Available online at http://jprm.sms.edu.pk/
Journal of Prime Research in Mathematics, 20(1) (2024), 1–14

Journal of Prime Research in Mathematics

Numerical solution of a multi-wing chaotic system with piecewise differential operators

Mehmet Akif Cetin[a,*], Selahattin Genc[b], Metin Araz[c]

[a]*ALTSO Vocational School, Alanya Alaaddin Keykubat University, Antalya, Turkey*
[b]*Yakutiye Public Education center, Erzurum, Turkey*
[c]*Siirt High School, Siirt, Turkey*

Abstract

In this study, a multi-wing chaotic system with classical derivative has been studied. The conditions under which the existence and uniqueness of the solution of this chaotic system exist are examined. Afterwards, this chaotic system has been modified using fractional differential operators, and in this case the behavior of the multi-wing chaotic system has been investigated. Moreover, the newly introduced piecewise differential operators is included in such a chaotic system and the piecewise chaotic system is solved by using Newton polynomial approach. The numerical simulations of piecewise chaotic system are performed for fractional order.

Keywords: Numerical scheme, multi-wing chaotic system, fractional calculus

1. Introduction and Preliminaries

The fractional derivative consept, which is a generalization of the classical derivative, has become the focus of attention of researchers and has been seccesfully applied to many real world problems. In this study, fractional differential operators such as Caputo, Caputo-Fabrizio and Atangana-Baleanu fractional derivatives will be considered [1-3]. Undoubtedly, the chaos theory, which has attracted the attention of researchers, has been reconsidered with fractional analysis and has opened new doors in research [4-11]. In [4], the authors have investigated the modified Chua attractor. Labyrinth chaotic system has been modeled with the fractional differential operators in [5]. In [6], Chua attractor with fractional and fractal-fractional operators has been solved by using a new numerical scheme based on Newton polynomial. In [7], chaotic systems have been modeled by using with fractal-fractional operators introduced by Atangana. In [8], chaotic attractors with many scrolls have been examined for fractional case. In [9], some chaotic attractors has been presented by using fractal-fractional differentiation and integration. In [10], Irving-Mullineux oscillator has

*Corresponding author
Email addresses: akif.cetin@alanya.edu.tr (Mehmet Akif Cetin), selahattingenc1980@gmail.com (Selahattin Genc), maraz250@hotmail.com (Metin Araz)

Received : 01 January 2024*; Accepted:* 11 January 2024*; Published Online:* 23 January 2024.

been modified by the fractional derivative with Mittag-Leffler kernel. In [11], the authors have examined the chaotic behavior in system of fractional ordinary differential equations.

Since chaotic systems are nonlinear, we need to solve such systems by using numerical methods. In this study, we present the numerical scheme based on Newton polynomial [6] to solve chaotic system that will be presented in the next section.

We now present some definitions about fractional differential operators that will be used in this study.

Definition 1.1. Let $u : \mathbb{R}^+ \to \mathbb{R}$. The left Caputo fractional derivative[2] of fractional order ρ of the function $u(t)$ is defined by

$$ {}_0^C D_t^\rho u(t) = \frac{1}{\Gamma(1-\rho)} \int_0^t (t-l)^{-\rho} u'(l)\, dl, \ t > 0 \tag{1.1} $$

where $\rho \in (0,1)$.

Definition 1.2. Let $u(t)$ be continuous and differentiable on $C^1[0,1]$. Then, the Caputo-Fabrizio fractional derivative[3] with fractional order ρ of the function $u(t)$ is given as follows

$$ {}_0^{CF} D_t^\rho u(t) = \frac{1}{1-\rho} \int_0^t \frac{du(l)}{dl} \exp\left[\frac{-\rho(t-l)}{1-\rho}\right] dl \tag{1.2} $$

where $0 < \rho \leq 1$.

Definition 1.3. Let $u(t) \in W_2^1(0,l)$, then for $\rho \in [0,1]$ Atangana-Baleanu fractional derivative[1] in Caputo sense of the function $u(t)$ is given by

$$ {}_0^{ABC} D_t^\rho u(t) = \frac{1}{1-\rho} \int_0^t \frac{d}{dl} u(l)\, E_\rho\left[-\frac{\rho}{1-\rho}(t-l)^\rho\right] dl. \tag{1.3} $$

Definition 1.4. The Caputo fractional integral of the function $u(t)$ is given as follows:

$$ {}_0^C I_t^\rho u(t) = \frac{1}{\Gamma(\rho)} \int_0^t u(l)(t-l)^{\rho-1}\, dl. \tag{1.4} $$

Definition 1.5. The Caputo-Fabrizio fractional integral[3] of the function $u(t)$ is in the form of:

$$ {}_0^{CF} I_t^\rho u(t) = (1-\rho) u(t) + \rho \int_0^t u(l)\, dl. \tag{1.5} $$

Definition 1.6. The Atangana-Baleanu fractional integral[1] of the function $u(t)$ is defined by the following:

$$ {}_0^{AB} I_t^\rho u(t) = (1-\rho) u(t) + \frac{\rho}{\Gamma(\rho)} \int_0^t u(l)(t-l)^{\rho-1}\, dl. \tag{1.6} $$

2. Statement of Problem

In this section, we consider a multi-wing chaotic system presented in [14] and we present conditions under which existence and uniqueness are ensured for such a system. To achieve this, we consider multi-wing chaotic system [14] with classical derivative

$$ \begin{aligned} u'(t) &= \kappa(\upsilon - u) \\ \upsilon'(t) &= \gamma\upsilon - uw \\ w'(t) &= -\xi + \delta(1 - \delta_1 \sin(\delta_2 \upsilon))\, \upsilon^2. \end{aligned} \tag{2.1} $$

Here, the parameters are

$$ \kappa = 2, \ \gamma = 1, \ \xi = 1, \ \delta = 0.4, \ \delta_1 = 3, \ \delta_2 = 4 \tag{2.2} $$

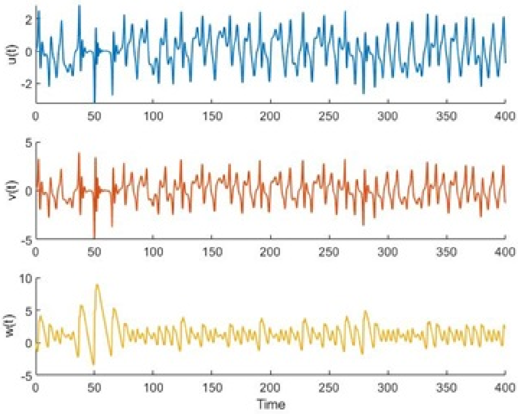

Figure 1: The graphical representation for multi-wing chaotic system.

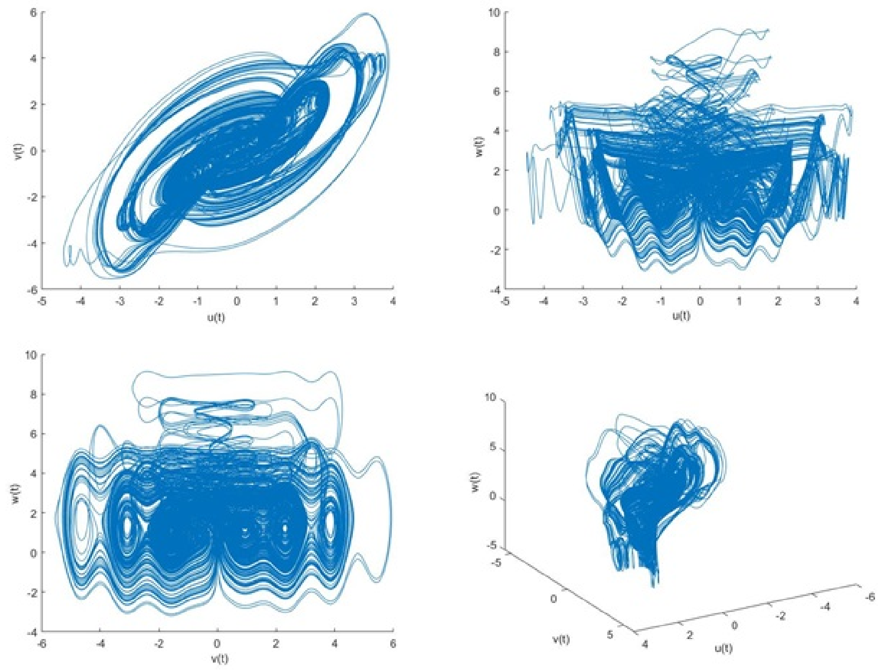

Figure 2: 3D simulation for multi-wing chaotic system with classical derivative for $\rho = 0.95$.

and initial data

$$u(0) = 0,\ \upsilon(0) = 0.1,\ w(0) = 0. \tag{2.3}$$

The numerical simulations for each function in the chaotic problem are decipted in Figure 1.

In Figure 2, we present the numerical simulations for chaotic problem with classical derivative.

Now, we prove the existence and uniqueness of the solutions for the considered chaotic problem. To do this, we present a theorem about existence and uniqueness of the system of equations **[15]**.

Theorem: Assume that there exists positive constants k_i, $\bar{k}_i$ such that

(i) $\forall i \in \{1, 2, 3\}$

$$\left|F_i(x_i, t) - F_i(x_i', t)\right|^2 \leq k_i \left|x_i - x_i'\right|^2, \tag{2.4}$$

(ii) $\forall (x, t) \in R^3 \times [0, T]$

$$|F_i(x_i, t)|^2 \leq \bar{k}_i \left(1 + |x_i|^2\right). \tag{2.5}$$

Proof. Before starting with the proof, we shall first define the following norm

$$\|\upsilon\|_\infty = \sup_{0 \leq t \leq T} |\upsilon(t)|. \tag{2.6}$$

We start with the function $F_1(t, u)$. Then, we will show that

$$\begin{aligned} |F_1(u_1, t) - F_1(u_2, t)|^2 &= |\kappa\upsilon(u_1 - u_2)|^2 \\ &= \kappa^2 |\upsilon(t)|^2 |u_1 - u_2|^2 \\ &\leq \kappa^2 \sup_{0 \leq t \leq T} |\upsilon(t)|^2 |u_1 - u_2|^2 \\ &\leq \kappa^2 \|\upsilon\|_\infty^2 |u_1 - u_2|^2 \\ &\leq k_1 |u_1 - u_2|^2, \end{aligned} \tag{2.7}$$

where $k_1 = \kappa^2 \left\|\upsilon^2\right\|_\infty$. Then, we write

$$\begin{aligned} |F_2(\upsilon_1, t) - F_2(\upsilon_2, t)|^2 &= |\gamma(\upsilon_1 - \upsilon_2)|^2 \\ &\leq \left(\gamma^2 + \varepsilon_1\right) |\upsilon_1 - \upsilon_2|^2 \\ &\leq k_2 |\upsilon_1 - \upsilon_2|^2, \end{aligned} \tag{2.8}$$

where $k_2 = \gamma^2 + \varepsilon_1$. Next, we write

$$|F_3(w_1, t) - F_3(w_2, t)|^2 \leq k_3 |w_1 - w_2|^2, \tag{2.9}$$

where $k_3 = \varepsilon_2$. We verified the first condition for all function. To verify the second condition for all function, we write the following

$$\begin{aligned} |F_1(u, t)|^2 &= |\kappa\upsilon - \kappa u|^2 \\ &\leq 2\kappa^2 \left(|\upsilon|^2 + |u|^2\right) \\ &\leq 2\kappa^2 \left(\sup_{0 \leq t \leq T} |\upsilon|^2 + |u|^2\right) \\ &\leq 2\kappa^2 \|\upsilon\|_\infty^2 \left(1 + \frac{1}{\|\upsilon\|_\infty^2} |u|^2\right) \\ &\leq \bar{k}_1 \left(1 + |u|^2\right), \end{aligned} \tag{2.10}$$

under the condition $\frac{1}{\|\upsilon^2\|_\infty} < 1$ such that $\bar{k}_1 = 2\kappa^2 \left\|\upsilon^2\right\|_\infty$. Later

$$\begin{aligned} |F_2(\upsilon,t)|^2 &= |\gamma\upsilon - uw|^2 \\ &\leq 2\left(\gamma^2|\upsilon|^2 + |u|^2|w|^2\right) \\ &\leq 2\left(\gamma^2|\upsilon|^2 + \sup_{0\leq t\leq T}|u|^2 \sup_{0\leq t\leq T}|w|^2\right) \\ &\leq 2\left(\gamma^2|\upsilon|^2 + \left\|u^2\right\|_\infty \left\|w^2\right\|_\infty\right) \\ &\leq 2\|u\|_\infty^2\|w\|_\infty^2\left(1 + \frac{\gamma^2}{\|u\|_\infty^2\|w\|_\infty^2}|\upsilon|^2\right) \\ &\leq \bar{k}_2\left(1+|\upsilon|^2\right), \end{aligned} \tag{2.11}$$

under the condition $\frac{\gamma^2}{\|u^2\|_\infty\|w^2\|_\infty} < 1$ such that $\bar{k}_2 = 2\|u\|_\infty^2\|w\|_\infty^2$. Finally, we have

$$\begin{aligned} |F_3(w,t)|^2 &= \left|\xi - u^2 - \upsilon^2 + \rho uw\right|^2 \\ &\leq 4\left(\xi^2 + \left|u^2\right|^2 + \left|\upsilon^2\right|^2 + \rho^2|u|^2|w|^2\right) \\ &\leq 4\left(\xi^2 + \sup_{0\leq t\leq T}\left|u^2\right|^2 + \sup_{0\leq t\leq T}\left|\upsilon^2\right|^2 + \rho^2\sup_{0\leq t\leq T}|u|^2|w|^2\right) \\ &\leq 4\left(\xi^2 + \left\|u^2\right\|_\infty^2 + \left\|\upsilon^2\right\|_\infty^2 + \rho^2\|u\|_\infty^2|w|^2\right) \\ &\leq 4\left(\xi^2 + \left\|u^2\right\|_\infty^2 + \left\|\upsilon^2\right\|_\infty^2\right)\left(1 + \frac{\rho^2\|u\|_\infty^2}{\xi^2 + \|u^2\|_\infty^2 + \|\upsilon^2\|_\infty^2}|w|^2\right) \\ &\leq \bar{k}_3\left(1+|w|^2\right), \end{aligned} \tag{2.12}$$

under the condition $\frac{\rho^2\|u\|_\infty^2}{\xi^2+\|u^2\|_\infty^2+\|\upsilon^2\|_\infty^2} < 1$ such that $\bar{k}_3 = 4\left(\xi^2 + \left\|u^2\right\|_\infty^2 + \left\|\upsilon^2\right\|_\infty^2\right)$. Then, the solution of our chaotic system is exist and unique when

$$\max\left\{\frac{1}{\|\upsilon^2\|_\infty}, \frac{\gamma^2}{\|u^2\|_\infty\|w^2\|_\infty}, \frac{\rho^2\left\|u^2\right\|_\infty}{\xi^2 + \|u^2\|_\infty^2 + \|\upsilon^2\|_\infty^2}\right\} < 1. \tag{2.13}$$

3. Multi-wing chaotic system with exponential decay kernel

In this section, we deal with the multi-wing chaotic system[14] including exponential decay kernel known as Caputo-Fabrizio fractional derivative

$$\begin{aligned} {}_0^{CF}D_t^\rho u(t) &= \kappa(\upsilon - u) \\ {}_0^{CF}D_t^\rho \upsilon(t) &= \gamma\upsilon - uw \\ {}_0^{CF}D_t^\rho w(t) &= -\xi + \delta\left(1 - \delta_1\sin(\delta_2\upsilon)\right)\upsilon^2. \end{aligned} \tag{3.1}$$

For brevity, we can write as follows:

$$\begin{aligned} {}_0^{CF}D_t^\rho u(t) &= U(t,u,\upsilon,w) \\ {}_0^{CF}D_t^\rho \upsilon(t) &= V(t,u,\upsilon,w) \\ {}_0^{CF}D_t^\rho w(t) &= W(t,u,\upsilon,w). \end{aligned} \tag{3.2}$$

When integrating the chaotic system, we can write the following equality

$$\begin{aligned} u(t) &= u_0 + (1-\rho) U(t,u,v,w) + \rho \int_0^t U(s,u,v,w)\,ds, \\ v(t) &= v_0 + (1-\rho) V(t,u,v,w) + \rho \int_0^t V(s,u,v,w)\,ds, \\ w(t) &= w_0 + (1-\rho) W(t,u,v,w) + \rho \int_0^t W(s,u,v,w)\,ds. \end{aligned} \tag{3.3}$$

If we write the above equations at points $t = t_n$ and $t = t_{n+1}$ and take the difference of the equations obtained, then the following equation can be achieved:

$$\begin{aligned} u(t_{n+1}) &= u(t_n) + (1-\rho)\left[U\left(t_{n+1}, \widetilde{u}^{n+1}, \widetilde{v}^{n+1}, \widetilde{w}^{n+1}\right) - U(t_n, u^n, v^n, w^n)\right] \\ &\quad +\rho \int_{t_n}^{t_{n+1}} U(s,u,v,w)\,ds, \\ v(t_{n+1}) &= v(t_n) + (1-\rho)\left[V\left(t_{n+1}, \widetilde{u}^{n+1}, \widetilde{v}^{n+1}, \widetilde{w}^{n+1}\right) - V(t_n, u^n, v^n, w^n)\right] \\ &\quad +\rho \int_{t_n}^{t_{n+1}} V(s,u,v,w)\,ds, \\ w(t_{n+1}) &= w(t_n) + (1-\rho)\left[W\left(t_{n+1}, \widetilde{u}^{n+1}, \widetilde{v}^{n+1}, \widetilde{w}^{n+1}\right) - W(t_n, u^n, v^n, w^n)\right] \\ &\quad +\rho \int_{t_n}^{t_{n+1}} W(s,u,v,w)\,ds. \end{aligned} \tag{3.4}$$

Replacing the functions $U(t,u,v,w)\,, V(t,u,v,w)$ and $W(t,u,v,w)$ by their two step Newton polynomial, then the following numerical scheme is obtained as

$$\begin{aligned} u^{n+1} &= u^n + (1-\rho)\left[U\left(t_{n+1}, \widetilde{u}^{n+1}, \widetilde{v}^{n+1}, \widetilde{w}^{n+1}\right) - U(t_n, u^n, v^n, w^n)\right] \\ &\quad +\rho h \left[\begin{array}{c} \frac{23}{12} U(t_n, u^n, v^n, w^n) - \frac{4}{3} U\left(t_{n-1}, u^{n-1}, v^{n-1}, w^{n-1}\right) \\ +\frac{5}{12} U\left(t_{n-2}, u^{n-2}, v^{n-2}, w^{n-2}\right) \end{array}\right], \\ v^{n+1} &= v^n + (1-\rho)\left[V\left(t_{n+1}, \widetilde{u}^{n+1}, \widetilde{v}^{n+1}, \widetilde{w}^{n+1}\right) - V(t_n, u^n, v^n, w^n)\right] \\ &\quad +\rho h \left[\begin{array}{c} \frac{23}{12} V(t_n, u^n, v^n, w^n) - \frac{4}{3} V\left(t_{n-1}, u^{n-1}, v^{n-1}, w^{n-1}\right) \\ +\frac{5}{12} V\left(t_{n-2}, u^{n-2}, v^{n-2}, w^{n-2}\right) \end{array}\right], \\ w^{n+1} &= w^n + (1-\rho)\left[W\left(t_{n+1}, \widetilde{u}^{n+1}, \widetilde{v}^{n+1}, \widetilde{w}^{n+1}\right) - W(t_n, u^n, v^n, w^n)\right] \\ &\quad +\rho h \left[\begin{array}{c} \frac{23}{12} W(t_n, u^n, v^n, w^n) - \frac{4}{3} W\left(t_{n-1}, u^{n-1}, v^{n-1}, w^{n-1}\right) \\ +\frac{5}{12} W\left(t_{n-2}, u^{n-2}, v^{n-2}, w^{n-2}\right) \end{array}\right], \end{aligned} \tag{3.5}$$

where the predictor terms are evaluated as

$$\begin{aligned} \widetilde{u}^{n+1} &= u_0 + (1-\rho) U(t_n, u^n, v^n, w^n) + \rho h \sum_{k=0}^{n} U\left(t_k, u^k, v^k, w^k\right), \\ \widetilde{v}^{n+1} &= v_0 + (1-\rho) V(t_n, u^n, v^n, w^n) + \rho h \sum_{k=0}^{n} V\left(t_k, u^k, v^k, w^k\right), \\ \widetilde{w}^{n+1} &= w_0 + (1-\rho) W(t_n, u^n, v^n, w^n) + \rho h \sum_{k=0}^{n} W\left(t_k, u^k, v^k, w^k\right). \end{aligned} \tag{3.6}$$

4. Multi-wing chaotic system with power-law kernel

In this section, we obtain numerical scheme for the following multi-wing chaotic model[14]

$$\begin{aligned} {}_0^C D_t^\rho u(t) &= \kappa(\upsilon - u) \\ {}_0^C D_t^\rho \upsilon(t) &= \gamma \upsilon - uw \\ {}_0^C D_t^\rho w(t) &= -\xi + \delta\left(1 - \delta_1 \sin(\delta_2 \upsilon)\right) \upsilon^2 \end{aligned} \tag{4.1}$$

where the derivative is the Caputo fractional derivative. Integrating above equation, we have the following equations

$$\begin{aligned} u(t) &= u_0 + \frac{1}{\Gamma(\rho)} \int_0^t U(s,u,v,w)(t-s)^{\rho-1} ds, \\ v(t) &= v_0 + \frac{1}{\Gamma(\rho)} \int_0^t V(s,u,v,w)(t-s)^{\rho-1} ds, \\ w(t) &= w_0 + \frac{1}{\Gamma(\rho)} \int_0^t W(s,u,v,w)(t-s)^{\rho-1} ds. \end{aligned} \tag{4.2}$$

If we replace the above functions by their Newton polynomials after taking $t = t_{n+1}$, we get the following

$$u^{n+1} = \left\{ \begin{array}{c} u_0 + \frac{h^\rho}{\Gamma(\rho+1)} \sum_{k=2}^n U\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \Pi_{n,k}^1 \\ + \frac{h^\rho}{\Gamma(\rho+2)} \sum_{k=2}^n \left[\begin{array}{c} U\left(t_{k-1}, u^{k-1}, v^{k-1}, w^{k-1}\right) \\ -U\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \end{array} \right] \Pi_{n,k}^2 \\ + \frac{h^\rho}{2\Gamma(\rho+3)} \sum_{k=2}^n \left[\begin{array}{c} U\left(t_k, u^k, v^k, w^k\right) \\ -2U\left(t_{k-1}, u^{k-1}, v^{k-1}, w^{k-1}\right) \\ +U\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \end{array} \right] \Pi_{n,k}^3 \end{array} \right. , \tag{4.3}$$

$$v^{n+1} = \left\{ \begin{array}{c} v_0 + \frac{h^\rho}{\Gamma(\rho+1)} \sum_{k=2}^n V\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \Pi_{n,k}^1 \\ + \frac{h^\rho}{\Gamma(\rho+2)} \sum_{k=2}^n \left[\begin{array}{c} V\left(t_{k-1}, u^{k-1}, v^{k-1}, w^{k-1}\right) \\ -V\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \end{array} \right] \Pi_{n,k}^2 \\ + \frac{h^\rho}{2\Gamma(\rho+3)} \sum_{k=2}^n \left[\begin{array}{c} V\left(t_k, u^k, v^k, w^k\right) \\ -2V\left(t_{k-1}, u^{k-1}, v^{k-1}, w^{k-1}\right) \\ +V\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \end{array} \right] \Pi_{n,k}^3 \end{array} \right. , \tag{4.4}$$

$$w^{n+1} = \left\{ \begin{array}{c} w_0 + \frac{h^\rho}{\Gamma(\rho+1)} \sum_{k=2}^n W\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \Pi_{n,k}^1 \\ + \frac{h^\rho}{\Gamma(\rho+2)} \sum_{k=2}^n \left[\begin{array}{c} W\left(t_{k-1}, u^{k-1}, v^{k-1}, w^{k-1}\right) \\ -W\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \end{array} \right] \Pi_{n,k}^2 \\ + \frac{h^\rho}{2\Gamma(\rho+3)} \sum_{k=2}^n \left[\begin{array}{c} W\left(t_k, u^k, v^k, w^k\right) \\ -2W\left(t_{k-1}, u^{k-1}, v^{k-1}, w^{k-1}\right) \\ +W\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \end{array} \right] \Pi_{n,k}^3 \end{array} \right. , \tag{4.5}$$

where

$$\begin{aligned} \Pi_{n,k}^1 &= \left[(n-k+1)^\rho - (n-k)^\rho\right], \\ \Pi_{n,k}^2 &= \left[\begin{array}{c} (n-k+1)^\rho (n-k+3+2\rho) \\ -(n-k)^\rho (n-k+3+3\rho) \end{array} \right], \\ \Pi_{n,k}^3 &= \left[\begin{array}{c} (n-k+1)^\rho \left(\begin{array}{c} 2(n-k)^2 + (3\rho+10)(n-k) \\ +2\rho^2 + 9\rho + 12 \end{array} \right) \\ -(n-k)^\rho \left(\begin{array}{c} 2(n-k)^2 + (5\rho+10)(n-k) \\ +6\rho^2 + 18\rho + 12 \end{array} \right) \end{array} \right]. \end{aligned} \tag{4.6}$$

5. Multi-wing chaotic system with Mittag-Leffler kernel

In this section, we establish numerical approximation for the considered model[14] including Mittag-Leffler kernel

$$\begin{aligned} {}_0^{AB}D_t^{\rho}u(t) &= \kappa(\upsilon-u) \\ {}_0^{AB}D_t^{\rho}u(t) &= \gamma\upsilon-uw \\ {}_0^{AB}D_t^{\rho}u(t) &= -\xi+\delta(1-\delta_1\sin(\delta_2\upsilon))\upsilon^2. \end{aligned} \tag{5.1}$$

When we integrate the above equation, we have the following

$$\begin{aligned} u(t) &= u_0+(1-\rho)U(t,u,v,w)+\frac{\rho}{\Gamma(\rho)}\int_0^t U(s,u,v,w)(t-s)^{\rho-1}ds, \\ v(t) &= v_0+(1-\rho)V(t,u,v,w)+\frac{\rho}{\Gamma(\rho)}\int_0^t V(s,u,v,w)(t-s)^{\rho-1}ds, \\ w(t) &= w_0+(1-\rho)W(t,u,v,w)+\frac{\rho}{\Gamma(\rho)}\int_0^t W(s,u,v,w)(t-s)^{\rho-1}ds. \end{aligned} \tag{5.2}$$

If we replace the above functions by their Newton polynomials after taking $t=t_{n+1}$, we get the following

$$\begin{aligned} u^{n+1} &= \begin{cases} u_0+(1-\rho)U\left(t_{n+1},\widetilde{u}^{n+1},\widetilde{\upsilon}^{n+1},\widetilde{w}^{n+1}\right)+\frac{\rho h^{\rho}}{\Gamma(\rho+1)}\sum_{k=2}^{n}U\left(t_{k-2},u^{k-2},\upsilon^{k-2},w^{k-2}\right)\Pi_{n,k}^{1} \\ +\frac{\rho h^{\rho}}{\Gamma(\rho+2)}\sum_{k=2}^{n}\left[U\left(t_{k-1},u^{k-1},\upsilon^{k-1},w^{k-1}\right)-U\left(t_{k-2},u^{k-2},\upsilon^{k-2},w^{k-2}\right)\right]\Pi_{n,k}^{2} \\ +\frac{\rho h^{\rho}}{2\Gamma(\rho+3)}\sum_{k=2}^{n}\left[\begin{array}{c} U\left(t_k,u^k,\upsilon^k,w^k\right)-2U\left(t_{k-1},u^{k-1},\upsilon^{k-1},w^{k-1}\right) \\ +U\left(t_{k-2},u^{k-2},\upsilon^{k-2},w^{k-2}\right)\end{array}\right]\Pi_{n,k}^{3} \end{cases}, \\ \upsilon^{n+1} &= \begin{cases} v_0+(1-\rho)V\left(t_{n+1},\widetilde{u}^{n+1},\widetilde{\upsilon}^{n+1},\widetilde{w}^{n+1}\right)+\frac{\rho h^{\rho}}{\Gamma(\rho+1)}\sum_{k=2}^{n}V\left(t_{k-2},u^{k-2},\upsilon^{k-2},w^{k-2}\right)\Pi_{n,k}^{1} \\ +\frac{\rho h^{\rho}}{\Gamma(\rho+2)}\sum_{k=2}^{n}\left[V\left(t_{k-1},u^{k-1},\upsilon^{k-1},w^{k-1}\right)-V\left(t_{k-2},u^{k-2},\upsilon^{k-2},w^{k-2}\right)\right]\Pi_{n,k}^{2} \\ +\frac{\rho h^{\rho}}{2\Gamma(\rho+3)}\sum_{k=2}^{n}\left[\begin{array}{c} V\left(t_k,u^k,\upsilon^k,w^k\right)-2V\left(t_{k-1},u^{k-1},\upsilon^{k-1},w^{k-1}\right) \\ +V\left(t_{k-2},u^{k-2},\upsilon^{k-2},w^{k-2}\right)\end{array}\right]\Pi_{n,k}^{3} \end{cases}, \\ w^{n+1} &= \begin{cases} w_0+(1-\rho)W\left(t_{n+1},\widetilde{u}^{n+1},\widetilde{\upsilon}^{n+1},\widetilde{w}^{n+1}\right)+\frac{\rho h^{\rho}}{\Gamma(\rho+1)}\sum_{k=2}^{n}W\left(t_{k-2},u^{k-2},\upsilon^{k-2},w^{k-2}\right)\Pi_{n,k}^{1} \\ +\frac{\rho h^{\rho}}{\Gamma(\rho+2)}\sum_{k=2}^{n}\left[W\left(t_{k-1},u^{k-1},\upsilon^{k-1},w^{k-1}\right)-W\left(t_{k-2},u^{k-2},\upsilon^{k-2},w^{k-2}\right)\right]\Pi_{n,k}^{2} \\ +\frac{\rho h^{\rho}}{2\Gamma(\rho+3)}\sum_{k=2}^{n}\left[\begin{array}{c} W\left(t_k,u^k,\upsilon^k,w^k\right)-2W\left(t_{k-1},u^{k-1},\upsilon^{k-1},w^{k-1}\right) \\ +W\left(t_{k-2},u^{k-2},\upsilon^{k-2},w^{k-2}\right)\end{array}\right]\Pi_{n,k}^{3} \end{cases} \end{aligned} \tag{5.3}$$

where the predictor terms are evaluated as

$$\begin{aligned} \widetilde{u}^{n+1} &= u_0+(1-\rho)U(t_n,u^n,\upsilon^n,w^n)+\frac{\rho}{\Gamma(\rho)}\sum_{k=0}^{n}U\left(t_k,u^k,\upsilon^k,w^k\right)\Pi_{n,k}^{1}, \\ \widetilde{\upsilon}^{n+1} &= v_0+(1-\rho)V(t_n,u^n,\upsilon^n,w^n)+\frac{\rho}{\Gamma(\rho)}\sum_{k=0}^{n}V\left(t_k,u^k,\upsilon^k,w^k\right)\Pi_{n,k}^{1}, \\ \widetilde{w}^{n+1} &= w_0+(1-\rho)W(t_n,u^n,\upsilon^n,w^n)+\frac{\rho}{\Gamma(\rho)}\sum_{k=0}^{n}W\left(t_k,u^k,\upsilon^k,w^k\right)\Pi_{n,k}^{1}. \end{aligned} \tag{5.4}$$

Example 1. We consider the folowing chaotic problem with exponential decay kernel

$$\begin{aligned} {}_0^{CF}D_t^{\rho}u(t) &= \kappa(\upsilon-u) \\ {}_0^{CF}D_t^{\rho}\upsilon(t) &= \gamma\upsilon-uw \\ {}_0^{CF}D_t^{\rho}w(t) &= -\xi+\delta(1-\delta_1\sin(\delta_2\upsilon))\upsilon^2. \end{aligned} \tag{5.5}$$

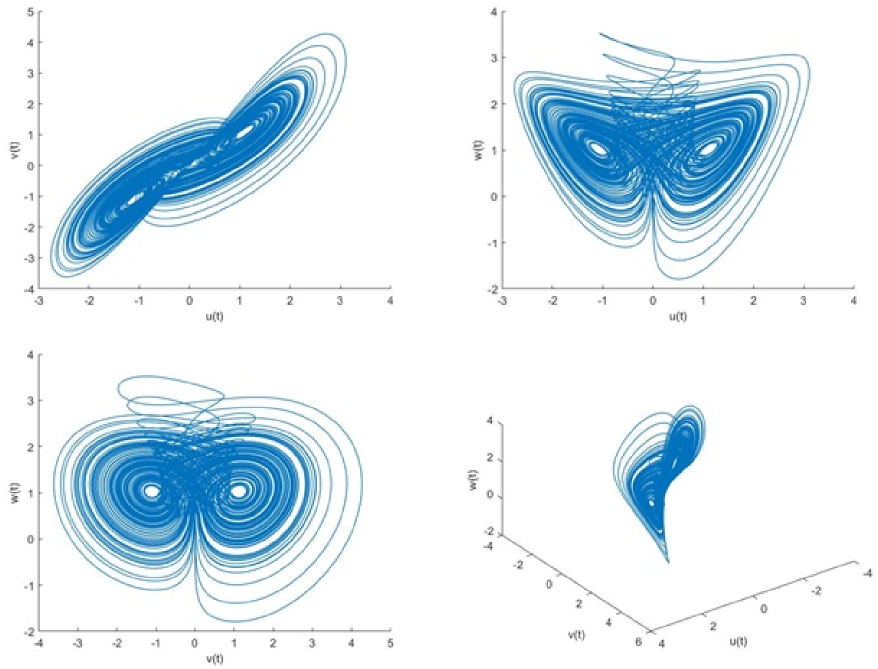

Figure 3: Simulation for multi-wing chaotic system with Caputo-Fabrizio fractional derivative for $\rho = 0.95$.

Here, the parameters are

$$\kappa = 1.8,\ \gamma = 1,\ \xi = 0.5,\ \delta = 0.4,\ \delta_1 = 0.03,\ \delta_2 = 2 \tag{5.6}$$

and initial data

$$u(0) = 0,\ \upsilon(0) = 0.1,\ w(0) = 0. \tag{5.7}$$

In Figure 3, the numerical simulations for chaotic problem with Caputo-Fabrizio fractional derivative are depicted using same parameters and initial conditions.

Example 2. We consider the folowing chaotic problem with Mittag-Leffler kernel

$$\begin{aligned} {}_0^{ABC}D_t^{\rho} u(t) &= \kappa(\upsilon - u) \\ {}_0^{ABC}D_t^{\rho} \upsilon(t) &= \gamma\upsilon - uw \\ {}_0^{ABC}D_t^{\rho} w(t) &= -\xi + \delta(1 - \delta_1 \sin(\delta_2 \upsilon))\,\upsilon^2. \end{aligned} \tag{5.8}$$

The parameters are

$$\kappa = 1.8,\ \gamma = 1,\ \xi = 0.5,\ \delta = 0.4,\ \delta_1 = 0.03,\ \delta_2 = 2 \tag{5.9}$$

and initial data

$$u(0) = 0,\ \upsilon(0) = 0.1,\ w(0) = 0. \tag{5.10}$$

In Figure 4, we perform the numerical simulations for chaotic problem with Atangana-Baleanu fractional derivative.

6. Chaotic model with piecewise derivative

We consider the chaotic model[14] with the scenario where the first part is with classical derivative and the second part is with Caputo fractional derivative. The piecewise system[16] under investigation is given

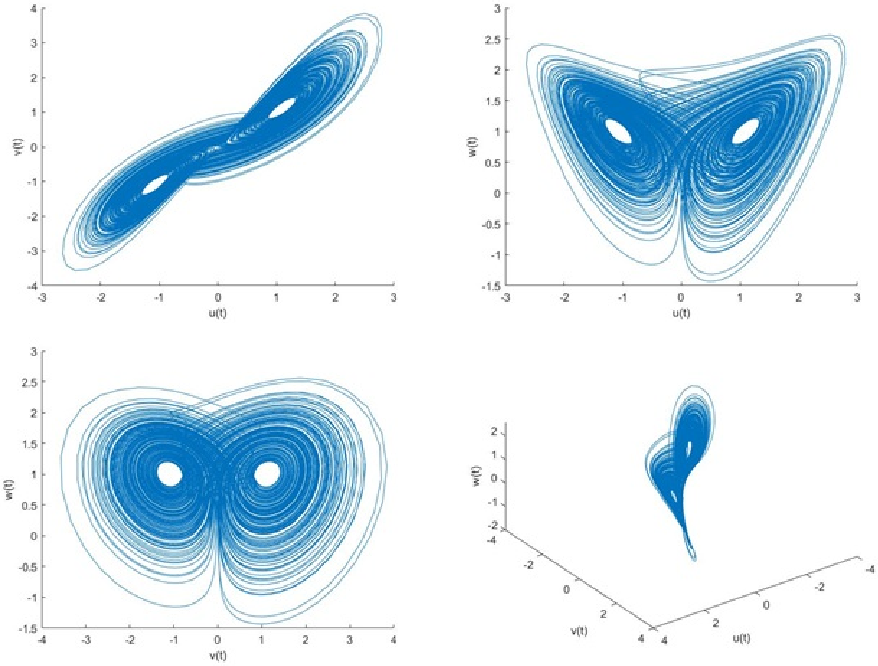

Figure 4: Simulation for multi-wing chaotic system with Atangana-Baleanu fractional derivative for $\rho = 0.95$.

by:

$$\begin{cases} {}_0^{ABC}D_t^{\rho}u\left(t\right) = U\left(t,u,v,w\right) \\ {}_0^{ABC}D_t^{\rho}v\left(t\right) = V\left(t,u,v,w\right) \\ {}_0^{ABC}D_t^{\rho}w\left(t\right) = W\left(t,u,v,w\right) \end{cases}, \quad 0 \leq t \leq t_0, \tag{6.1}$$

$$\begin{cases} u'\left(t\right) = U\left(t,u,v,w\right) \\ v'\left(t\right) = V\left(t,u,v,w\right) \\ w'\left(t\right) = W\left(t,u,v,w\right) \end{cases}, t_0 \leq t \leq T.$$

Applying the associated integral, we obtain

$$\begin{aligned} u\left(t\right) &= \begin{cases} u\left(0\right) + \left(1-\rho\right)U\left(t,u,v,w\right) + \dfrac{\rho}{\Gamma\left(\rho\right)}\int_0^t U\left(s,u,v,w\right)ds, 0 \leq t \leq t_0 \\ u\left(t_0\right) + \int_{t_0}^t U\left(s,u,v,w\right)\left(t-s\right)^{\rho-1}ds, t_0 \leq t \leq T \end{cases}, \\ v\left(t\right) &= \begin{cases} v\left(0\right) + \left(1-\rho\right)V\left(t,u,v,w\right) + \dfrac{\rho}{\Gamma\left(\rho\right)}\int_0^t V\left(s,u,v,w\right)ds, 0 \leq t \leq t_0 \\ v\left(t_0\right) + \int_{t_0}^t V\left(s,u,v,w\right)\left(t-s\right)^{\rho-1}ds, t_0 \leq t \leq T \end{cases}, \ . \\ v\left(t\right) &= \begin{cases} w\left(0\right) + \left(1-\rho\right)W\left(t,u,v,w\right) + \dfrac{\rho}{\Gamma\left(\rho\right)}\int_0^t W\left(s,u,v,w\right)ds, 0 \leq t \leq t_0 \\ w\left(t_0\right) + \int_{t_0}^t W\left(s,u,v,w\right)\left(t-s\right)^{\rho-1}ds, t_0 \leq t \leq T \end{cases} \end{aligned} \tag{6.2}$$

We consider above at $t = t_{n+1}$, and replace the function $f(x,t)$ by its Newton polynomial [16]. Thus, we obtain

$$u^{n+1} = \begin{cases} \begin{cases} u(0) + (1-\rho) U\left(t_{n+1}, \widetilde{u}^{n+1}, \widetilde{v}^{n+1}, \widetilde{w}^{n+1}\right) \\ +\frac{\rho h^{\rho}}{\Gamma(\rho+2)} \sum_{k=2}^{m} U\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \Pi_{n,k}^{1} \\ +\frac{\rho h^{\rho}}{\Gamma(\rho+2)} \sum_{k=2}^{m} \begin{bmatrix} U\left(t_{k-1}, u^{k-1}, v^{k-1}, w^{k-1}\right) \\ -U\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \end{bmatrix} \Pi_{n,k}^{2} \\ +\frac{\rho h^{\rho}}{\Gamma(\rho+3)} \sum_{k=2}^{m} \begin{bmatrix} U\left(t_{k}, u^{k}, v^{k}, w^{k}\right) \\ -2U\left(t_{k-1}, u^{k-1}, v^{k-1}, w^{k-1}\right) \\ +U\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \end{bmatrix} \Pi_{n,k}^{3}, 0 \le t \le t_0, \end{cases} \\ \begin{cases} u(t_0) + h \sum_{k=m+3}^{n} \begin{bmatrix} \frac{23}{12} U\left(t_k, u^k, v^k, w^k\right) - \frac{4}{3} U\left(t_{k-1}, u^{k-1}, v^{k-1}, w^{k-1}\right) \\ +\frac{5}{12} U\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \end{bmatrix}, \ t_0 \le t \le T \end{cases} \end{cases}, \quad (6.3)$$

$$v^{n+1} = \begin{cases} \begin{cases} v(0) + (1-\rho) V\left(t_{n+1}, \widetilde{u}^{n+1}, \widetilde{v}^{n+1}, \widetilde{w}^{n+1}\right) \\ +\frac{\rho h^{\rho}}{\Gamma(\rho+2)} \sum_{k=2}^{m} V\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \Pi_{n,k}^{1} \\ +\frac{\rho h^{\rho}}{\Gamma(\rho+2)} \sum_{k=2}^{m} \begin{bmatrix} V\left(t_{k-1}, u^{k-1}, v^{k-1}, w^{k-1}\right) \\ -V\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \end{bmatrix} \Pi_{n,k}^{2} \\ +\frac{\rho h^{\rho}}{\Gamma(\rho+3)} \sum_{k=2}^{m} \begin{bmatrix} V\left(t_{k}, u^{k}, v^{k}, w^{k}\right) \\ -2V\left(t_{k-1}, u^{k-1}, v^{k-1}, w^{k-1}\right) \\ +V\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \end{bmatrix} \Pi_{n,k}^{3}, 0 \le t \le t_0, \end{cases} \\ \begin{cases} v(t_0) + h \sum_{k=m+3}^{n} \begin{bmatrix} \frac{23}{12} V\left(t_k, u^k, v^k, w^k\right) - \frac{4}{3} V\left(t_{k-1}, u^{k-1}, v^{k-1}, w^{k-1}\right) \\ +\frac{5}{12} V\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \end{bmatrix}, \ t_0 \le t \le T \end{cases} \end{cases}, \quad (6.4)$$

$$w^{n+1} = \begin{cases} \begin{cases} w(0) + (1-\rho) W\left(t_{n+1}, \widetilde{u}^{n+1}, \widetilde{v}^{n+1}, \widetilde{w}^{n+1}\right) \\ +\frac{\rho h^{\rho}}{\Gamma(\rho+2)} \sum_{k=2}^{m} W\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \Pi_{n,k}^{1} \\ +\frac{\rho h^{\rho}}{\Gamma(\rho+2)} \sum_{k=2}^{m} \begin{bmatrix} W\left(t_{k-1}, u^{k-1}, v^{k-1}, w^{k-1}\right) \\ -W\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \end{bmatrix} \Pi_{n,k}^{2} \\ +\frac{\rho h^{\rho}}{\Gamma(\rho+3)} \sum_{k=2}^{m} \begin{bmatrix} W\left(t_{k}, u^{k}, v^{k}, w^{k}\right) \\ -2W\left(t_{k-1}, u^{k-1}, v^{k-1}, w^{k-1}\right) \\ +W\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \end{bmatrix} \Pi_{n,k}^{3}, 0 \le t \le t_0, \end{cases} \\ \begin{cases} w(t_0) + h \sum_{k=m+3}^{n} \begin{bmatrix} \frac{23}{12} W\left(t_k, u^k, v^k, w^k\right) - \frac{4}{3} W\left(t_{k-1}, u^{k-1}, v^{k-1}, w^{k-1}\right) \\ +\frac{5}{12} W\left(t_{k-2}, u^{k-2}, v^{k-2}, w^{k-2}\right) \end{bmatrix}, \ t_0 \le t \le T \end{cases} \end{cases},$$

where the predictor components are obtained as:

$$\begin{aligned} \widetilde{u}^{n+1} &= u_0 + (1-\rho) U(t_n, u^n, v^n, w^n) + \rho \sum_{k=0}^{n} U\left(t_k, u^k, v^k, w^k\right) \Pi_{n,k}^{1}, \quad (6.5) \\ \widetilde{v}^{n+1} &= v_0 + (1-\rho) V(t_n, u^n, v^n, w^n) + \rho \sum_{k=0}^{n} V\left(t_k, u^k, v^k, w^k\right) \Pi_{n,k}^{1}, \\ \widetilde{w}^{n+1} &= w_0 + (1-\rho) W(t_n, u^n, v^n, w^n) + \rho \sum_{k=0}^{n} W\left(t_k, u^k, v^k, w^k\right) \Pi_{n,k}^{1}. \end{aligned}$$

We next consider the chaotic model with piecewise derivative [16]

$$\begin{cases} u'(t) = U(t,u,v,w) \\ v'(t) = V(t,u,v,w) \\ w'(t) = W(t,u,v,w) \end{cases}, \ 0 \le t \le t_0, \quad (6.6)$$

$$\begin{cases} {}_{t_0}^{CF}D_t^{\rho} u(t) = U(t,u,v,w) \\ {}_{t_0}^{CF}D_t^{\rho} v(t) = V(t,u,v,w) \\ {}_{t_0}^{CF}D_t^{\rho} w(t) = W(t,u,v,w) \end{cases}, t_0 \le t \le T.$$

where the first part is with classical derivative and the second part is with Caputo-Fabrizio fractional derivative. Applying the associated integral, we obtain

$$u(t) = \begin{cases} u(0) + \int_0^t U(s,u,v,w)\,ds, 0 \le t \le t_0 \\ u(t_0) + (1-\rho)U(t,u,v,w) + \rho\int_{t_0}^t U(s,u,v,w)\,ds, t_0 \le t \le T \end{cases},$$
$$v(t) = \begin{cases} v(0) + \int_0^t V(s,u,v,w)\,ds, 0 \le t \le t_0 \\ v(t_0) + (1-\rho)V(t,u,v,w) + \rho\int_{t_0}^t V(s,u,v,w)\,ds, t_0 \le t \le T \end{cases}, \quad . \tag{6.7}$$
$$w(t) = \begin{cases} w(0) + \int_0^t W(s,u,v,w)\,ds, 0 \le t \le t_0 \\ w(t_0) + (1-\rho)W(t,u,v,w) + \rho\int_{t_0}^t W(s,u,v,w)\,ds, t_0 \le t \le T \end{cases}$$

We consider above at $t = t_{n+1}$, and replace the function $f(x,t)$ by its Newton polynomial [16]. Thus, we obtain

$$u^{n+1} = \begin{cases} \left\{ u(0) + h\sum_{k=0}^{m} \left[\begin{array}{c} \frac{23}{12}U\left(t_k,u^k,v^k,w^k\right) - \frac{4}{3}U\left(t_{k-1},u^{k-1},v^{k-1},w^{k-1}\right) \\ +\frac{5}{12}U\left(t_{k-2},u^{k-2},v^{k-2},w^{k-2}\right) \end{array} \right],\ 0 \le t \le t_0, \right. \\ \left\{ \begin{array}{c} u(t_0) + (1-\rho)U\left(t_{n+1},\widetilde{u}^{n+1},\widetilde{v}^{n+1},\widetilde{w}^{n+1}\right) \\ +\rho h\sum_{k=m+3}^{n} \left[\begin{array}{c} \frac{23}{12}U\left(t_k,u^k,v^k,w^k\right) - \frac{4}{3}U\left(t_{k-1},u^{k-1},v^{k-1},w^{k-1}\right) \\ +\frac{5}{12}U\left(t_{k-2},u^{k-2},v^{k-2},w^{k-2}\right) \end{array} \right], t_0 \le t \le T \end{array} \right. \end{cases}, \tag{6.8}$$

$$v^{n+1} = \begin{cases} \left\{ v(0) + h\sum_{k=0}^{m} \left[\begin{array}{c} \frac{23}{12}V\left(t_k,u^k,v^k,w^k\right) - \frac{4}{3}V\left(t_{k-1},u^{k-1},v^{k-1},w^{k-1}\right) \\ +\frac{5}{12}V\left(t_{k-2},u^{k-2},v^{k-2},w^{k-2}\right) \end{array} \right],\ 0 \le t \le t_0,\ , \right. \\ \left\{ \begin{array}{c} v(t_0) + (1-\rho)V\left(t_{n+1},\widetilde{u}^{n+1},\widetilde{v}^{n+1},\widetilde{w}^{n+1}\right) \\ +\rho h\sum_{k=m+3}^{n} \left[\begin{array}{c} \frac{23}{12}V\left(t_k,u^k,v^k,w^k\right) - \frac{4}{3}V\left(t_{k-1},u^{k-1},v^{k-1},w^{k-1}\right) \\ +\frac{5}{12}V\left(t_{k-2},u^{k-2},v^{k-2},w^{k-2}\right) \end{array} \right], t_0 \le t \le T \end{array} \right. \end{cases},$$

$$w^{n+1} = \begin{cases} \left\{ w(0) + h\sum_{k=0}^{m} \left[\begin{array}{c} \frac{23}{12}W\left(t_k,u^k,v^k,w^k\right) - \frac{4}{3}W\left(t_{k-1},u^{k-1},v^{k-1},w^{k-1}\right) \\ +\frac{5}{12}W\left(t_{k-2},u^{k-2},v^{k-2},w^{k-2}\right) \end{array} \right],\ 0 \le t \le t_0, \right. \\ \left\{ \begin{array}{c} u(t_0) + (1-\rho)W\left(t_{n+1},\widetilde{u}^{n+1},\widetilde{v}^{n+1},\widetilde{w}^{n+1}\right) \\ +\rho h\sum_{k=m+3}^{n} \left[\begin{array}{c} \frac{23}{12}W\left(t_k,u^k,v^k,w^k\right) - \frac{4}{3}W\left(t_{k-1},u^{k-1},v^{k-1},w^{k-1}\right) \\ +\frac{5}{12}W\left(t_{k-2},u^{k-2},v^{k-2},w^{k-2}\right) \end{array} \right], t_0 \le t \le T \end{array} \right. \end{cases},$$

where the predictor components are obtained as:

$$\begin{aligned} \widetilde{u}^{n+1} &= u_0 + (1-\rho)U(t_n,u^n,v^n,w^n) + \rho h\sum_{k=0}^{n} U\left(t_k,u^k,v^k,w^k\right), \\ \widetilde{v}^{n+1} &= v_0 + (1-\rho)V(t_n,u^n,v^n,w^n) + \rho h\sum_{k=0}^{n} V\left(t_k,u^k,v^k,w^k\right), \\ \widetilde{w}^{n+1} &= w_0 + (1-\rho)W(t_n,u^n,v^n,w^n) + \rho h\sum_{k=0}^{n} W\left(t_k,u^k,v^k,w^k\right). \end{aligned} \tag{6.9}$$

The numerical simulation for chaotic problem with piecewise derivative are performed using same parameters and initial data that we took for fractional case in Figure 5 and 6.

7. Conclusion

In this study, a multi-wing chaotic system with different differential operators are considered. It has been investigated under which conditions the existence and the uniqueness of the solution of the chaotic system with the classical derivative are guaranteed. Afterwards, this newly introduced chaotic system is modeled with fractional differential operators. A numerical method based on Newton polynomial for the

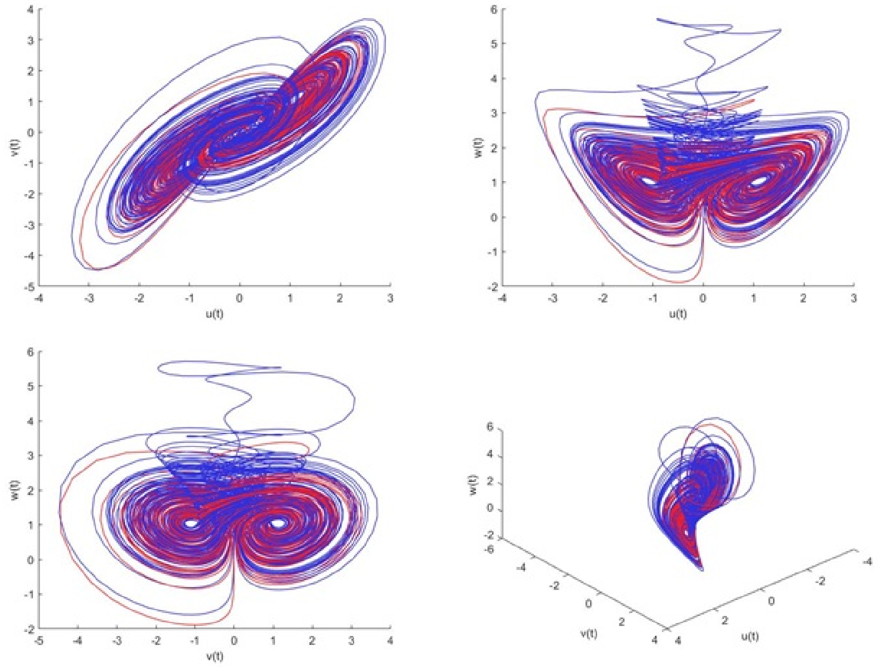

Figure 5: The graphical representation piecewise chaotic model with first scenario $\rho = 0.95$.

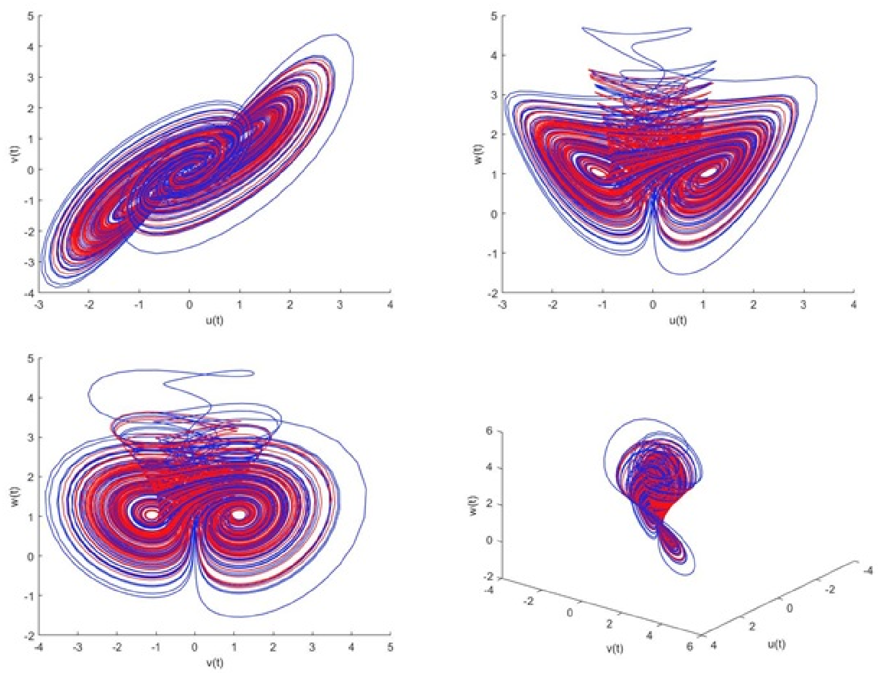

Figure 6: The graphical representation piecewise chaotic model with second scenario for $\rho = 0.99$.

numerical solution of this chaotic model with Caputo, Caputo-Fabrizio and Atangana-Baleanu fractional derivative has been presented. The numerical scheme based on Newton polynomial has been created to be used in solving chaotic model with piecewise derivative. Numerical simulations are depicted for different values of fractional orders.

References

[1] Atangana A., Baleanu D. (2016). *New fractional derivatives with non-local and non-singular kernel*, Theory and Application to Heat Transfer Model, Thermal Science, **20:2**, 763-769. 1.3, 1.6

[2] Caputo M. (1967). Linear model of dissipation whose Q is almost frequency independent-II, Geo-physical Journal International,**13:5**, 529-539. 1.1

[3] Caputo M., Fabrizio M. (2015). A new definition of fractional derivative without singular kernel, Progress in Fractional Differentiation and Applications, **1:2**, p. 73-85. 1.2, 1.5

[4] Atangana A., Igret Araz S. (2019). Fractional stochastic modelling illustration with modified Chua attractor, European Physical Journal Plus, **134:4**. 1

[5] Atangana A., Igret Araz S. (2020). Atangana-Seda numerical scheme for Labyrinth attractor with new differential and integral operators, Fractals. 1

[6] Atangana A., Igret Araz S. (2020). New numerical approximation for Chua attractor with fractional and fractal-fractional operators, Alexandria Engineering Journal. 1

[7] Atangana A., Qureshi S. (2019). Modeling attractors of chaotic dynamical systems with fractal–fractional operators, Chaos, Solitons & Fractals, **123**, 320-337. 1

[8] Goufo EFD. (2019). Multi-directional and saturated chaotic attractors with many scrolls for fractional dynamical systems. Discrete Continuous Dyn. Syst-S., **1**, 241–252. 1

[9] Gómez-Aguilar J.F., Atangana A. (2020).New chaotic attractors: Application of fractal-fractional differentiation and integration. Math Meth Appl Sci. 1–30. 1

[10] Gomez-Aguilar J.F. (2017). Irving–Mullineux oscillator via fractional derivatives with Mittag-Leffler kernel, Chaos, Solitons & Fractals, **95**, 179-186. 1

[11] Owolabi K.M., Atangana A. (2018). Chaotic behavior in system of noninteger-order ordinary differential equations, Chaos, Solitons & Fractals, **115**, 362–370. 1

[12] Hammouch Z., Mekkaoui T. (2015). Control of a new chaotic fractional-order system using Mittag-Leffler stability, Nonlinear Studies, **22:4**, 565-577.

[13] Alkahtani BST. (2019). A new numerical scheme based on Newton polynomial with application to Fractional nonlinear differential equations, Alexandria Engineering Journal.

[14] Sahoo S., Roy BK. (2022). A new multi-wing chaotic attractor with unusual variation in the number of wings, Chaos, Solitons & Fractals, **164**. 2, 3, 4, 5, 6

[15] Steele, M.J., 2010. Stochastic Calculus and Financial Applications, Springer. 2

[16] Atangana A., Igret Araz S. New concept in calculus: Piecewise differential and integral operators, Chaos, Solitons & Fractals, **145**, 110638, 2021.