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Complexity of Monad graphs generated by the function $f(g) = g^5$

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Abstract

A Monad graph is a graph Γ in which each of its vertices belongs to a finite group G and connects with its image under the action of a linear map f. This kind of graph was introduced by V. Arnold in 2003. In this paper, we compute the Monad graphs in which G is isomorphic to a cyclic group C_n of order n and f the fifth power function, i.e. $f(g) = g^5$. Furthermore, some algebraic and dynamical properties of the studied Monad graphs are obtained. The proofs of our results are based on various tools and results with regard to the fields of number theory, algebra and graph theory.

Keywords: Cyclic group, directed graph, Monad graph, Euler Phi function.
2010 MSC: 11A07, 11A41, 11P83, 05C25.

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1. Introduction and Preliminaries

In 2003, V.I. Arnold [1] introduced a very motivating phenomena termed by *Monad.* that concerns a dynamical systems defined on group actions. More precisely, we may recall the definition of Monad graph. Suppose that G is a group, $f : G \longrightarrow G$ is a map, $f(g) = g^k$ for $g \in G$ with $k \geq 0$, the Monad graph $\Gamma_k(G)$ is defined under the following properties:

1. $V(G) = \{g | g \in G\}$, which represents the set of the vertices,
2. $E(G) = \{a \to a^k | f(a) = a^k \text{ or } f_k(a) \to a^k\}$, which represents the set of edges.

Suppose that G is an arbitrary finite set, f is a function that maps each element in G into itself, and Γ is Monad graph that connects each of the vertices of the corresponding G with its image by directed connected edge under the action of map f, then a discrete dynamical system has the form of triple (G, Γ, f_g) or (G, Γ_g, f_g).

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A special case of a discrete dynamical system $(G, \Gamma, f_g) = (G, \Gamma, g^2)$ was considered by Arnold [1] in which he showed the following result: Each connected component of a Monad graph is a forest of rooted trees directed towards their roots, which lie on a directed cycle (topologically, a circle) formed by the edges connecting the roots.

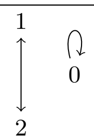
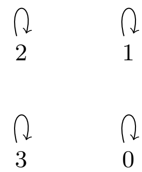
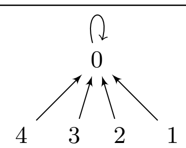
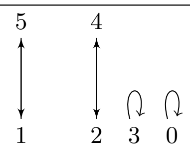
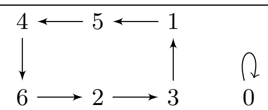
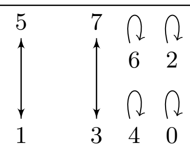
On the other hand, it's well-known that *the finite cycle orbit* of an element is the action of element g under the map f_g (see e.g. [7]) i.e.

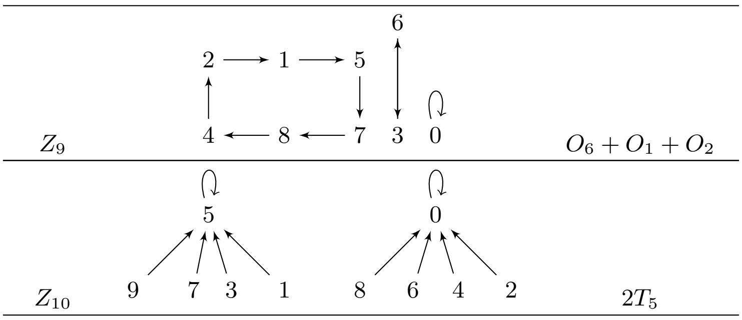
$$orb_c(g, f_g) = \{g, f(g), f^2(g), \cdots, f^{n-1}(g) : \ n \in \mathbb{N}\}.$$

For $g \in G$, we remind $f^n(g) = g$ for $n \in \mathbb{N}$ is called *periodic point*. We denote by $Pr(g)$ the set of periodic points. For more information about such graphs, see e.g. [2], [3], [8] and [9].

In this paper, we study the Monad graph of discrete dynamical system (G, Γ, f_g), where G is isomorphic to cyclic group C_n of order n and f_g is given by the map $f(g) = g^5$. Furthermore, we show the dynamical properties of each element in the group C_n and prove our results using elementary results regarding the fields of number theory, graph theory and algebra.

To show the action of this map on certain groups, in the following table we show the Monad graphs generated by the map $f(g) = g^5$ for the group Z_n with $2 \leq n \leq 10$ under the additive operation:

Group	*Monad Graph*	*Graph's symbol*
Z_2		$O_1 + O_1$
Z_3		$O_2 + O_1$
Z_4		$4O_1$
Z_5		T_5
Z_6		$2O_2 + 2O_1$
Z_7		$O_6 + O_1$
Z_8		$2O_2 + 4O_1$



where O_n denotes the directed cycle of length n and T_{5^m} is the rooted tree with 5^m vertices with m branches.

An interesting result regarding the directed cycle O_n (due to [1]) stated that the product of cyclic Monads graph O_n and O_m is a sum of identical cyclic Monads; namely

$$O_n * O_m = dO_c,$$

where $d = gcd(n, m)$ and $c = \frac{nm}{d}$.

For later use and in the proof of our main results, we need to recall an important concept in number theory called by the Euler Phi Function, which is denoted by $\varphi(n)$, that is given by

$$\varphi(n) = \prod_{1\leq i\leq s} \varphi(p_i^{\alpha_i}) = \prod_{1\leq i\leq s} (p_i^{\alpha_i}(1 - \frac{1}{p_i})) = n \prod_{1\leq i\leq s} (1 - \frac{1}{p_i}),$$

where $n = \prod_{1\leq i\leq s} p_i^{\alpha_i}$ represent a positive integer number with p denotes a prime number and the integer $\alpha_i \geq 1$. Moreover, $\Delta(n)$ represents the set of all divisors of n. More precisely, the Euler Phi Function counts the positive integers less than a given integer n that are relatively prime to n. For more details about the Euler Phi Function and its application, one can see e.g. [4], [5] and [6].

2. Main results

In this section, we introduce our results concerning to the Monad graphs in case of the mapping $f : g \to g^t$ and t is an odd prime number represented by 5.

Firstly, we present the results of Monad graphs when G be isomorphic to C_n and $n = 2^r$, $r \geq 1$.

Proposition 2.1. *Let C_n be a cyclic group and $n = 2^r, r \geq 3$, then, there exist exactly 2 of O_2.*

Proof. In fact, the elements of the group will act under the Monad mapping $f(g) = g^5$ under modulo $n = 2^r$. We can see the elements

$$a^{2^{r-3}} \equiv a^{5\cdot 2^{r-3}} \to a^{2^{r-3}+2^{r-1}}$$

and

$$a^{2^{r-3}+2^{r-1}} \equiv a^{5(2^{r-3}+2^{r-1})} \equiv a^{2^{r-3}+2^{r-1}} a^{5\cdot 2^{r-1}} \equiv a^{2^{r-3}+2\cdot 2^{r-1}} \to a^{2^{r-3}}.$$

Similarly, we can easily see that

$$a^{2^{r-3}+2^{r-2}} \to a^{2^{r-3}+2^{r-2}+2^{r-1}}$$

and

$$a^{2^{r-3}+2^{r-2}+2^{r-1}} \to a^{2^{r-3}+2^{r-2}}.$$

Hence, the proposition is proved. □

Corollary 2.2. *Let C_n be a cyclic group and $n = 2^r, r \geq 1$, the following are held:*

1. *If $r = 1$, then there exist exactly 2 of O_1;*

2. *If $r \geq 2$, then there exist exactly 4 of O_1 and representation by $\{a^{2^{r-1}}, a^{2^{r-2}}, a^{2^{r-1}+2^{r-2}}, e\}$.*

Proof. The proof of this corollary can be achieved by following the proof of Proposition 2.1. Thus, we omit the details of the proof. □

Proposition 2.3. *Let C_n be a cyclic group and $n = 2^r, r \geq 4$, then, there exist exactly 2 of O_4.*

Proof. The elements of the group will act under the Monad mapping $f(g) = g^5$ (or sometimes we denoted by $f_5(g) = g^5$) under modulo $n = 2^r$. We indeed can see the elements:

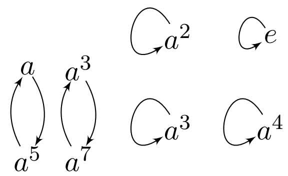
$$\begin{aligned}
f_5(a^{2^{r-4}}) &= a^{5\cdot 2^{r-4}} \\
&= a^{(2^2+1)\cdot 2^{r-4}} \\
&= a^{2^{r-2}+2^{r-4}}. \\
f_5(a^{2^{r-2}+2^{r-4}}) &= a^{5\cdot 2^{r-2}+2^{r-4}} \\
&= a^{(2^2+1)\cdot 2^{r-2}+2^{r-4}} \\
&= a^{2^{r-1}+2^{r-4}}. \\
f_5(a^{2^{r-1}+2^{r-4}}) &= a^{5\cdot 2^{r-1}+2^{r-4}} \\
&= a^{(2+2+1)\cdot 2^{r-1}+2^{r-4}} \\
&= a^{2^{r-1}+2^{r-2}+2^{r-4}}. \\
f_5(a^{2^{r-1}+2^{r-2}+2^{r-4}}) &= a^{5\cdot 2^{r-1}+2^{r-2}+2^{r-4}} \\
&= a^{(2+2+1)\cdot 2^{r-1}+2^{r-2}+2^{r-4}} \\
&= a^{2^{r-4}}.
\end{aligned}$$

A similar approach can be followed for the element $a^{2^{r-3}+2^{r-4}}$. Hence, the proposition is proved. □

Proposition 2.4. *The Monad graph of the cyclic group C_n when $n = 2^r, r \geq 1$ is given by the following:*

1. *If $r = 1$, then $\Gamma(C_2) = 2O_1$;*
2. *If $r = 2$, then $\Gamma(C_{2^2}) = 4O_1$;*
3. *If $r = 3$, then $\Gamma(C_{2^3}) = 2O_2 + 4O_1$;*
4. *If $r \geq 4$, then $\Gamma(C_{2^r}) = \sum_{i=2}^{r-1} 2O_{2^{r-i}} + 4O_1$.*

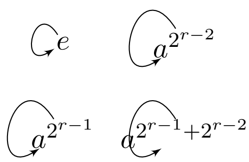
Proof. One can easily see that the first two cases are trivial, so we start proving Case **3** with which we have the group $C_8 = \{e, a, a^2, \ldots, a^7\}$ with a generates e, that denotes the identity of C_8. Starting with the identity element, one easily case see it goes to itself under the Monad mapping $f(g) = g^5$. We also see that the element a^2 has the order 4, and with the mapping $f(g) = g^5$ it goes to itself, i.e. $a^2 \equiv a^{10} \to a^2$. Let's next consider the elements a^4 and a^6, that also go to themselves under the Monad mapping, namely $a^4 \to a^4$ and $a^6 \to a^6$, respectively. Thus, we have $4O_1$. Finally, we consider the other elements $a \to a^5$, $a^5 \to a^1$, $a^3 \to a^7$ and $a^7 \to a^3$. These give the $2O_2$. These mappings can be represented in the following graph:



Finally, we consider Case **(4)**. Indeed, there exist exactly four elements loop to themselves, namely e, $a^{2^{r-2}}$,$a^{2^{r-1}}$ and $a^{2^{r-2}+2^{r-1}}$ since $f_5(e) = e$ and

$$\begin{aligned} f_5(a^{2^{r-1}}) &= a^{52^{r-1}} \\ &= a^{(2+2+1)2^{r-1}} \\ &= a^{2^r}a^{2^r}a^{2^{r-1}mod(2^r)} \\ &= a^{2^{r-1}}. \\ f_5(a^{2^{r-2}}) &= a^{52^{r-2}} \\ &= a^{(2^2+1)2^{r-2}} \\ &= a^{2^r}a^{2^{r-1}mod(2^r)} \\ &= a^{2^{r-2}}. \\ f_5(a^{2^{r-2}+2^{r-1}}) &= a^{5(2^{r-2}+2^{r-1})} \\ &= a^{(2^2+1)2^{r-2}+(2+2+1)2^{r-1}} \\ &= a^{2^r+2^{r-2}+2^r+2^r+2^{r-1}mod(2^r)} \\ &= a^{2^{r-2}+2^{r-1}}. \end{aligned}$$

In other words, these mappings can be represented as follows:



Also, for the elements $a^{2^{r-3}}$ and $a^{5\cdot 2^{r-3}}$ we see the following:

$$\begin{aligned} f_5(a^{2^{r-3}}) &= a^{52^{r-3}} = a^{(2^2+1)2^{r-3}} = a^{2^{r-1}+2^{r-3}}. \\ f_5(a^{2^{r-1}+2^{r-3}}) &= a^{5*(2^{r-1}+2^{r-3})} = a^{(2^{r-1}+2^{r-1}+2^{r-3})} = a^{2^{r-3}}. \end{aligned}$$

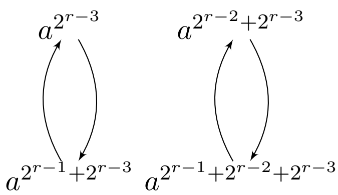
Also, for the elements $a^{2^{r-2}+2^{r-3}}$ and $a^{5\cdot 2^{r-2}+2^{r-3}}$ we see the following:

$$\begin{aligned} f_5(a^{2^{r-2}+2^{r-3}}) &= a^{2^{r-1}+2^{r-2}+2^{r-3}}, \\ f_5(a^{2^{r-1}+2^{r-2}+2^{r-3}}) &= a^{2^{r-2}+2^{r-3}}, \end{aligned}$$

which means there exists an edge from $v_{a^{2^{r-2}+2^{r-3}}}$ into $v_{a^{5\cdot 2^{r-2}+2^{r-3}}}$. For

$$\begin{aligned} f_5(a^{2^{r-2}+2^{r-3}}) &= a^{5\cdot(2^{r-2}+2^{r-3})} \\ &= a^{2^{r-1}+2^{r-2}+2^{r-3}} \\ &= f_5(a^{2^{r-1}+2^{r-2}+2^{r-3}}) \\ &= a^{2^{r-1}+2^{r-2}+2^{r-3}}, \end{aligned}$$

that also means there exists an edge from $v_{a^{32^{r-3}}}$ into $v_{a^{2^{r-3}}}$. Similar idea goes to the elements $a^{2^{r-2}}$, $a^{6\cdot 2^{r-3}}$, $a^{5\cdot 2^{r-3}}$ and $a^{7\cdot 2^{r-3}}$, and they are represented in the following figure:



Now, it remains to consider the remaining 2^{r-1} elements. The number 2^{r-1} is clearly an odd number. Here, this leads to two cyclic graphs of order $O_{2^{r-2}}$ since these elements are clearly splitted up to two cyclic graphs of order $O_{2^{r-2}}$. To finish the proof of this case, it is enough to show that the order of the group is equal to the order of the Monad graph. This can be done as follows:

$$2\sum_{t=2}^{r-1}|V(O_{2^{r-t}})|+4|V(O_1)|=2\sum_{t=2}^{r-1}2^{r-t}+4.$$

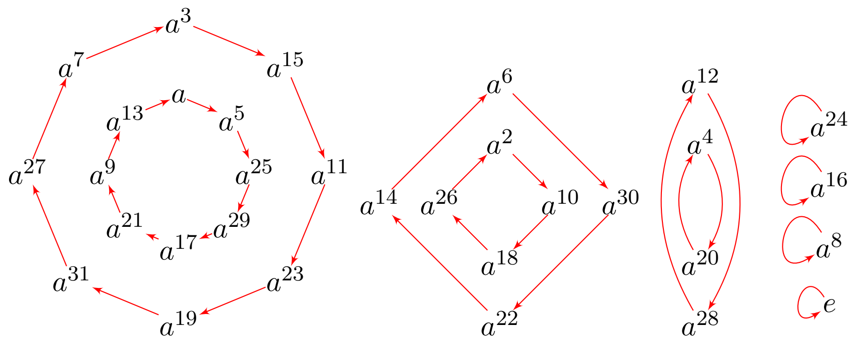
Let's now consider the right hand side of the latter equation, we obtain that

$$\begin{aligned}
2\sum_{t=2}^{r-1}2^{r-t}+4 &= 2(2^{r-2}+2^{r-3}+\cdots+2^2+2)+2^2\\
&= 2^2(2^{r-3}+2^{r-4}+\cdots+2+1)+2^2\\
&= 2^2[(2^{r-3}+2^{r-4}+\cdots+2+1)+1)]\\
&= 2^2[2^{r-2}-1+1]\\
&= 2^r\\
&= |C_n|=|V(C_n)|,
\end{aligned}$$

and this completes the proof of the fourth case. □

Example 2.5. Let's take $n=2^5$, then the Monad graph $\Gamma_5(C_{2^7})$ is isomorphic to $2O_{32}+2O_{16}+2O_8+2O_4+2O_2+4O_1$, which is obtained as follows: the set of all vertices is $V(\Gamma_5)=\{a^i : 1\le i\le 128\}$ and the set of all edges is given by $E(\Gamma_5)=\{\{a^i,f_5(a^i)\} : 1\le i\le 128\}$. Therefore, the corresponding Monad graph is represented by

n	$\Gamma_5(C_n)$
4	$2O_4+2O_2+4O_1$
5	$2O_8+2O_4+2O_2+4O_1$
6	$2O_{16}+2O_8+2O_4+2O_2+4O_1$
7	$2O_{32}+2O_{16}+2O_8+2O_4+2O_2+4O_1$
8	$2O_{64}+2O_{32}+2O_{16}+2O_8+2O_4+2O_2+4O_1$



Proposition 2.6. *Let C_n be a cyclic group and $n=3^\alpha, 1\le\alpha\le4$, the following are hold:*

1. *If $\alpha=1$, then there exist exactly 1 of O_1 and 1 of O_2;*
2. *If $\alpha=2$, then there exist exactly 1 of O_1, 1 of O_2 and 1 of O_6;*
3. *If $\alpha=3$, then there exist exactly 1 of O_1, 1 of O_2, 1 of O_6 and 1 of O_{18};*
4. *If $\alpha=4$, then there exist exactly 1 of O_1, 1 of O_2, 1 of O_6, 1 of O_{18} and 1 of O_{54}.*

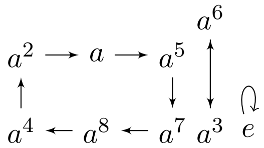
Proof. The first case can be easily achieved, so we may start considering the second case in which we have $C_9 = \{e, a, a^2, \ldots, a^8\}$ with a generates e, that denotes the identity of C_9. It is clear that the identity element goes to itself under the Monad mapping $f(g) = g^5$. We also see that the element a^3 with the mapping $f(g) = g^5$ goes to a^6, which also goes to a^3, i.e.

$$a^3 \equiv a^{15} \to a^6 \equiv a^{30} \to a^3.$$

Also, under the same mapping, we see that

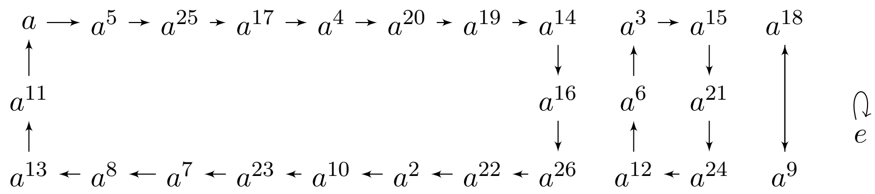
$$a^1 \to a^5 \to a^7 \to a^8 \to a^4 \to a^2 \to a^1.$$

We also can represent this case with the following graph:



Thus, we have exactly 1 of O_1, 1 of O_2 and 1 of O_6.

Next, we deal with the third case in which we have that $C_{27} = \{e, a, a^2, \ldots, a^{26}\}$. Under the Monad mapping $f(g) = g^5$, we summarize the details of computations for the elements of C_{27} as follows:



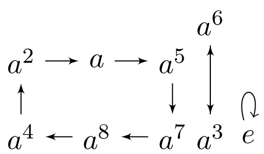
This proves that with $\alpha = 3$, there exist exactly 1 of O_1, 1 of O_2, 1 of O_6 and 1 of O_{18}.

It remains to consider the last case with $\alpha = 4$, namely we deal with the elements of the group $C_{81} = \{e, a, a^2, \ldots, a^{80}\}$ with the Monad mapping $g \to g^5$. In fact, in a similar way done with above cases we can easily obtain that there are only 1 of O_1, 1 of O_2, 1 of O_6, 1 of O_{18} and 1 of O_{54}. Thus, Proposition 2.6 is completely proved. □

Proposition 2.7. *The Monad graph of the cyclic group C_n when $n = 3^\alpha, \alpha \geq 1$ is given by the following:*

1. *If $\alpha = 1$, then $\Gamma(C_3) = O_2 + O_1$;*
2. *If $\alpha = 2$, then $\Gamma(C_{3^2}) = O_6 + O_2 + O_1$;*
3. *If $\alpha \geq 3$, then $\Gamma(C_{3^\alpha}) = \sum_{t=0}^{\alpha-1} O_{23^t} + O_1$.*

Proof. Here, the proof of these three cases can be achieved in a similar way as done in the proofs of Propositions 2.4 and 2.6, so we only consider proving the second case and the others are proved similarly. We have the group $C_9 = \{e, a, a^2, \ldots, a^8\}$ with a generates e, that denotes the identity of C_9. From the proof of the second case of Proposition 2.6, we have from this mapping the following graph:



That has exactly 1 of O_1, 1 of O_2 and 1 of O_6. Thus, with $\alpha = 2$, we obtain that $\Gamma(C_{3^2}) = O_6 + O_2 + O_1$. This proves the second case of Proposition 2.7, and the remaining cases will be achieved similarly. So, we omit the details of computations. Hence, Proposition 2.7 is proved.

□

Proposition 2.8. *The Monad graph of the cyclic group C_n when $n = 5^\alpha, \alpha \geq 1$ is given by the following:*

$$\Gamma(C_n) \cong T_{5^\alpha}.$$

Proof. Suppose that $n = 5^\alpha$. Consider the following five elements in the group that are $a^{5*i}{}_{0\leq i\leq 4}$ which goes to e, $a^{5*i+1}{}_{0\leq i\leq 4}$ goes to a^5, $a^{5*i+2}{}_{0\leq i\leq 4}$ goes to a^{10}, $a^{5*i+3}{}_{0\leq i\leq 4}$ goes to a^{15} and $a^{5*i+4}{}_{0\leq i\leq 4}$ goes to a^{20}, which imply that

$$\begin{aligned} f_5(a^{5*i}) &= a^{25i} = e \\ f_5(a^{10*i}) &= a^{50i} = e \\ f_5(a^{15*i}) &= a^{75i} = e \\ f_5(a^{20*i}) &= a^{100i} = e, \end{aligned}$$

$$\begin{aligned} f_5(a^{5*i+1}) &= a^{25i}a^5 = a^5 \\ f_5(a^{10*i+1}) &= a^{50i}a^5 = a^5 \\ f_5(a^{15*i+1}) &= a^{75i}a^5 = a^5 \\ f_5(a^{20*i+1}) &= a^{100i}a^5 = a^5, \end{aligned}$$

$$\begin{aligned} f_5(a^{5*i+2}) &= a^{25i}a^{10} = a^{10} \\ f_5(a^{10*i+2}) &= a^{50i}a^{10} = a^{10} \\ f_5(a^{15*i+2}) &= a^{75i}a^{10} = a^{10} \\ f_5(a^{20*i+2}) &= a^{100i}a^{10} = a^{10}, \end{aligned}$$

$$\begin{aligned} f_5(a^{5*i+3}) &= a^{25i}a^{15} = a^{15} \\ f_5(a^{10*i+3}) &= a^{50i}a^{15} = a^{15} \\ f_5(a^{15*i+3}) &= a^{75i}a^{15} = a^{15} \\ f_5(a^{20*i+3}) &= a^{100i}a^{15} = a^{15} \end{aligned}$$

and

$$\begin{aligned} f_5(a^{5*i+4}) &= a^{25i}a^{20} = a^{20} \\ f_5(a^{10*i+4}) &= a^{50i}a^{20} = a^{20} \\ f_5(a^{15*i+4}) &= a^{75i}a^{20} = a^{20} \\ f_5(a^{20*i+4}) &= a^{100i}a^{20} = a^{20}. \end{aligned}$$

In general, let $a^{5^{r-t}\cdot i+k}$, where $n = 5^r$, $1 \leq t \leq r, 0 \leq i, k \leq 4$. Thus,

$$\begin{aligned} f_5(a^{5^{r-t}\cdot i+k}) &= a^{55^{r-t}\cdot i+5k} \\ &= a^{5^{r-t+1}\cdot i+5k}. \end{aligned}$$

Therefore, we have the following cases that complete the proof:

1. If $k = 0$, then
$$f_5(a^{5^{r-t}\cdot 0}) = a^{5^{r-t+1}}.$$

2. If $k = 1$, then
$$f_5(a^{5^{r-t}\cdot 1}) = a^{5^{r-t+1}+5}.$$

3. If $k = 2$, then
$$f_5(a^{5^{r-t}\cdot 2}) = a^{5^{r-t+1}+10}.$$

4. If $k = 3$, then
$$f_5(a^{5^{r-t}\cdot 2}) = a^{5^{r-t+1}+15}.$$

5. If $k = 4$, then
$$f_5(a^{5^{r-t}\cdot 4}) = a^{5^{r-t+1}+20}.$$

Therefore, Proposition 2.8 is completely proved. □

Remark 2.9. As a problem that can be left to the reader to prove, we propose the following conjecture:
Conjecture: Let C_n be a cyclic group and $n = p^{\alpha}$ with $\alpha \geq 1$ and $p \neq 2, 3, 5$ is prime, then the following is held:
$$\Gamma(C_n) = kO_{\frac{p-1}{k}} + O_1,$$
where k is some positive integer. As a hint, this conjecture could be proved using the theory of the Euler Phi function.

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