$-\left\lfloor \frac{1}{2}\left(t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3\right)\right\rfloor \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 1\right)\right\rceil$$
$$-\left\lceil \frac{1}{2}\left(t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3\right)\right\rceil \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil\right)\right\rceil - j.$$
$$= 1.$$

For $i = 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 1$; $j = 1, 2, \ldots, t.$

$$W_{\xi(1,1,1)}\left(f_i^{j+1}\right) - W_{\xi(1,1,1)}\left(f_i^{j}\right) = \xi\left(v_i^{j+1}\right) + \xi\left(v_i^{j+2}\right) + \xi\left(v_{i+1}^{j+1}\right) + \xi\left(v_{i+1}^{j+2}\right) + \xi\left(v_i^{j+1}v_i^{j+2}\right)$$
$$+\xi\left(v_i^{j+1}v_{i+1}^{j+1}\right) + \xi\left(v_{i+1}^{j+1}v_{i+1}^{j+2}\right) + \xi\left(v_i^{j+2}v_{i+1}^{j+2}\right) - \xi\left(v_i^{j}\right) - \xi\left(v_i^{j+1}\right) - \xi\left(v_{i+1}^{j}\right) - \xi\left(v_{i+1}^{j+1}\right)$$
$$-\xi\left(v_i^{j}v_i^{j+1}\right) - \xi\left(v_i^{j}v_{i+1}^{j}\right) - \xi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right) - \xi\left(v_i^{j+1}v_{i+1}^{j+1}\right) + \xi\left(f_i^{j+1}\right) - \xi\left(f_i^{j}\right).$$
$$W_{\xi(1,1,1)}\left(f_i^{j+1}\right) - W_{\xi(1,1,1)}\left(f_i^{j}\right) = \rho + \rho + \rho + \rho + 1$$
$$+\left\lfloor \frac{1}{2}\left(t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3\right)\right\rfloor \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 1\right)\right\rceil$$
$$+\left\lceil \frac{1}{2}\left(t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3\right)\right\rceil \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil\right)\right\rceil$$
$$+j + 1 - \rho - \rho - \rho - \rho - 1$$
$$-\left\lfloor \frac{1}{2}\left(t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3\right)\right\rfloor \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 1\right)\right\rceil$$
$$-\left\lceil \frac{1}{2}\left(t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3\right)\right\rceil \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil\right)\right\rceil - j.$$
$$= 1.$$

For $i = 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 2, \cdots, 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 1 + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil}{\lceil \frac{t}{2} \rceil} \right\rfloor$; $j = 1, 2, \ldots, t.$

$$W_{\xi(1,1,1)}\left(f_i^{j+1}\right) - W_{\xi(1,1,1)}\left(f_i^{j}\right) = \xi\left(v_i^{j+1}\right) + \xi\left(v_i^{j+2}\right) + \xi\left(v_{i+1}^{j+1}\right) + \xi\left(v_{i+1}^{j+2}\right) + \xi\left(v_i^{j+1}v_i^{j+2}\right)$$
$$+ \xi\left(v_i^{j+1}v_{i+1}^{j+1}\right) + \xi\left(v_{i+1}^{j+1}v_{i+1}^{j+2}\right) + \xi\left(v_i^{j+2}v_{i+1}^{j+2}\right) - \xi\left(v_i^{j}\right) - \xi\left(v_i^{j+1}\right) - \xi\left(v_{i+1}^{j}\right) - \xi\left(v_{i+1}^{j+1}\right)$$
$$- \xi\left(v_i^{j}v_i^{j+1}\right) - \xi\left(v_i^{j}v_{i+1}^{j}\right) - \xi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right) - \xi\left(v_i^{j+1}v_{i+1}^{j+1}\right) + \xi\left(f_i^{j+1}\right) - \xi\left(f_i^{j}\right).$$
$$W_{\xi(1,1,1)}\left(f_i^{j+1}\right) - W_{\xi(1,1,1)}\left(f_i^{j}\right) = \rho + \rho + \rho + \rho + 1 + \left\lfloor \frac{t}{2} \right\rfloor \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil - 1\right) \right\rceil$$
$$+ \left\lceil \frac{t}{2} \right\rceil \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil - 2\right) \right\rceil + t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3 + j + 1$$
$$- \rho - \rho - \rho - \rho - 1 - \left\lfloor \frac{t}{2} \right\rfloor \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil - 1\right) \right\rceil$$
$$- \left\lceil \frac{t}{2} \right\rceil \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil - 2\right) \right\rceil - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil - 3\left\lfloor \frac{t+1}{3} \right\rfloor + t + 3\rho - 3 - j.$$
$$= 1.$$

For $i = 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil}{\lceil \frac{t}{2} \rceil} \right\rfloor + 2$; $j = 1, 2, \ldots, t.$

$$W_{\xi(1,1,1)}\left(f_i^{j+1}\right) - W_{\xi(1,1,1)}\left(f_i^{j}\right) = \xi\left(v_i^{j+1}\right) + \xi\left(v_i^{j+2}\right) + \xi\left(v_{i+1}^{j+1}\right) + \xi\left(v_{i+1}^{j+2}\right) + \xi\left(v_i^{j+1}v_i^{j+2}\right)$$
$$+ \xi\left(v_i^{j+1}v_{i+1}^{j+1}\right) + \xi\left(v_{i+1}^{j+1}v_{i+1}^{j+2}\right) + \xi\left(v_i^{j+2}v_{i+1}^{j+2}\right) - \xi\left(v_i^{j}\right) - \xi\left(v_i^{j+1}\right) - \xi\left(v_{i+1}^{j}\right) - \xi\left(v_{i+1}^{j+1}\right)$$
$$- \xi\left(v_i^{j}v_i^{j+1}\right) - \xi\left(v_i^{j}v_{i+1}^{j}\right) - \xi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right) - \xi\left(v_i^{j+1}v_{i+1}^{j+1}\right) + \xi\left(f_i^{j+1}\right) - \xi\left(f_i^{j}\right).$$
$$W_{\xi(1,1,1)}\left(f_i^{j+1}\right) - W_{\xi(1,1,1)}\left(f_i^{j}\right) = \rho + \rho + \rho + \rho + \rho + j + 1$$
$$+ \left(9 + t\left(2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil}{\lceil \frac{t}{2} \rceil} \right\rfloor + 1\right) - 8\rho - 1\right) - \rho - \rho - \rho$$
$$- \rho - \rho - j - \left(9 + t\left(2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil}{\lceil \frac{t}{2} \rceil} \right\rfloor + 1\right) - 8\rho - 1\right).$$
$$= 1.$$

For $i = 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil}{\lceil \frac{t}{2} \rceil} \right\rfloor + 3, \cdots, s\,;\ j = 1, 2, \ldots, t.$

$$W_{\xi(1,1,1)}\left(f_i^{j+1}\right) - W_{\xi(1,1,1)}\left(f_i^{j}\right) = \xi\left(v_i^{j+1}\right) + \xi\left(v_i^{j+2}\right) + \xi\left(v_{i+1}^{j+1}\right) + \xi\left(v_{i+1}^{j+2}\right) + \xi\left(v_i^{j+1}v_i^{j+2}\right)$$
$$+ \xi\left(v_i^{j+1}v_{i+1}^{j+1}\right) + \xi\left(v_{i+1}^{j+1}v_{i+1}^{j+2}\right) + \xi\left(v_i^{j+2}v_{i+1}^{j+2}\right) - \xi\left(v_i^{j}\right) - \xi\left(v_i^{j+1}\right) - \xi\left(v_{i+1}^{j}\right) - \xi\left(v_{i+1}^{j+1}\right)$$
$$- \xi\left(v_i^{j}v_i^{j+1}\right) - \xi\left(v_i^{j}v_{i+1}^{j}\right) - \xi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right) - \xi\left(v_i^{j+1}v_{i+1}^{j+1}\right) + \xi\left(f_i^{j+1}\right) - \xi\left(f_i^{j}\right).$$
$$W_{\xi(1,1,1)}\left(f_i^{j+1}\right) - W_{\xi(1,1,1)}\left(f_i^{j}\right) = \rho + \rho + \rho + \rho + \rho + j + 1 + 9$$
$$+ t\left(2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil}{\lceil \frac{t}{2} \rceil} \right\rfloor + 1\right) - 8\rho - 1$$
$$+ t\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil - \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil}{\lceil \frac{t}{2} \rceil} \right\rfloor - 2\right) - \rho - \rho - \rho - \rho - \rho - j$$
$$- \left(9 + t\left(2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil}{\lceil \frac{t}{2} \rceil} \right\rfloor + 1\right) - 8\rho - 1\right).$$
$$= 1.$$

Now let us move to the calculation of vertical weights differences.

For $i = 1,2,3,\ldots,2\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil - 2\ ;\ j = 1,2,\ldots,t.$

$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right) - W_{\xi(1,1,1)}\left(f_{i}^{j}\right) = \xi\left(v_{i+1}^{j}\right) + \xi\left(v_{i+1}^{j+1}\right) + \xi\left(v_{i+2}^{j}\right) + \xi\left(v_{i+2}^{j+1}\right) + \xi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right)$
$+\xi\left(v_{i+1}^{j}v_{i+2}^{j}\right) + \xi\left(v_{i+2}^{j}v_{i+2}^{j+1}\right) + \xi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right) + \xi\left(f_{i+1}^{j}\right) - \xi\left(v_{i}^{j}\right) - \xi\left(v_{i}^{j+1}\right) - \xi\left(v_{i+1}^{j}\right)$
$-\xi\left(v_{i+1}^{j+1}\right) - \xi\left(v_{i}^{j}v_{i}^{j+1}\right) - \xi\left(v_{i}^{j}v_{i+1}^{j}\right) - \xi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right) - \xi\left(v_{i}^{j+1}v_{i+1}^{j+1}\right) - \xi\left(f_{i}^{j}\right).$
$= 1 + \left\lfloor\frac{i+1}{2}\right\rfloor\left\lfloor\frac{t+1}{3}\right\rfloor + 1 + \left\lfloor\frac{i+1}{2}\right\rfloor\left\lfloor\frac{t+1}{3}\right\rfloor + 1 + 1 + \left\lfloor\frac{i+1}{2}\right\rfloor\left\lfloor\frac{t+1}{3}\right\rfloor + 1 + j$
$-1 - \left\lfloor\frac{i-1}{2}\right\rfloor\left\lfloor\frac{t+1}{3}\right\rfloor - 1 - \left\lfloor\frac{i-1}{2}\right\rfloor\left\lfloor\frac{t+1}{3}\right\rfloor - 1 - \left\lfloor\frac{i-1}{2}\right\rfloor\left\lfloor\frac{t+1}{3}\right\rfloor - 1 - 1 - j$
$= 3\left\lfloor\frac{i+1}{2}\right\rfloor\left\lfloor\frac{t+1}{3}\right\rfloor - 3\left\lfloor\frac{i-1}{2}\right\rfloor\left\lfloor\frac{t+1}{3}\right\rfloor.$
$= 3\left\lfloor\frac{t+1}{3}\right\rfloor\left(\left\lfloor\frac{i+1}{2}\right\rfloor - \left\lfloor\frac{i-1}{2}\right\rfloor\right)$

Since $\left(\left\lfloor\frac{i+1}{2}\right\rfloor - \left\lfloor\frac{i-1}{2}\right\rfloor\right) = 1,$ for every value of $i,$ so we have
$= 3\left\lfloor\frac{t+1}{3}\right\rfloor.$
$= t.$

For $i = 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil - 1\ ;\ j = 1, 2, \ldots, t.$

$$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right) - W_{\xi(1,1,1)}\left(f_{i}^{j}\right) = \xi\left(v_{i+1}^{j}\right) + \xi\left(v_{i+1}^{j+1}\right) + \xi\left(v_{i+2}^{j}\right) + \xi\left(v_{i+2}^{j+1}\right) + \xi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right)$$
$$+ \xi\left(v_{i+1}^{j}v_{i+2}^{j}\right) + \xi\left(v_{i+2}^{j}v_{i+2}^{j+1}\right) + \xi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right) + \xi\left(f_{i+1}^{j}\right) - \xi\left(v_{i}^{j}\right) - \xi\left(v_{i}^{j+1}\right) - \xi\left(v_{i+1}^{j}\right)$$
$$- \xi\left(v_{i+1}^{j+1}\right) - \xi\left(v_{i}^{j}v_{i}^{j+1}\right) - \xi\left(v_{i}^{j}v_{i+1}^{j}\right) - \xi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right) - \xi\left(v_{i}^{j+1}v_{i+1}^{j+1}\right) - \xi\left(f_{i}^{j}\right).$$

$$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right) - W_{\xi(1,1,1)}\left(f_{i}^{j}\right) = \rho + \rho + \rho + 1 + 1$$
$$+ \left\lceil \frac{1}{2}\left(i + 1 - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 1\right) \right\rceil$$
$$\left\{ \left\lfloor \frac{\left(t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3\right)}{2} \right\rfloor + \left\lceil \frac{\left(t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3\right)}{2} \right\rceil \right\}$$
$$+ j - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - 1 - j.$$
$$= 3\rho + 2 + (1)\left(t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3\right) - 5 - 3\left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor.$$
$$= 3\rho + 2 + t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3 - 5 - 3\left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor.$$
$$= t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor.$$
$$= t.$$

For $i = 2\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil$; $j = 1, 2, \ldots, t$.

$$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right) - W_{\xi(1,1,1)}\left(f_{i}^{j}\right) = \xi\left(v_{i+1}^{j}\right) + \xi\left(v_{i+1}^{j+1}\right) + \xi\left(v_{i+2}^{j}\right) + \xi\left(v_{i+2}^{j+1}\right) + \xi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right)$$
$$+ \xi\left(v_{i+1}^{j}v_{i+2}^{j}\right) + \xi\left(v_{i+2}^{j}v_{i+2}^{j+1}\right) + \xi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right) + \xi\left(f_{i+1}^{j}\right) - \xi\left(v_{i}^{j}\right) - \xi\left(v_{i}^{j+1}\right) - \xi\left(v_{i+1}^{j}\right)$$
$$- \xi\left(v_{i+1}^{j+1}\right) - \xi\left(v_{i}^{j}v_{i}^{j+1}\right) - \xi\left(v_{i}^{j}v_{i+1}^{j}\right) - \xi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right) - \xi\left(v_{i}^{j+1}v_{i+1}^{j+1}\right) - \xi\left(f_{i}^{j}\right).$$

$$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right) - W_{\xi(1,1,1)}\left(f_{i}^{j}\right) = \rho + \rho + \rho + 2 + \left\lceil \frac{1}{2}\left(i + 1 - 2\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil + 1\right) \right\rceil$$
$$\left\{ \left\lfloor \frac{\left(t\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil + 3\lfloor\frac{t+1}{3}\rfloor - t - 3\rho + 3\right)}{2} \right\rfloor + \left\lceil \frac{\left(t\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil + 3\lfloor\frac{t+1}{3}\rfloor - t - 3\rho + 3\right)}{2} \right\rceil \right\}$$
$$+ \left\lceil \frac{1}{2}\left(i + 1 - 2\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil\right) \right\rceil$$
$$\left\{ \left\lceil \frac{\left(t\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil + 3\lfloor\frac{t+1}{3}\rfloor - t - 3\rho + 3\right)}{2} \right\rceil + \left\lfloor \frac{\left(t\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil + 3\lfloor\frac{t+1}{3}\rfloor - t - 3\rho + 3\right)}{2} \right\rfloor \right\}$$
$$+ j - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 2$$
$$- \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil + 1\right) \right\rceil$$
$$\left\{ \left\lfloor \frac{\left(t\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil + 3\lfloor\frac{t+1}{3}\rfloor - t - 3\rho + 3\right)}{2} \right\rfloor + \left\lceil \frac{\left(t\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil + 3\lfloor\frac{t+1}{3}\rfloor - t - 3\rho + 3\right)}{2} \right\rceil \right\}$$
$$- \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil\right) \right\rceil$$
$$\left\{ \left\lceil \frac{\left(t\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil + 3\lfloor\frac{t+1}{3}\rfloor - t - 3\rho + 3\right)}{2} \right\rceil + \left\lfloor \frac{\left(t\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil + 3\lfloor\frac{t+1}{3}\rfloor - t - 3\rho + 3\right)}{2} \right\rfloor \right\}$$
$$- j.$$
$$= 3\rho + (1)\left(t\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3\right)$$
$$+ (1)\left(t\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3\right) - 3 - 3\left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor$$
$$- (1)\left(t\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3\right).$$
$$= 3\rho + t\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3 + t\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3$$
$$- 3 - 3\left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - t\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil - 3\left\lfloor \frac{t+1}{3} \right\rfloor + t + 3\rho - 3.$$
$$= t\left\lceil \frac{\rho}{\lfloor\frac{t+1}{3}\rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor.$$

For $i=2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+1\ ;\ j=1,2,\ldots,t.$

$$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right)-W_{\xi(1,1,1)}\left(f_{i}^{j}\right)=\xi\left(v_{i+1}^{j}\right)+\xi\left(v_{i+1}^{j+1}\right)+\xi\left(v_{i+2}^{j}\right)+\xi\left(v_{i+2}^{j+1}\right)+\xi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right)$$
$$+\xi\left(v_{i+1}^{j}v_{i+2}^{j}\right)+\xi\left(v_{i+2}^{j}v_{i+2}^{j+1}\right)+\xi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right)+\xi\left(f_{i+1}^{j}\right)-\xi\left(v_{i}^{j}\right)-\xi\left(v_{i}^{j+1}\right)-\xi\left(v_{i+1}^{j}\right)$$
$$-\xi\left(v_{i+1}^{j+1}\right)-\xi\left(v_{i}^{j}v_{i}^{j+1}\right)-\xi\left(v_{i}^{j}v_{i+1}^{j}\right)-\xi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right)-\xi\left(v_{i}^{j+1}v_{i+1}^{j+1}\right)-\xi\left(f_{i}^{j}\right).$$
$$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right)-W_{\xi(1,1,1)}\left(f_{i}^{j}\right)=\rho+\rho+\rho+2+\left\lceil\frac{1}{2}\left(i+1-2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil-1\right)\right\rceil\left\{\left\lfloor\frac{t}{2}\right\rfloor+\left\lceil\frac{t}{2}\right\rceil\right\}$$
$$+\left\lceil\frac{1}{2}\left(i+1-2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil-2\right)\right\rceil\left\{\left\lceil\frac{t}{2}\right\rceil+\left\lfloor\frac{t}{2}\right\rfloor\right\}+2t\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+6\left\lfloor\frac{t+1}{3}\right\rfloor-2t-6\rho+6$$
$$+j-\rho-\rho-\rho-2-\left\lceil\frac{1}{2}\left(i-2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+1\right)\right\rceil$$
$$\left\{\left\lfloor\frac{\left(t\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+3\left\lfloor\frac{t+1}{3}\right\rfloor-t-3\rho+3\right)}{2}\right\rfloor+\left\lceil\frac{\left(t\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+3\left\lfloor\frac{t+1}{3}\right\rfloor-t-3\rho+3\right)}{2}\right\rceil\right\}$$
$$-\left\lceil\frac{1}{2}\left(i-2\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil\right)\right\rceil$$
$$\left\{\left\lceil\frac{\left(t\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+3\left\lfloor\frac{t+1}{3}\right\rfloor-t-3\rho+3\right)}{2}\right\rceil+\left\lfloor\frac{\left(t\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+3\left\lfloor\frac{t+1}{3}\right\rfloor-t-3\rho+3\right)}{2}\right\rfloor\right\}-j.$$
$$=3\rho+2+(1)\,t+0+2t\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+6\left\lfloor\frac{t+1}{3}\right\rfloor-2t-6\rho+6-3\rho-2$$
$$-(1)\left(t\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+3\left\lfloor\frac{t+1}{3}\right\rfloor-t-3\rho+3\right)-(1)\left(t\left\lceil\frac{\rho}{\lfloor\frac{t+1}{3}\rfloor}\right\rceil+3\left\lfloor\frac{t+1}{3}\right\rfloor-t-3\rho+3\right).$$
$$=3\rho+2+t-3\rho-2.$$
$$=t.$$

For $i = 2\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil}{\left\lceil \frac{t}{2} \right\rceil} \right\rfloor + 2$; $j = 1, 2, \ldots, t.$

$$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right) - W_{\xi(1,1,1)}\left(f_{i}^{j}\right) = \xi\left(v_{i+1}^{j}\right) + \xi\left(v_{i+1}^{j+1}\right) + \xi\left(v_{i+2}^{j}\right) + \xi\left(v_{i+2}^{j+1}\right) + \xi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right)$$
$$+ \xi\left(v_{i+1}^{j}v_{i+2}^{j}\right) + \xi\left(v_{i+2}^{j}v_{i+2}^{j+1}\right) + \xi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right) + \xi\left(f_{i+1}^{j}\right) - \xi\left(v_{i}^{j}\right) - \xi\left(v_{i}^{j+1}\right) - \xi\left(v_{i+1}^{j}\right)$$
$$- \xi\left(v_{i+1}^{j+1}\right) - \xi\left(v_{i}^{j}v_{i}^{j+1}\right) - \xi\left(v_{i}^{j}v_{i+1}^{j}\right) - \xi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right) - \xi\left(v_{i}^{j+1}v_{i+1}^{j+1}\right) - \xi\left(f_{i}^{j}\right).$$
$$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right) - W_{\xi(1,1,1)}\left(f_{i}^{j}\right) = \rho + \rho + \rho + \rho + \rho + j$$
$$+ \left(9 + t\left(2\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil}{\left\lceil \frac{t}{2} \right\rceil} \right\rfloor + 1\right) - 8\rho - 1\right) - \rho - \rho - \rho - \rho$$
$$- \rho - j - \left(9 + t\left(2\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil}{\left\lceil \frac{t}{2} \right\rceil} \right\rfloor\right) - 8\rho - 1\right).$$
$$= 5\rho + j + \left(9 + t\left(2\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil}{\left\lceil \frac{t}{2} \right\rceil} \right\rfloor + 1 + 1 - 1\right) - 8\rho - 1\right)$$
$$- 5\rho - j - \left(9 + t\left(2\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil}{\left\lceil \frac{t}{2} \right\rceil} \right\rfloor + 1 - 1\right) - 8\rho - 1\right)$$
$$= (9 + t(i + 1 - 1) - 8\rho - 1) - (9 + t(i - 1) - 8\rho - 1).$$
$$= (9 + t(i) - 8\rho - 1) - (9 + t(i - 1) - 8\rho - 1).$$
$$= 9 + mi - 8\rho - 1 - 9 - mi + t + 8\rho + 1.$$
$$= t.$$

For $i = 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil}{\lceil \frac{t}{2} \rceil} \right\rfloor + 3, \cdots, s\ ;\ j = 1, 2, \ldots, t.$

$$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right) - W_{\xi(1,1,1)}\left(f_{i}^{j}\right) = \xi\left(v_{i+1}^{j}\right) + \xi\left(v_{i+1}^{j+1}\right) + \xi\left(v_{i+2}^{j}\right) + \xi\left(v_{i+2}^{j+1}\right) + \xi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right)$$
$$+ \xi\left(v_{i+1}^{j}v_{i+2}^{j}\right) + \xi\left(v_{i+2}^{j}v_{i+2}^{j+1}\right) + \xi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right) + \xi\left(f_{i+1}^{j}\right) - \xi\left(v_{i}^{j}\right) - \xi\left(v_{i}^{j+1}\right) - \xi\left(v_{i+1}^{j}\right)$$
$$- \xi\left(v_{i+1}^{j+1}\right) - \xi\left(v_{i}^{j}v_{i}^{j+1}\right) - \xi\left(v_{i}^{j}v_{i+1}^{j}\right) - \xi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right) - \xi\left(v_{i}^{j+1}v_{i+1}^{j+1}\right) - \xi\left(f_{i}^{j}\right)$$
$$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right) - W_{\xi(1,1,1)}\left(f_{i}^{j}\right) = \xi\left(v_{i+2}^{j}\right) + \xi\left(v_{i+2}^{j+1}\right) + \xi\left(v_{i+1}^{j}v_{i+2}^{j}\right) + \xi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right) + \xi\left(v_{i+2}^{j}v_{i+2}^{j+1}\right)$$
$$+ \xi\left(f_{i+1}^{j}\right) - \xi\left(v_{i}^{j}\right) - \xi\left(v_{i}^{j+1}\right) - \xi\left(v_{i}^{j}v_{i}^{j+1}\right) - \xi\left(v_{i}^{j}v_{i+1}^{j}\right) - \xi\left(v_{i}^{j+1}v_{i+1}^{j+1}\right) - \xi\left(f_{i}^{j}\right).$$
$$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right) - W_{\xi(1,1,1)}\left(f_{i}^{j}\right) = \rho + \rho + \rho + \rho + \rho + j + 9$$
$$+ t\left(2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil}{\lceil \frac{t}{2} \rceil} \right\rfloor + 1\right) - 8\rho - 1$$
$$+ t\left(i + 1 - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil - \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil}{\lceil \frac{t}{2} \rceil} \right\rfloor - 2\right) - \rho - \rho - \rho - \rho - \rho - j - 9$$
$$- t\left(2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil}{\lceil \frac{t}{2} \rceil} \right\rfloor + 1\right) + 8\rho + 1$$
$$- t\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil - \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil}{\lceil \frac{t}{2} \rceil} \right\rfloor - 2\right).$$
$$= t\left(i + 1 - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil - \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil}{\lceil \frac{t}{2} \rceil} \right\rfloor - 2\right)$$
$$- t\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil - \left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil}{\lceil \frac{t}{2} \rceil} \right\rfloor - 2\right).$$
$$= mi + t - 2t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil - t\left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil}{\lceil \frac{t}{2} \rceil} \right\rfloor - 2t - mi + 2t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil$$
$$+ t\left\lfloor \frac{4\rho - t - t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil}{\lceil \frac{t}{2} \rceil} \right\rfloor + 2t.$$
$$= t.$$

□

It can be clearly seen that the conditions regarding differences in weights are fulfilled and so we have

$$Efs_{(1,1,1)}(G_s^t) = \left\lceil \frac{mn+8}{9} \right\rceil.$$

Example 2.7. Investigate the exact value of entire FIS of G_{18}^{15} by using $\rho-$lableing of type $(1, 1, 1)$.

Proof. We will label a grid graph G_{18}^{15} under the supervision of generalized results about vertices, horizontal edges, vertical edges and faces.

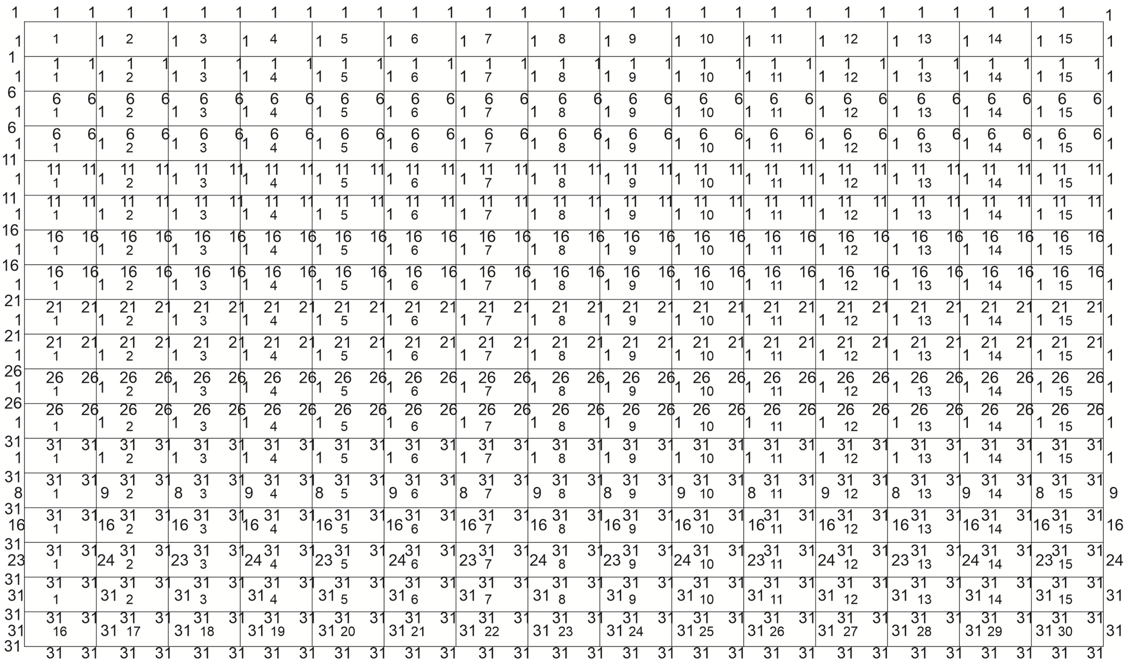

Figure1
Entire Face Irregularity Strength of G_{18}^{15} under 31−labeling of type $(1, 1, 1)$

In G_{18}^{15}, the value of s is 18 and the value of t is 15. From Figure1, we see that the minimum integer for which the face weights are distinct is 31. So, $\rho = 31$ and hence, we have

$$\left\lfloor \frac{t}{2} \right\rfloor = \left\lfloor \frac{15}{2} \right\rfloor = 7 \ , \left\lceil \frac{t}{2} \right\rceil = \left\lceil \frac{15}{2} \right\rceil = 8 \ , \left\lfloor \frac{t+1}{3} \right\rfloor = \left\lfloor \frac{16}{3} \right\rfloor = 5 \ , \left\lceil \frac{\rho}{\left\lfloor \frac{t+1}{3} \right\rfloor} \right\rceil = \left\lceil \frac{31}{5} \right\rceil = 7.$$

To investigate the exact value of $\rho = Efs_{(1,1,1)}(G_{18}^{15})$, we will calculate the horizontal and vertical differences in face weights.

Horizontal differences in face weights for different intervals of i and j can be calculated by the following method.

For $i = 1, 2, 3, \dots, 12,\ ;\ j = 1, 2, \dots, 15.$

$$W_{\xi(1,1,1)}\left(f_i^{j+1}\right) - W_{\xi(1,1,1)}\left(f_i^{j}\right) = 1 + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + 1 + \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + 1$$
$$+ \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + 1 + \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + 1 + j + 1 - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor$$
$$- 1 - \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - j.$$

For $i = 1$, we have

$$W_{\xi(1,1,1)}\left(f_i^{j+1}\right) - W_{\xi(1,1,1)}\left(f_i^{j}\right) = 1 + 0 + 1 + 0 + 1 + 0 + 1 + 0 + 1 + j + 1 - 1$$
$$- 0 - 1 - 0 - 1 - 0 - 1 - 0 - 1 - j.$$
$$= 1.$$

For $i = 2$, we have

$$W_{\xi(1,1,1)}\left(f_i^{j+1}\right) - W_{\xi(1,1,1)}\left(f_i^{j}\right) = 1 + 0 + 1 + 5 + 1 + 0 + 1 + 5 + 1 + j + 1 - 1 - 0$$
$$- 1 - 5 - 1 - 0 - 1 - 5 - 1 - j.$$
$$= 1.$$

Similarly for $i = 3, 4, 5, 6, 7, 8, 9, 10, 11$ and 12, we have

$$W_{\xi(1,1,1)}\left(f_i^{j+1}\right) - W_{\xi(1,1,1)}\left(f_i^{j}\right) = 1.$$

For $i = 13\ ;\ j = 1, 2, \dots, 15.$

$$W_{\xi(1,1,1)}\left(f_i^{j+1}\right) - W_{\xi(1,1,1)}\left(f_i^{j}\right) =$$
$$= 1 + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + 1 + \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + 1 + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + 1 + \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor$$
$$+ 1 + j + 1 - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1$$
$$- \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - j.$$
$$= 1 + (6)(5) + 1 + (6)(5) + 1 + (6)(5) + 1 + (6)(5) + 1 + j + 1 - 1 - (6)(5) - 1 - (6)(5).$$
$$- 1 - (6)(5) - 1 - (6)(5) - 1 - j.$$
$$= 1.$$

For $i = 14$; $j = 1, 2, \dots, 15$.

$$W_{\xi(1,1,1)}\left(f_i^{j+1}\right) - W_{\xi(1,1,1)}\left(f_i^{j}\right) =$$

$$= 1 + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + \rho + 1 + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + \rho + 1$$

$$+ \left\lfloor \frac{1}{2}\left(t \left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3 \right) \right\rfloor \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 1 \right) \right\rceil$$

$$+ \left\lceil \frac{1}{2}\left(t \left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3 \right) \right\rceil \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil \right) \right\rceil$$

$$+ j + 1 - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - \rho - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - \rho - 1$$

$$- \left\lfloor \frac{1}{2}\left(t \left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3 \right) \right\rfloor \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 1 \right) \right\rceil$$

$$- \left\lceil \frac{1}{2}\left(t \left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3 \right) \right\rceil \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil \right) \right\rceil - j.$$

$$= 1 + (6)(5) + 31 + 1 + (6)(5) + 31 + 1 + (7)(1) + 0 + j + 1 - 1 - (6)(5) - \rho$$

$$- 1 - (6)(5) - 31 - 1 - (7)(1) - 0 - j.$$

$$= 1.$$

By using the similar method, reader can verify the upcoming horizontal differences in face weights, the answer will always be 1 for all values of i and j.
Now we investigate the vertical differences in face weights by the following method.

For $i = 1, 2, 3, \dots, 12$; $j = 1, 2, \dots, 15$.

$$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right) - W_{\xi(1,1,1)}\left(f_i^{j}\right)$$

$$= 1 + \left\lfloor \frac{i+1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + 1 + \left\lfloor \frac{i+1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + 1 + 1 + \left\lfloor \frac{i+1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor + 1 + j$$

$$- 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - 1 - j.$$

For $i = 1$, we have

$$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right) - W_{\xi(1,1,1)}\left(f_i^{j}\right)$$

$$= 1 + (1)(5) + 1 + (1)(5) + 1 + 1 + (1)(5) + 1 + j - 1 - 0 - 1 - 0 - 1 - 0 - 1 - 1 - j.$$

$$= 15.$$

$$= t.$$

For $i = 2$, we have

$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right) - W_{\xi(1,1,1)}\left(f_{i}^{j}\right)$

$= 1 + (1)(5) + 1 + (1)(5) + 1 + 1 + (1)(5) + 1 + j - 1 - 0 - 1 - 0 - 1 - 0 - 1 - 1 - j.$

$= 15.$

$= t.$

For $i = 3$, we have

$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right) - W_{\xi(1,1,1)}\left(f_{i}^{j}\right)$

$= 1 + (2)(5) + 1 + (2)(5) + 1 + 1 + (2)(5) + 1 + j - 1 - (1)(5) - 1 - (1)(5) - 1 - (1)(5)$

$- 1 - 1 - j.$

$= 15.$

$= t.$

Similarly for $i = 4, 5, 6, 7, 8, 9, 10, 11$ and 12, we have

$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right) - W_{\xi(1,1,1)}\left(f_{i}^{j}\right) = 15.$

For $i = 13$; $j = 1, 2, \ldots, 15.$

$$W_{\xi(1,1,1)}\left(f_{i+1}^{j}\right) - W_{\xi(1,1,1)}\left(f_{i}^{j}\right) = \rho + \rho + \rho + 1 + 1 + \left\lceil \frac{1}{2}\left(i + 1 - 2\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 1\right)\right\rceil$$

$$\left\{\left\lfloor \frac{\left(t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3\right)}{2} \right\rfloor + \left\lceil \frac{\left(t\left\lceil \frac{\rho}{\lfloor \frac{t+1}{3} \rfloor} \right\rceil + 3\left\lfloor \frac{t+1}{3} \right\rfloor - t - 3\rho + 3\right)}{2} \right\rceil\right\}$$

$$+ j - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{t+1}{3} \right\rfloor - 1 - 1 - j.$$

$= 31 + 31 + 31 + 1 + 1 + (1)(15) + j - 1 - (6)(5) - 1 - (6)(5) - 1 - (6)(5) - 1 - 1 - j.$

$= 15.$

Similarly, the upcoming differences can be calculated. The answer will always be t for all values of i and j. Hence, the horizontal and vertical differences of face weights are verified and so the exact value of $Efs_{(1,1,1)}(G_{18}^{15}) = 31$. □

3. Conclusion

In this paper, we have applied new graph parameters $(\alpha_1, \beta_1, \gamma_1)$ on grid graphs. Problems in this article are based on face labeling of plane graphs. We investigated the tight lower bound for the face irregular strength of vertex $(1, 0, 0)$, edge $(0, 1, 0)$, face $(0, 0, 1)$, vertex-face $(1, 0, 1)$, edge-face $(0, 1, 1)$ and vertex-edge-face $(1, 1, 1)$ of G_s^t with the help of $\rho-$lableing of class $(\alpha_1, \beta_1, \gamma_1)$ under some labeling conditions. Since a plane graph can be labelled in many ways but we focus on obtaining a minimum number from the set of integers, so in this kind of calculation, graphs may undergo some conditions of labeling. We generalized formulas to prove results and applied results to solve examples. The results of labeling depend on the size of the graph. Sometimes, it happens that graphs do not provide exact results. In order to get exact and precise results, we recommend the reader to work on bigger graphs. Face irregularity strength of graphs under $\rho-$labeling of type $(\alpha_1, \beta_1, \gamma_1)$ is an emerging topic in recent times. Many authors like Martin Baca, Aleem Mughal and Noshad jamil have done valuable work on face labeling of graphs [15, 21, 22].

References

[1] A. Rosa, *On certain valuations of the vertices of a graph*, Journal of Graph Theory. **01** (1967), 349–355. 1

[2] Nierhoff T. *A tight bound on the irregularity strength of graphs*, SIAM J. Discrete Math. **13** (2000), 313–323.

[3] Frieze A, Gould RJ, Karonski M, Pfender F. *On graph irregularity strength*, J. Graph Theory. **41** (2002), 120–137. 1, 1

[4] Bača M, Jendrol' S, Miller M, Ryan J. *On irregular total labellings*, Discrete Math. **307** (2007), 1378-–1388. 1

[5] Anholcer M, Kalkowski M, Przybylo J. *A new upper bound for the total vertex irregularity strength of graphs*, Discrete Math. **309** (2009), 6316–6317. 1

[6] Jendrol S, Miskuf J, Sotak R. *Total edge irregularity strength of complete graphs and complete bipartite graphs*, Discrete Math. **310** (2010), 400–407. 1

[7] Nurdin ET, Baskoro, ANM, Salman NN, Gaos. *On the total vertex irregularity strength of trees*, Discrete Math. **310** (2010), 3043–3048. 1

[8] Kalkowski M, Karonski M, Pfender F. *A new upper bound for the irregularity strength of graphs*, SIAM J. Discrete Math. **25** (2011), 1319–1321.

[9] Mominul K, Haque M. *Irregular total labellings of generalized Petersen graphs*, Theory Comput. Syst. **50** (2012), 537–544.

[10] Ahmad A, Al-Mushayt OBS, Bača M. *On edge irregularity strength of graphs*, Appl. Math. Comput. **243** (2014), 607–610. 1

[11] Bača M, Jendrol S, Kathiresan K, Muthugurupackiam K. *Entire labelling of plane graphs*, Appl. Math. Inf. Sci. **9** (2015), 263-–267.

[12] Chen, Z, Virk, A. R, Habib, M, Zia, T, Ahmed, I, Shi, C, Nazeer, W. *Irregularity indices of dendrimer structures used as molecular disrupter in QSAR study.*, Journal of Chemistry, (2019), 1-21.

[13] Bača M, Ovais A, Semaničová-Feňovčíková A, Suparta N. *On face irregular evaluations of plane graphs*, Discussiones Mathematicae Graph Theory. In press, doi:10.7151/dmgt.2294. 1

[14] Chartrand G. Jacobson MS, Lehel J, Oellermann OR, Ruiz S, Saba F. *Irregular networks*, Congr. Numer. **64** (1988), 187–192. 1

[15] Bača M, Jendrol S, Kathiresan K, Muthugurupackiam K, Semaničová-Feňovčíková A. *A survey of irregularity strength*, Electron. Notes Discrete Math. **48** (2015), 19—26. 1, 3

[16] Virk, A. R, Rehman, M. A, Shi, C, Nazeer, W. *Useful Irregularity Indices in QSPR Study for Bismuth Tri-Iodide.*, Journal of Chemistry, (2019), 1-17.

[17] Butt S.I, Numan M, Shah I.A, Ali S. *Face labelings of class (1,1,1) for generalized prism*, ARS Combinatoria. **137** (2018), 41–52. 1

[18] Butt S.I, Numan M, Qaisar S. *Labelings of class (1,1,1) for Klein bottle fullerenes*, J. Math. Chem. **54** (2016), 428–441. 1

[19] Imran M, Siddiqui M.K, Numan M. *Super d-antimagic labelling of uniform subdivision of wheel*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **77** (2015), 227–240. 1

[20] Gallian J. *A dynamic survey of graph labelling*, Electron. J. Comb. **16** (2009), 1-–442.

[21] Aleem A, Jamil RN. *Total Face Irregularity Strength of Grid and Wheel Graph under K-Labeling of Type (1, 1, 0)*, Journal of Mathematics. Article ID 1311269 (2021). 1, 1, 1, 3

[22] Aleem A, Jamil RN, Rehman AU, Cancan M. *Real Life Applications of Tight Face Irregularity Strength under Wheel Graphs Including Vertices, Edges and Faces*, Journal of Tianjin University Science and Technology. **56** (2023), Issue: 04:2023 DOI10.17605/OSF.IO/7D64K.