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Journal of Prime Research in Mathematics, 20(1) (2024), 58–64

Journal of Prime Research in Mathematics

Metric Dimension and Some Related Parameters of Different Classes of Benzenoid System

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Abstract

The resolving set for connected graphs has become one of the most important concept due to its applicability in networking, robotics and computer sciences. Let G be a simple and connected graph, an ordered-subset B of $V(G)$ is called resolving set of G, if every distinct vertex of G have different metric code w.r.t B. Smallest resolving set of G is known as basis of G and size of basis set is called as metric dimension(MD) of graph G. A resolving set B' of G is known as fault-tolerant resolving set(FTRS), if $B' \setminus \{v\}$ is also resolving set, $\forall\ v \in B'$. Such set B' with smallest size is termed as fault-tolerant metric basis and the cardinality of this set is called fault-tolerant metric dimension(FTMD) of graph G. A FTMD set B' for which the system failure at vertex location v of any station still provide us a resolving set. In this article, we have provided the MD and FTMD for triangular benzenoid system and hourglass benzenoid system.

Keywords: Metric dimension, Resolving set, FTMD, Triangular benzenoid system, Hourglass benzenoid system.
2010 MSC: 47H10, 54H25.

1. Introduction and Preliminaries

In graph theory, the designs of computer networks and of related network systems are treated as graphs, in which every node expresses as vertex of the graph, and each edge describes the relationship between nodes. In any computer network, one is concerned to allot a unique address to each vertex to recognize the failure of any vertex. To control this type of situations, the concept of resolving set(RS) is derived. First time in 1975, P. J. Slater [1] and independently, Melter and Harary established the idea of resolving set [2]. The terminology of resolving set for Euclidean spaces was first appeared in [3] by Blumenthal. Resolving sets have significance importance in several fields, such as digital geometry, image processing, master mind

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Received : 16 April 2024*; Accepted:* 3 July 2024*; Published Online:* 05 July 2024.

games [4], drug designs, pharmaceutical chemistry [6], pattern recognition ,robot navigation and telecommunication.
For a graph G, $E(G)$ and $V(G)$ are the edge and vertex set respectively. A shortest path $d(v,u)$ between a vertex v to a vertex u is called distance between u and v. Assume an ordered set $B = \{b_1, b_2, ..., b_t\} \subseteq V(G)$ and let w in $V(G)$. Then the $t-$tuple, $r(w|B) = (d(w,b_1), d(w,b_2), ..., d(w,b_t))$ is representation of w w.r.t B. Such B is termed as resolving set for G, if $r(w|B)$ is distinguish for all distinct $w \in G$ [8]. A resolving set of G is called basis if its cardinality is minimum and its cardinality considered as metric dimension(MD) of G, which is denoted as $\beta(G)$ or dim(G).
The metric dimension of many classes of graphs and different families of graphs is computed by several researchers. Tomescu et. al. in [7] proved that the MD of Jahangir graph $J_{2k}$is $[\frac{2k}{3}]$, for $k \geq 4$. Imran et. al. [8] computed the metric dimension of generalized Petersen multigraphs, they also calculated the metric dimension of of a Mobius ladder graph by its barycentric subdivisions. Aurandha et. al. [9] investigate the metric dimension of split graph and pizza graph. Shao et. al. [**?**] gave the metric dimension of some generalized petersen graph. For all $m \geq 1$ there exist a family of connected graphs which have bounded metric dimension G_n, such that $\exists$ a constant $S > 0$ such as $\beta(G_m) \leq S$. Zhu et. al. in [**?**] computed the metric dimension of some hex derived networks.
Recently, the idea of metric dimension has expanded for a new related parameter known as fault-tolerance. In censors network, if one of the nodes, may not work correct, then we have not adequate details to treat with the interloper. To get control on such kind of issues fault-tolerance concept was introduced in [12]. Consequently Hernando et. al. established the concept of fault-tolerance metric dimension [13]. Fault-tolerant resolving sets provides the detail of problems in a system, if one of the censors is not working appropriately in that system. Fault-tolerant designs are being successfully used in computer sciences and in many network related engineering.

A resolving set B' of a one component simple graph G is termed as fault-tolerance resolving set(FTRS), if $B' \backslash \{v\}$ is also resolving set, $\forall v \in B'$. Such FTRS B' with minimum cardinality termed as its basis and cardinality of the set of basis is called fault-tolerant metric dimension(FTMD), denoted as $\beta'(G)$. Clearly $\beta'(G) \geq \beta(G) + 1$, also MD and FTMD satisfy the following inequality $\beta'(G) \leq \beta(G)(1 + (2.5)^{\beta(G)-1})$ [13]. Raza et. al. [14], investigated the FTMD of convex polytopes. Vietz et. al. in [15] studied about the FTMD of co-graphs.
Liu et. al. [16] studied the FTMD of wheel related graphs. For comprehensive overview about fault-tolerant, we refer the reader to [17], [18] and [19].

In this article, we calculated the metric dimension and FTMD for triangular benzenoid system and hourglass benzenoid system. From numerous years these graphs are under discussion. For further detailed about triangular and hourglass benzenod system, we refer the reader to [20] and [21].

2. Metric dimension of triangular and hourglass benzenoid system

Let T_q represents the triangular benzenoid system, the number layers of hexagons in graph is represented by q. The total number of vertices in T_q is given as,

$$|V(T_q)| = \Sigma_{k=1}^{q}(2k+1) + 2q + 1$$
$$= (q+1)^2 - 1 + 2q + 1.$$

We give the sequential labeling to the nodes or vertices of T_q as shown in Figure 1.

Theorem 2.1. *For all $q \geq 1$, we have $dim(T_q) = 2$.*

Proof. The set of vertices of T_q can be partitioned as

$$\begin{aligned} V(T_q) =& \{v_{p,k}\ ,\ p = 1, 2, ..., q\ ,\ 1 \leq k \leq 2p + 1\} \\ & \cup \{v_{p,k}\ ,\ p = q + 1\ ,\ 2 \leq k \leq 2p\}. \end{aligned}$$

Let $B = \{v_{1,1}, v_{q+1,2}\}$. We shall prove that B is resolving set for T_q.
The vector representation of vertices for $p = 1$

$$r(v_{p,k}|B) = \begin{cases} \text{(0,2q-1)}, & \text{for } k = 1 \\ \text{(1,2q)}, & \text{for } k = 2 \\ \text{(2,2q+1)}, & \text{for } k = 3. \end{cases}$$

Also the vector representation of vertices for $2 \leq p \leq q + 1$

$$r(v_{p,k}|B) = \begin{cases} \text{(2p-3,2q-2p+k)}, & \text{for } 2 \leq k \leq 2p - 2\ ,\ k \textit{ is even} \\ \text{(2p-2,2q-2p+k)}, & \text{for } 1 \leq k \leq 2p - 1\ ,\ k \textit{ is odd} \\ \text{(k-1 ,2q-2p+k)}, & \text{for } 2p \leq k \leq 2p + 1. \end{cases}.$$

From above representations, we exclude the cases when $p = q + 1$ and $k = 1$, $k = 2p + 1$ because no vertices are there with labeling $v_{q+1,1}, v_{q+1,2q+1}$.
It is also important to observe that no two different vertices or nodes have the same representation.
$\Rightarrow dim(T_q) \leq 2$. Clearly $\dim(T_q) \geq 2$ as T_q is not a path. Consequently

$$dim(T_q) = 2$$

□

Given two copies of the triangular benzenoid system T_q, overlap their external hexagons to obtain the hourglass benzenoid system X_q. The number of vertices for X_q is

$$|V(X_q)| = 2q^2 + 8q - 4.$$

We give the sequential labeling of the nodes or vertices of X_q as shown in Figure 2.

Theorem 2.2. *For all $q \geq 2$, we have $dim(X_q) = 2$*

Proof. The set of vertices of X_q can be partitioned as

$$\begin{aligned} V(X_q) =& \{v_{p,k}\ ,\ j = 1, 2, ..., q - 1\ ,\ k = 1, 2, ..., 2p + 3\} \\ & \cup \{w_{p,k}\ ,\ p = 1, 2, ..., q - 1\ ,\ k = 1, 2, ..., 2p + 3\} \\ & \cup \{v_{q,k}\ ,\ k = 2, 3, ..., 2q + 2\} \\ & \cup \{w_{q,k}\ ,\ k = 2, 3, ..., 2q + 2\}. \end{aligned}$$

Let $B = \{v_{q,2}, w_{q,2}\}$. We shall prove that B is resolving set for X_q.

The vector representation of vertices for $1 \leq j \leq q$, when $v_{p,k} \in V_1(X_q) \cup V_3(X_q)$ is.

$$r(v_{j,k}|B) = \begin{cases} \text{(2q-2p+k-2,2q+2p-2)}, & \text{for } 1 \leq k \leq 2p + 1\ ,\ k \textit{ is odd} \\ \text{(2q-2p+k-2,2q+2p-3)}, & \text{for } 2 \leq k \leq 2p\ ,\ k \textit{ is even} \\ \text{(2q-2p+k-2,2q+k-3)}, & \text{for } 2j + 2 \leq k \leq 2p + 3 \end{cases}.$$

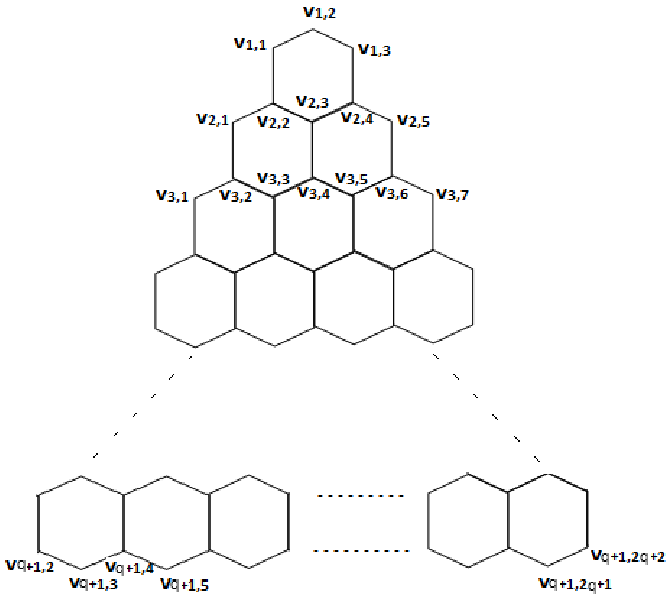

Figure 1: Triangular Benzenoid System

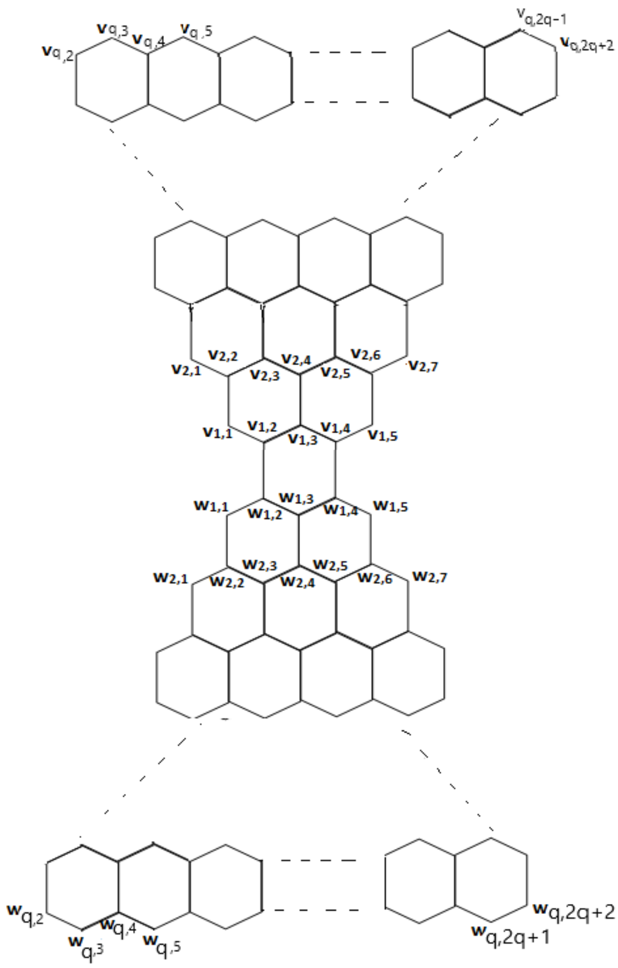

Figure 2: Hourglass Benzenoid System

The vector representation of vertices for $1 \leq p \leq q$, when $w_{p,k} \in V_2(X_q) \cup V_4(X_q)$ is

$$r(w_{j,k}|B) = \begin{cases} (2q+2p-2,2q-2p+k-2), & \text{for } 1 \leq k \leq 2p+1 \text{ , } k \text{ is odd} \\ (2q+2j-3,2q-2j+k-2), & \text{for } 2 \leq k \leq 2p \text{ , } k \text{ is even} \\ (2q+k-3\ ,2q-2p+k-2), & \text{for } 2p+2 \leq k \leq 2p+3 \end{cases} .$$

From above representations, we exclude the cases when $k = 1$ and $k = 2p + 3$ for $p = q$ because no vertices are there with labeling $v_{q,1}, v_{q,2q+3}$. There is same process for the representations of w's.
It is also to be noted the point that the representation of all vertices is different.
$\Rightarrow dim(X_q) \leq 2$. Clearly $\dim(X_q) \geq 2$, as X_q is not a path. Consequently

$$dim(X_q) = 2$$

□

3. Fault-tolerant metric dimension of triangular and hourglass benzenoid system

Theorem 3.1. *For all $q \geq 1$, we have $\beta'(T_q) = 3$.*

Proof. The set of vertices of T_q can be partitioned as

$$V(T_q) = \{v_{p,k}\ ,\ p = 1, 2, ..., q\ ,\ 1 \leq k \leq 2p+1\} \\ \cup \{v_{p,k}\ ,\ p = q+1\ ,\ 2 \leq k \leq 2p\}.$$

Let $B' = \{v_{1,1}, v_{q+1,2}, v_{q+1,2q+2}\}$. To show that B' is fault-tolerant resolving set for triangular benzenoid system. We compute the distance vectors for all vertices of triangular benzenoid system w.r.t B', which are different for at least two coordinates for distinct vertices.
The vector representations of vertices for $p = 1$

$$r(v_{p,k}|B') = \begin{cases} (0,2q-1,2q+1), & \text{for } k = 1 \\ (1,2q,2q), & \text{for } k = 2 \\ (2,2q+1,2q-1), & \text{for } k = 3. \end{cases}$$

Also the vector representations of vertices for $2 \leq p \leq q+1$

$$r(v_{p,k}|B') = \begin{cases} (2p-3,2q-2p+k,2q-k+2), & \text{for } 2 \leq k \leq 2p-2 \text{ , } k \text{ is even} \\ (2p-2,2q-2p+k,2q-k+2), & \text{for } 1 \leq k \leq 2p-1 \text{ , } k \text{ is odd} \\ (k-1,2q-2p+k,2q-k+2), & \text{for } 2j \leq k \leq 2p+1 \end{cases} .$$

These vector representations are different in at least two coordinates. So B' is a fault-tolerant resolving set(FTRS), which means that $\beta'(T_q) \leq 3$. Since $\beta(T_q) = 2$ so $\beta'(T_q) > 2$. Hence

$$\beta'(T_q) = 3$$

□

Theorem 3.2. *For all $q \geq 2$, we have $3 \leq \beta'(X_q) \leq 4$.*

Proof. The set of vertices of X_q can be partitioned as

$$V(X_q) = \{v_{p,k}\ ,\ p = 1, 2, ..., q-1\ ,\ k = 1, 2, ..., 2p+3\} \\ \cup \{w_{p,k}\ ,\ p = 1, 2, ..., q-1\ ,\ k = 1, 2, ..., 2p+3\} \\ \cup \{v_{q,k}\ ,\ k = 2, 3, ..., 2q+2\} \\ \cup \{w_{q,k}\ ,\ k = 2, 3, ..., 2q+2\}.$$

Let $B' = \{v_{q,2}, v_{q,2q+2}, w_{q,2}, w_{q,2q+2}\}$. To show that B' is fault-tolerant resolving set for hourglass benzenoid system. We compute the distance vectors for all vertices of hourglass benzenoid system w.r.t B', which are different for two or more coordinates for distinct vertices.

The vector representations of vertices for $1 \le p \le n$, when $v_{p,k} \in V_1(X_q) \cup V_3(X_q)$

$$r(v_{p,k}|B') = \begin{cases} (2q\text{-}2p\text{+}k\text{-}2, 2q\text{-}k\text{+}2, 2q\text{+}2p\text{-}k\text{-}1, 2q\text{+}2p\text{-}k\text{+}1), & \text{for } 1 \le k \le 2 \\ (2q\text{-}2p\text{+}k\text{-}2, 2q\text{-}k\text{+}2, 2q\text{+}2p\text{-}2, 2q\text{+}2p\text{-}2), & \text{for } 3 \le k \le 2p+1 \text{ , } k \text{ is odd} \\ (2q\text{-}2p\text{+}k\text{-}2, 2q\text{-}k\text{+}2, 2q\text{+}2p\text{-}3, 2q\text{+}2p\text{-}3), & \text{for } 4 \le k \le 2j \text{ , } i \text{ is even} \\ (2q\text{-}2p\text{+}k\text{-}2, 2q\text{-}k\text{+}2, 2q\text{+}k\text{-}3, 2q\text{+}k\text{-}5), & \text{for } 2p+2 \le k \le 2p+3. \end{cases}$$

The vector representation of vertices for $1 \le p \le q$, when $w_{p,k} \in V_2(X_q) \cup V_4(X_q)$

$$r(w_{p,k}|B') = \begin{cases} (2q\text{+}2p\text{-}k\text{-}1, 2q\text{+}2p\text{-}k\text{+}1, 2q\text{-}2p\text{+}k\text{-}2, 2q\text{-}k\text{+}2), & \text{for } 1 \le k \le 2 \\ (2q\text{+}2p\text{-}2, 2q\text{+}2p\text{-}2, 2q\text{-}2p\text{+}k\text{-}2, 2q\text{-}k\text{+}2), & \text{for } 3 \le k \le 2p+1 \text{ , } k \text{ is odd} \\ (2q\text{+}2p\text{-}3, 2q\text{+}2p\text{-}3, 2q\text{-}2p\text{+}k\text{-}2, 2q\text{-}k\text{+}2), & \text{for } 4 \le k \le 2p \text{ , } k \text{ is even} \\ (2q\text{+}k\text{-}3, 2q\text{+}k\text{-}5, 2q\text{-}2p\text{+}k\text{-}2, 2q\text{-}k\text{+}2), & \text{for } 2p+2 \le k \le 2p+3. \end{cases}$$

It gives that at least two vectors coordinates have different representations, which provides that B' is a resolving set of FTMD. So, it is concluded that $\beta'(X_q) \le 4$. Since $\beta(X_q) = 2$, so $\beta'(T_q) > 2$. Consequently

$$3 \le \beta'(X_q) \le 4$$

□

4. Conclusion

In this work, first we provided the MD of triangular benzenoid system represented by T_q and hourglass benzenoid system represented by X_q. We also calculated the (FTMD) of triangular benzenoid system T_q and hourglass benzenoid system X_q.

1. For all $q \ge 1$, we have $dim(T_q) = 2$
2. For all $q \ge 2$, we have $dim(X_q) = 2$
3. For all $q \ge 1$, we have $\beta'(T_q) = 3$
4. For all $q \ge 2$, we have $3 \le \beta'(X_q) \le 4$.

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