Available online at http://jprm.sms.edu.pk/
Journal of Prime Research in Mathematics, 20(2) (2024), 30–47

Journal of Prime Research in Mathematics

A common generalization of Dickson polynomials, Fibonacci polynomials, and Lucas polynomials and applications

Said Zriaa*, Mohammed Mouçouf

Department of Mathematics, University Chouaib Doukkali, Eljadida Morocco.

Abstract

In this work, we define a more general family of polynomials in several variables satisfying a linear recurrence relation. We provide explicit formulas and determinantal expressions. Our results are then applied to second-order recurrent polynomials, presenting several relationships and identities involving Fibonacci polynomials of order 2, Lucas polynomials of order 2, classical Fibonacci polynomials, classical Lucas polynomials, Fibonacci numbers, Lucas numbers, and both kinds of Dickson polynomials. Our findings offer a unified generalization of various existing works, with several well-known results emerging as special cases.

Keywords: Dickson polynomials, Fibonacci numbers, Lucas numbers, Fibonacci polynomials, Lucas polynomials.
2010 MSC: 11B39, 11B83.

1. Introduction

The classical Fibonacci polynomials $F_n(x)$ and the classical Lucas polynomials $L_n(x)$ are defined by

$$F_0(x) = 0, \ \ F_1(x) = 1, \ \ F_{n+2}(x) = xF_{n+1}(x) + F_n(x), n \geq 0,$$

$$L_0(x) = 2, \ \ L_1(x) = x, \ \ L_{n+2}(x) = xL_{n+1}(x) + L_n(x), n \geq 0.$$

It is well known that the explicit expressions of the sequences $\{F_n(x)\}_{n\geq 0}$ and $\{L_n(x)\}_{n\geq 0}$ are given by the Binet formula,

$$F_n(x) = \frac{1}{\alpha - \beta}(\alpha^n - \beta^n), \ \text{ and } \ L_n(x) = \alpha^n + \beta^n, \ \text{ for all } \ n \geq 0,$$

*Corresponding author
Email addresses: saidzriaa1992@gmail.com (Said Zriaa), moucouf@hotmail.com (Mohammed Mouçouf)

Received : 23 January 2024*; Accepted:* 24 September 2024*; Published Online:* 26 September 2024.

where $\alpha = \frac{1}{2}(x + \sqrt{x^4+4})$ and $\beta = \frac{1}{2}(x - \sqrt{x^4+4})$.

These polynomials are of great importance in the study of various topics such as number theory, algebra, combinatorics, statistics, geometry, approximation theory, and other areas. It is not easy to describe versatile applications that rely on the classical Fibonacci polynomials and the classical Lucas polynomials. For specific references to some applications, the reader can consult for example [4, 5, 18, 24, 26]. The classical Fibonacci and Lucas polynomials have always attracted the attention of several researchers. Therefore, many generalizations and research works have been considered in recent studies on the subject; some of them can be found in [6, 8, 11, 14, 19, 21, 23, 25, 27]. Regardless of the fact that they are of great importance, to the best of our knowledge, no one has yet considered and studied the general case.

The theory of sequence polynomials can be applied to have powerful results on certain integer sequences including Fibonacci numbers, Lucas numbers, and a wide range other of sequences.
It is clear that when $x = 1$, the classical Fibonacci polynomials turn into the well known Fibonacci numbers, $F_n(1) = F_n$. Also, it is clear that when $x = 1$, the classical Lucas polynomials turn into the well known Lucas numbers, $L_n(1) = L_n$.
The Fibonacci and Lucas numbers are of intrinsic interest and have various fascinating properties. They too continue to amaze mathematicians with their splendid beauty, applicability, and ubiquity. They provide delightful opportunities to explore, experiment, conjecture, and problem-solve. The Fibonacci and Lucas numbers form a unifying thread intertwining geometry, analysis, trigonometry, and numerous areas of discrete mathematics such as combinatorics, linear algebra, number theory, and graph theory. For a deep and extensive survey of the theory and applications of the Fibonacci and Lucas numbers, we refer the reader to the book [10]. That research monograph contains not only a comprehensive treatise on this topic but contains in one location all currently known results concerning these numbers and their numerous applications and an extensive bibliography.

In this paper, we study general polynomial sequences in several variables of high order. Further, we investigate these polynomials from different points of view, and we provide interesting explicit formulas and elegant determinantal expressions. Using these results, we obtain numerous interesting identities involving the Fibonacci polynomials order 2 and the Lucas polynomials of order 2. Consequently, several relationships between the classical Fibonacci and Lucas polynomials are presented. Also, relationships between the Dickson polynomials of the first and the second kind are provided. The results presented in this work can be used to recuperate, generalize, and develop various essential works on this important topic.

2. Recurrent polynomials in several variables over unitary commutative ring

Let R be a commutative ring with identity. In the commutative ring $R[x_1, x_2, \cdots, x_k]$ of polynomials in the variables $x_1, x_2, \cdots, x_k$, we consider the sequence of polynomials $(P_n(x_1, x_2, \cdots, x_k))_{n\geq 0}$ satisfying the following general recurrence relation

$$P_{n+k}(x_1, x_2, \cdots, x_k) = \sum_{i=1}^{k} c_i P_{n+k-i}(x_1, x_2, \cdots, x_k), \quad n \geq 0, \tag{2.1}$$

where for $i = 1, 2, \ldots, k$, we have $c_i = q_i(x_1, x_2, \cdots, x_k)$ is a polynomial in $x_1, x_2, \cdots, x_k$.
For $j = 0, 1, 2, \ldots, k-1$, we consider the elements $P_n^{(j)}(x_1, x_2, \cdots, x_k)$ satisfying the recurrence relation (2.1) with the initial conditions $P_i^{(j)}(x_1, x_2, \cdots, x_k) = \delta_{ij}$ for $i = 0, 1, 2, \ldots, k-1$.
The polynomial

$$g(X) = X^k - c_1 X^{k-1} - c_2 X^{k-2} - \cdots - c_k, \tag{2.2}$$

is called the characteristic polynomial of each elements solution to this recurrence relation.
In general, we call the polynomials solution to the equation (2.1) the recurrent polynomials of order k, and it is clear that they can be written as

$$P_n(x_1, x_2, \cdots, x_k) = \sum_{i=0}^{k-1} P_i(x_1, x_2, \cdots, x_k) P_n^{(i)}(x_1, x_2, \cdots, x_k), \text{ for } n \geq 0. \tag{2.3}$$

Example 2.1. In the case where $c_i = (-1)^{i+1}$, the recurrent polynomials are equivalent to a set of polynomials termed the Dickson polynomials in several variables, which have occurred in many applications in the theory of finite fields [16].

Example 2.2. In the case of one variable, that is when $x_1 = x_2 = \cdots = x_k = x$, and $c_i = x^{k-i}$, the recurrent polynomials of order k turn into the k-step Lucas polynomials studied and extended recently in [13] with the aid of some matrices.

For the remainder of this section, we only consider $P_n^{(k-1)}(x_1, x_2, \cdots, x_k)$ instead of $P_n^{(l)}(x_1, x_2, \cdots, x_k), l = 0, \ldots, k-2$, because of its simplicity in calculus and the fact that each polynomial $P_n^{(l)}(x_1, x_2, \cdots, x_k), l = 0, \ldots, k-2$, can be represented in terms of this important recurrent polynomial.

For proving some main results, we need the following lemma, which is a consequence of Theorem1.3 and Theorem1.4 of [28].

Lemma 2.3. *Let $c_1, c_2, \ldots, c_k$ be the elements of a commutative ring with unity. Then*

$$\left(1 - \sum_{i=1}^{k} c_i X^i\right)^{-1} = 1 + \sum_{n \geq 1} b_n X^n,$$

where

$$b_n = \sum_{i_1+2i_2+\cdots+ki_k=n} \frac{(i_1 + i_2 + \cdots + i_k)!}{i_1! i_2! \cdots i_k!} c_1^{i_1} c_2^{i_2} \cdots c_k^{i_k},$$

and

$$b_n = (-1)^n \begin{vmatrix} -c_1 & -c_2 & \cdots & \cdots & \cdots & -c_n \\ 1 & \ddots & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & -c_2 \\ 0 & \cdots & \cdots & 0 & 1 & -c_1 \end{vmatrix}.$$

with $c_n = 0$ if $n \geq k+1$.

Remark 2.4. It is important to note that this lemma has clear potential for broad applications and further developments, offering a flexible tool that can be adapted to various contexts and contribute to future research in multiple fields. We encourage interested readers to refer to some applications treated recently in [20].

In the following theorem, we derive a formula of $P_n^{(k-1)}(x_1, x_2, \cdots, x_k)$ in terms of $c_1, c_2, \ldots, c_k$ the coefficients of the polynomial (2.2) which are polynomials in $x_1, x_2, \cdots, x_k$.

Theorem 2.5. *For $n \geq 0$, we have*

$$P_{n+k-1}^{(k-1)}(x_1, x_2, \cdots, x_k) = \sum_{i_1+2i_2+\cdots+ki_k=n} \frac{(i_1 + i_2 + \cdots + i_k)!}{i_1! i_2! \cdots i_k!} c_1^{i_1} c_2^{i_2} \cdots c_k^{i_k},$$

and

$$P_{n+k-1}^{(k-1)}(x_1, x_2, \cdots, x_k) = \begin{vmatrix} c_1 & c_2 & \cdots & \cdots & \cdots & c_n \\ -1 & \ddots & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & c_2 \\ 0 & \cdots & \cdots & 0 & -1 & c_1 \end{vmatrix},$$

with $c_n = 0$ *if* $n \geq k+1$.

Proof. It is a routine to check that

$$\sum_{n=0} P_{n+k-1}^{(k-1)}(x_1, x_2, \cdots, x_k)X^n = \Big(1 - (c_1X + \cdots + c_kX^k)\Big)^{-1}.$$

Then the rest follows easily from Lemma 2.3. □

Now, we recall an important class of recurrent polynomials of order k.

Definition 2.6. The generalized Lucas polynomials in several variables of order $k \geq 2$ are defined by

$$L_{n+k}(x_1, x_2, \cdots, x_k) = \sum_{i=1}^{k}(-1)^{i+1}x_iL_{n+k-i}(x_1, x_2, \cdots, x_k),$$

with $L_j(x_1, x_2, \cdots, x_k) = \delta_{j,k-1}$ for $0 \leq j \leq k-1$.

R. Barakat and E. Baumann [3] indicated the great importance of the generalized Lucas polynomials in several variables in many physical problems and asked for obtaining them in a closed-form formula. Here, we provide the solution to this problem in the following interesting theorem.

Theorem 2.7. *For all* $n \geq 0$*, the generalized Lucas polynomials of order* $k \geq 2$ *satisfy the following results*

$$L_{n+k-1}(x_1, x_2, \ldots, x_k) = \sum_{i_1+2i_2+\cdots+ki_k=n} \frac{(i_1+i_2+\cdots+i_k)!}{i_1!i_2!\cdots i_k!}(-1)^{k+i_1+i_2+\cdots+i_k}x_1^{i_1}x_2^{i_2}\cdots x_k^{i_k},$$

and

$$L_{n+k-1}(x_1, x_2, \ldots, x_k) = \begin{vmatrix} x_1 & -x_2 & \cdots & \cdots & \cdots & (-1)^{n+1}x_n \\ -1 & \ddots & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & -x_2 \\ 0 & \cdots & \cdots & 0 & -1 & x_1 \end{vmatrix}_{n\times n}.$$

with $x_n = 0$ *if* $n \geq k+1$.

Proof. It is obvious that when $c_i = (-1)^{i+1}x_i$, we have $L_n(x_1, x_2, \ldots, x_k) = P_n^{(k-1)}(x_1, x_2, \cdots, x_k)$. The rest can be deduced immediately from Theorem 2.5. □

3. Linear recurrence sequences via formal power series

In this section, we derive explicit formulas of the recurrent polynomial $P_n^{(k-1)}(x_1, x_2, \cdots, x_k)$ in terms of the roots of its characteristic polynomial (2.2). In the sequel, we consider only $R = \mathbb{C}$ the field of complex numbers.

Theorem 3.1. *Let* $g(X) = \prod_{i=1}^{k}(X - \alpha_i) = X^k - c_1X^{k-1} - c_2X^{k-2} - \cdots - c_k$ *be the characteristic polynomial of* $P_n^{(k-1)}(x_1, x_2, \cdots, x_k)$ *having different roots* $\alpha_1 = \alpha_1(x_1, x_2, \cdots, x_k), \alpha_2 = \alpha_2(x_1, x_2, \cdots, x_k), \ldots, \alpha_k = \alpha_k(x_1, x_2, \cdots, x_k)$*. Then for* $n \geq 0$*, we have*

$$P_{n+k-1}^{(k-1)}(x_1, x_2, \cdots, x_k) = \sum_{i_1+i_2+\cdots+i_k=n} \alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_k^{i_k}.$$

Proof. By algebraic manipulations, we can easily show that

$$\begin{aligned}\sum_{n\geq 0} P_{n+k-1}^{(k-1)}(x_1,x_2,\cdots,x_k)X^n &= \left(1-c_1X-\cdots-c_kX^k\right)^{-1}\\ &= \prod_{i=1}^{k}(1-\alpha_iX)^{-1}\\ &= \prod_{i=1}^{k}(\sum_{n\geq 0}\alpha_i^nX^n)\\ &= \sum_{n\geq 0}\sum_{i_1+i_2+\cdots+i_k=n}\alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_k^{i_k}X^n\end{aligned}$$

Consequently

$$P_{n+k-1}^{(k-1)}(x_1,x_2,\cdots,x_k) = \sum_{i_1+i_2+\cdots+i_k=n}\alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_k^{i_k}.$$

as claimed in the result. □

The previous result was obtained by C. Levesque in [12]. The general case when the characteristic polynomial of $P_n^{(k-1)}(x_1,x_2,\cdots,x_k)$ has multiple roots is the following general result.

Theorem 3.2. *Let $g(X)=\prod_{i=1}^{s}(X-\alpha_i)^{m_i}=X^k-c_1X^{k-1}-c_2X^{k-2}-\cdots-c_k$ be the characteristic polynomial of $P_n^{(k-1)}(x_1,x_2,\cdots,x_k)$ having different roots $\alpha_1=\alpha_1(x_1,x_2,\cdots,x_k),\alpha_2=\alpha_2(x_1,x_2,\cdots,x_k),\ldots,\alpha_s=\alpha_s(x_1,x_2,\cdots,x_k)$ with multiplicities $m_1,m_2,\ldots,m_s$ respectively. Then for $n\geq 0$, we have the following general formula*

$$P_{n+k-1}^{(k-1)}(x_1,x_2,\cdots,x_k) = \sum_{i_1+i_2+\cdots+i_s=n}\binom{i_1+m_1-1}{i_1}\alpha_1^{i_1}\cdots\binom{i_s+m_s-1}{i_s}\alpha_s^{i_s}.$$

Proof. We clearly have

$$\begin{aligned}\sum_{n\geq 0} P_{n+k-1}^{(k-1)}(x_1,x_2,\cdots,x_k)X^n &= (1-c_1X-\cdots-c_kX^k)^{-1}\\ &= \prod_{i=1}^{s}(1-\alpha_iX)^{-m_i}.\end{aligned}$$

On the other hand, it is not difficult to show that

$$(1-\alpha_iX)^{-m_i} = \sum_{n\geq 0}\binom{n+m_i-1}{n}\alpha_i^nX^n,$$

this gives

$$\begin{aligned}\sum_{n\geq 0} P_{n+k-1}^{(k-1)}(x_1,x_2,\cdots,x_k)X^n &= \prod_{i=1}^{s}\left(\sum_{n\geq 0}\binom{n+m_i-1}{n}\alpha_i^nX^n\right)\\ &= \sum_{n\geq 0}\left(\sum_{i_1+i_2+\cdots+i_s=n}\binom{i_1+m_1-1}{i_1}\alpha_1^{i_1}\cdots\binom{i_s+m_s-1}{i_s}\alpha_s^{i_s}\right)X^n.\end{aligned}$$

Therefore

$$P_{n+k-1}^{(k-1)}(x_1, x_2, \cdots, x_k) = \sum_{i_1+i_2+\cdots+i_s=n} \binom{i_1+m_1-1}{i_1} \alpha_1^{i_1} \cdots \binom{i_s+m_s-1}{i_s} \alpha_s^{i_s}.$$

Then we reach the desired result. □

In the case of a single root, we have

Corollary 3.3. *Let $g(X) = (X-\alpha)^k = X^k - c_1 X^{k-1} - c_2 X^{k-2} - \cdots - c_k$ be the characteristic polynomial of $P_n^{(k-1)}(x_1, x_2, \cdots, x_k)$ having exactly one root $\alpha = \alpha(x_1, x_2, \cdots, x_k)$. Then for $n \geq 0$, we have*

$$P_{n+k-1}^{(k-1)}(x_1, x_2, \cdots, x_k) = \binom{n+k-1}{n} \alpha^n.$$

4. The Fibonacci polynomials of order 2 and the Lucas polynomials of order 2

This section is devoted to recurrent polynomials of order 2, which are arguably the most important because they have various remarkable properties and a large number of applications in mathematics, computer sciences, physics, and other related topics. For some references of these applications see [1, 2, 3, 7, 9, 13, 15, 17] and the references therein.

The results of this section can be seen as the generalization and unification of various previously obtained results on sequence polynomials satisfying a second order linear recurrence relation. Taking into account some results of the previous sections, we will split the statements into numerous theorems. Now we are ready to present and prove our results. Let us first start by defining two class of polynomials.

Definition 4.1. Let $q_1(x, y)$ and $q_2(x, y)$ be two polynomials in x and y. The Fibonacci polynomials of order 2 are defined by the recurrence relation

$$F_{n+2}(x, y) = q_1(x, y) F_{n+1}(x, y) + q_2(x, y) F_n(x, y), \;\; n \geq 0,$$

with initial conditions $F_0(x, y) = 0$ and $F_1(x, y) = 1$.

Definition 4.2. Let $q_1(x, y)$ and $q_2(x, y)$ be two polynomials in x and y. The Lucas polynomials of order 2 are defined by the recurrence relation

$$L_{n+2}(x, y) = q_1(x, y) L_{n+1}(x, y) + q_2(x, y) L_n(x, y), \;\; n \geq 0,$$

with initial conditions $L_0(x, y) = 2$ and $L_1(x, y) = q_1(x, y)$.

Remark 4.3. It is clear that the Fibonacci polynomials and the Lucas polynomials of order 2 are a natural generalization of the classical Fibonacci polynomials and Lucas polynomials treated recently in [13].

The following lemma is a useful result which will be required for proving some statements.

Lemma 4.4. *Every recurrent polynomial $P_n(x, y)$ of order 2 with arbitrary initial conditions $P_0(x, y), P_1(x, y)$ can be written in terms of the Fibonacci polynomials of order 2 as*

$$P_n(x, y) = q_2(x, y) P_0(x, y) F_{n-1}(x, y) + P_1(x, y) F_n(x, y), \;\; n \geq 1.$$

Proof. Obviously these polynomials verify the same recurrence relation and have the same initial conditions. □

Theorem 4.5. *For $n \geq 1$, the following identity holds*

$$L_n(x, y) = 2 q_2(x, y) F_{n-1}(x, y) + q_1(x, y) F_n(x, y).$$

Proof. According to Lemma 4.4, we have

$$L_n(x,y) = q_2(x,y)L_0(x,y)F_{n-1}(x,y) + L_1(x,y)F_n(x,y).$$

In fact this gives the desired formula. □

An easy consequence is the following.

Corollary 4.6. *For $n \geq 1$, the following identity holds*

$$L_n(x) = 2F_{n-1}(x) + xF_n(x) = F_{n-1}(x) + F_{n+1}(x).$$

In particular

$$L_n = 2F_{n-1} + F_n = F_{n-1} + F_{n+1}.$$

Theorem 4.7. *For all $n \geq 0$, we have*

$$2L_{n+1}(x,y) - q_1(x,y)L_n(x,y) = \Big\{q_1^2(x,y) + 4q_2(x,y)\Big\}F_n(x,y).$$

Proof. Using the formula (2.3), we obtain

$$\begin{cases} L_n(x,y) &=& L_0(x,y)P_n^{(0)}(x,y) + L_1(x,y)P_n^{(1)}(x,y), \\ L_{n+1}(x,y) &=& L_1(x,y)P_n^{(0)}(x,y) + L_2(x,y)P_n^{(1)}(x,y). \end{cases}$$

This entails the matrix identity

$$\begin{pmatrix} L_n(x,y) \\ L_{n+1}(x,y) \end{pmatrix} = \begin{pmatrix} L_0(x,y) & L_1(x,y) \\ L_1(x,y) & L_2(x,y) \end{pmatrix} \begin{pmatrix} P_n^{(0)}(x,y) \\ P_n^{(1)}(x,y)) \end{pmatrix}.$$

Consequently

$$\begin{cases} u(x,y)P_n^{(0)}(x,y) &=& L_2(x,y)L_n(x,y) - L_1(x,y)L_{n+1}(x,y), \\ u(x,y)P_n^{(1)}(x,y) &=& L_0(x,y)L_{n+1}(x,y) - L_1(x,y)L_n(x,y), \end{cases}$$

where $u(x,y) = L_0(x,y)L_2(x,y) - (L_1(x,y))^2$. From this, we deduce the result. □

We easily deduce the following corollary.

Corollary 4.8. *For all $n \geq 0$, we have*

$$2L_{n+1}(x) - xL_n(x) = (x^2 + 4)F_n(x).$$

In particular

$$2L_{n+1} - L_n = 5F_n.$$

Theorem 4.9. *For $n \geq 1$ and $p \geq 0$, we have*

$$L_{n+p}(x,y) = q_2(x,y)L_p(x,y)F_{n-1}(x,y) + L_{p+1}(x,y)F_n(x,y).$$

In particular

$$L_{2n}(x,y) = q_2(x,y)L_n(x,y)F_{n-1}(x,y) + L_{n+1}(x,y)F_n(x,y).$$

Proof. This is result can be easily deduced from Lemma 4.4. □

An immediate consequence is obtained in the following result.

Corollary 4.10. *For $n \geq 1$ and $p \geq 0$, we have the following relations*

$$L_{n+p}(x) = L_p(x)F_{n-1}(x) + L_{p+1}(x)F_n(x),$$

and

$$L_{2n}(x) = L_n(x)F_{n-1}(x) + L_{n+1}(x)F_n(x).$$

In particular

$$L_{n+p} = L_pF_{n-1} + L_{p+1}F_n,$$

and

$$L_{2n} = L_nF_{n-1} + L_{n+1}F_n.$$

In the following theorem, we express $L_{2n}(x, y)$ only in terms of the Fibonacci polynomials of order 2.

Theorem 4.11. *For any $n \geq 1$, we have the following identity*

$$L_{2n}(x, y) = F_{n+1}^2(x, y) + 2q_2(x, y)F_n^2(x, y) + q_2^2(x, y)F_{n-1}^2(x, y).$$

Proof. Using Theorem 4.9, we have

$$L_{2n}(x, y) = q_2(x, y)L_n(x, y)F_{n-1}(x, y) + L_{n+1}(x, y)F_n(x, y).$$

and by Theorem 4.5, we have

$$\begin{cases} L_n(x, y) &= 2q_2(x, y)F_{n-1}(x, y) + q_1(x, y)F_n(x, y), \\ L_{n+1}(x, y) &= q_1(x, y)q_2(x, y)F_{n-1}(x, y) + \Big\{2q_2(x, y) + q_1^2(x, y)\Big\}F_n(x, y). \end{cases}$$

It follows that

$$\begin{aligned} L_{2n}(x, y) &= q_2(x, y)\Big\{2q_2(x, y)F_{n-1}(x, y) + q_1(x, y)F_n(x, y)\Big\}F_{n-1}(x, y) \\ &+ \Big\{q_1(x, y)q_2(x, y)F_{n-1}(x, y) + \Big(2q_2(x, y) + q_1^2(x, y)\Big)F_n(x, y)\Big\}F_n(x, y) \\ &= \Big(2q_2(x, y) + q_1^2(x, y)\Big)F_n^2(x, y) + 2q_2^2(x, y)F_{n-1}^2(x, y) \\ &+ 2q_1(x, y)q_2(x, y)F_{n-1}(x, y)F_n(x, y) \\ &= q_1^2(x, y)F_n^2(x, y) + q_2^2(x, y)F_{n-1}^2(x, y) \\ &+ 2q_1(x, y)q_2(x, y)F_{n-1}(x, y)F_n(x, y) \\ &+ 2q_2(x, y)F_n^2(x, y) + q_2^2(x, y)F_{n-1}^2(x, y) \\ &= \Big(q_1(x, y)F_n(x, y) + q_2(x, y)F_{n-1}(x, y)\Big)^2 + 2q_2(x, y)F_n^2(x, y) \\ &+ q_2^2(x, y)F_{n-1}^2(x, y) \\ &= F_{n+1}^2(x, y) + 2q_2(x, y)F_n^2(x, y) + q_2^2(x, y)F_{n-1}^2(x, y). \end{aligned}$$

as claimed. □

The last theorem can be used to establish further identities for the classical Fibonacci and Lucas numbers or polynomials. For example, we state the following corollary.

Corollary 4.12. *For any $n \geq 1$, we have the following identity*

$$L_{2n}(x) = F_{n+1}^2(x) + 2F_n^2(x) + F_{n-1}^2(x).$$

In particular

$$L_{2n} = F_{n+1}^2 + 2F_n^2 + F_{n-1}^2.$$

In the following theorem, we show that the Fibonacci polynomials of order 2 have an elegant determinantal expression.

Theorem 4.13. *For any two polynomials $q_1(x,y)$ and $q_2(x,y)$ and for every $n \geq 1$, the following determinantal identity holds*

$$F_{n+1}(x,y) = \begin{vmatrix} q_1(x,y) & q_2(x,y) & 0 & \cdots & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & q_2(x,y) \\ 0 & \cdots & \cdots & 0 & -1 & q_1(x,y) \end{vmatrix}_{n\times n}.$$

Proof. Using standard algebraic techniques, we can write

$$\sum_{n\geq 0} F_{n+1}(x,y)X^n = (1 - q_1(x,y)X - q_2(x,y)X^2)^{-1}.$$

and the result follows easily by Lemma 2.3. □

From this theorem, we easily deduce the following.

Corollary 4.14. *For all $n \geq 0$, the classical Fibonacci polynomials $F_n(x)$ can be represented in the following determinantal expression*

$$F_{n+1}(x) = \begin{vmatrix} x & 1 & 0 & \cdots & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & -1 & x \end{vmatrix}_{n\times n}.$$

In particular, we can express the Fibonacci numbers F_n in terms of a tridiagonal determinant as follows:

$$F_{n+1} = \begin{vmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{vmatrix}_{n\times n}.$$

for $n \geq 0$.

Theorem 4.15. *For all $n \geq 0$, there is a relationship between the Fibonacci polynomials of order 2 and the Lucas polynomials of order 2 given by*

$$L_n(x,y) = 2F_{n+1}(x,y) - q_1(x,y)F_n(x,y).$$

Proof. An easy computation shows that

$$\sum_{n\geq 0} L_n(x,y)X^n = \Big(2 - q_1(x,y)X\Big)\Big(1 - q_1(x,y)X - q_2(x,y)X^2\Big)^{-1}.$$

This implies

$$\sum_{n\geq 0} L_n(x,y)X^n = \Big(2 - q_1(x,y)X\Big)\sum_{n\geq 0} F_{n+1}(x,y)X^n.$$

Thus

$$\sum_{n\geq 0} L_n(x,y)X^n = 2 + \sum_{n\geq 1}(2F_{n+1}(x,y) - q_1(x,y)F_n(x,y))X^n.$$

Consequently, the identification of coefficients gives

$$L_n(x,y) = 2F_{n+1}(x,y) - q_1(x,y)F_n(x,y), \quad n \geq 0.$$

this proves the result. □

Corollary 4.16. *For $n \geq 0$, we have*

$$L_n(x) = 2F_{n+1}(x) - xF_n(x).$$

In particular

$$L_n = 2F_{n+1} - F_n.$$

In the following result, we develop an interesting explicit formula for $F_n(x,y)$.

Theorem 4.17. *For any two polynomials $q_1(x,y)$ and $q_2(x,y)$, the Fibonacci polynomials of order 2 can be expressed in terms of powers of $q_1(x,y)$ and $q_2(x,y)$ as follows:*

$$F_{n+1}(x,y) = \sum_{i=0}^{[\frac{n}{2}]} \binom{n-i}{i} (q_1(x,y))^{n-2i}(q_2(x,y))^i, \quad n \geq 0.$$

Proof. Since

$$F_n(x,y) = P_n^{(1)}(x,y).$$

Then using Theorem 2.5, we have

$$\begin{aligned} F_{n+1}(x,y) &= \sum_{i_1+2i_2=n} \frac{(i_1+i_2)!}{i_1!i_2!}(q_1(x,y))^{i_1}(q_2(x,y))^{i_2} \\ &= \sum_{i=0}^{[\frac{n}{2}]} \binom{n-i}{i} (q_1(x,y))^{n-2i}(q_2(x,y))^i. \end{aligned}$$

Therefore the formula is proved. □

As a consequence, we obtain explicit formulas for $F_n(x)$ and F_n.

Corollary 4.18. *For any $n \geq 0$, the classical Fibonacci polynomials can be expressed as follows:*

$$F_{n+1}(x) = \sum_{i=0}^{[\frac{n}{2}]} \binom{n-i}{i} x^{n-2i},$$

and the Fibonacci numbers can be written as:

$$F_{n+1} = \sum_{i=0}^{[\frac{n}{2}]} \binom{n-i}{i}.$$

The following theorem provides an explicit formula for $L_n(x, y)$.

Theorem 4.19. *For any two polynomials $q_1(x, y)$ and $q_2(x, y)$, the Lucas polynomials of order 2 can be expressed in terms of powers of $q_1(x, y)$ and $q_2(x, y)$ as follows:*

$$L_n(x, y) = \sum_{i=0}^{[\frac{n}{2}]} \frac{n}{n-i} \binom{n-i}{i} (q_1(x, y))^{n-2i} (q_2(x, y))^i, \quad n \geq 1.$$

Proof. Theorem4.15 tells us

$$L_n(x, y) = 2F_{n+1}(x, y) - q_1(x, y)F_n(x, y).$$

And by Theorem 4.17, we have

$$\begin{aligned}
L_n(x, y) = {} & 2\sum_{i=0}^{[\frac{n}{2}]} \binom{n-i}{i} (q_1(x, y))^{n-2i} (q_2(x, y))^i \\
& - q_1(x, y) \sum_{i=0}^{[\frac{n-1}{2}]} \binom{n-i-1}{i} (q_1(x, y))^{n-2i-1} (q_2(x, y))^i \\
= {} & \sum_{i=0}^{[\frac{n}{2}]} \left\{ (2q_1(x, y)\binom{n-i}{i} - q_1(x, y)\binom{n-i-1}{i} \right\} (q_1(x, y))^{n-2i-1} (q_2(x, y))^i \\
= {} & \sum_{i=0}^{[\frac{n}{2}]} \left\{ 2\binom{n-i}{i} - \binom{n-i-1}{i} \right\} (q_1(x, y))^{n-2i} (q_2(x, y))^i \\
= {} & \sum_{i=0}^{[\frac{n}{2}]} \frac{n}{n-i} \binom{n-i}{i} (q_1(x, y))^{n-2i} (q_2(x, y))^i.
\end{aligned}$$

This completes the proof. □

Correspondingly, we can derive explicit formulas for the classical Lucas numbers and polynomials.

Corollary 4.20. *For any $n \geq 0$, the classical Lucas polynomials can be expressed as follows:*

$$L_n(x) = \sum_{i=0}^{[\frac{n}{2}]} \frac{n}{n-i} \binom{n-i}{i} x^{n-2i}.$$

As a consequence, the Lucas numbers can be written as:

$$L_n = \sum_{i=0}^{[\frac{n}{2}]} \frac{n}{n-i} \binom{n-i}{i}, \quad n \geq 0.$$

Theorem 4.21. *For any two polynomials $q_1(x,y)$ and $q_2(x,y)$. Let*

$$\alpha(x,y) = \frac{1}{2}\Big(q_1(x,y) + \sqrt{(q_1(x,y))^2 + 4q_2(x,y)}\Big)$$

and

$$\beta(x,y) = \frac{1}{2}\Big(q_1(x,y) - \sqrt{(q_1(x,y))^2 + 4q_2(x,y)}\Big)$$

be the roots of the polynomial $P(t) = t^2 - q_1(x,y)t - q_2(x,y)$, then

1) *For every $n \geq 0$, we have*

$$F_n(x,y) = \frac{\alpha^n(x,y) - \beta^n(x,y)}{\alpha(x,y) - \beta(x,y)}.$$

2) *For every $n \geq 0$, we have*

$$L_n(x,y) = \alpha^n(x,y) + \beta^n(x,y).$$

3) *For every $n \geq 0$ and $m \geq 0$, we have*

$$\Big(L_n(x,y) + \sqrt{(q_1(x,y))^2 + 4q_2(x,y)}F_n(x,y)\Big)^m =$$
$$2^{m-1}\Big(L_{nm}(x,y) + \sqrt{(q_1(x,y))^2 + 4q_2(x,y)}F_{nm}(x,y)\Big).$$

4) *For every $n \geq 0$ and $m \geq 0$, we have*

$$\Big(L_n(x,y) - \sqrt{(q_1(x,y))^2 + 4q_2(x,y)}F_n(x,y)\Big)^m =$$
$$2^{m-1}\Big(L_{nm}(x,y) - \sqrt{(q_1(x,y))^2 + 4q_2(x,y)}F_{nm}(x,y)\Big).$$

5) *For $n \geq 0$ and $m \geq 0$, we have*

$$\Big(L_n(x,y) + \sqrt{(q_1(x,y))^2 + 4q_2(x,y)}F_n(x,y)\Big)^m$$
$$+\Big(L_n(x,y) - \sqrt{(q_1(x,y))^2 + 4q_2(x,y)}F_n(x,y)\Big)^m = 2^m L_{nm}(x,y).$$

Proof. Using (2.3), we can easily obtain

$$\begin{cases} \alpha^n(x,y) &= q_2(x,y)F_{n-1}(x,y) + \alpha(x,y)F_n(x,y), \\ \beta^n(x,y) &= q_2(x,y)F_{n-1}(x,y) + \beta(x,y)F_n(x,y). \end{cases}$$

This implies

$$F_n(x,y) = \frac{\alpha^n(x,y) - \beta^n(x,y)}{\alpha(x,y) - \beta(x,y)}.$$

On the other hand, we have by Theorem 4.5

$$L_n(x,y) = 2q_2(x,y)F_{n-1}(x,y) + q_1(x,y)F_n(x,y).$$

Using the last identity and the fact that $\alpha(x,y)\beta(x,y) = -q_2(x,y)$, we have

$$\begin{aligned} L_n(x,y) &= 2q_2(x,y)\frac{\alpha^{n-1}(x,y) - \beta^{n-1}(x,y)}{\alpha(x,y) - \beta(x,y)} + q_1(x,y)\frac{\alpha^n(x,y) - \beta^n(x,y)}{\alpha(x,y) - \beta(x,y)} \\ &= \frac{q_1(x,y)\alpha(x,y) + 2q_2(x,y)}{\alpha(x,y) - \beta(x,y)}\alpha^{n-1}(x,y) \\ &- \frac{q_1(x,y)\beta(x,y) + 2q_2(x,y)}{\alpha(x,y) - \beta(x,y)}\beta^{n-1}(x,y) \\ &= \frac{q_1(x,y) - 2\beta(x,y)}{\alpha(x,y) - \beta(x,y)}\alpha^n(x,y) + \frac{2\alpha(x,y) - q_1(x,y)}{\alpha(x,y) - \beta(x,y)}\beta^n(x,y). \end{aligned}$$

Since $\alpha(x,y) + \beta(x,y) = q_1(x,y)$, then

$$L_n(x,y) = \alpha^n(x,y) + \beta^n(x,y).$$

Now, it is clear that

$$\begin{aligned} \Big(L_n(x,y) + \sqrt{(q_1(x,y))^2 + 4q_2(x,y)}F_n(x,y)\Big)^m &= \Big(\alpha^n(x,y) + \beta^n(x,y) + \alpha^n(x,y) - \beta^n(x,y)\Big)^m \\ &= 2^{m-1}(2\alpha^{nm}(x,y) + \beta^{nm}(x,y) - \beta^{nm}(x,y)) \\ &= 2^{m-1}(L_{nm}(x,y) + \sqrt{(q_1(x,y))^2 + 4q_2(x,y)}F_{nm}(x,y)). \end{aligned}$$

The rest of the proof is similar. □

Using Theorem 4.21, we can immediately obtain the following interesting corollary.

Corollary 4.22. *For every $n \geq 0$ and $m \geq 0$, the classical Fibonacci polynomials $F_n(x)$ and the classical Lucas polynomials $L_n(x)$ satisfy*

1)

$$\Big(L_n(x) + \sqrt{x^2+4}F_n(x)\Big)^m = 2^{m-1}\Big(L_{nm}(x) + \sqrt{x^2+4}F_{nm}(x)\Big).$$

2)

$$\Big(L_n(x) - \sqrt{x^2+4}F_n(x)\Big)^m = 2^{m-1}\Big(L_{nm}(x) - \sqrt{x^2+4}F_{nm}(x)\Big).$$

3)

$$\Big(L_n(x) + \sqrt{x^2+4}F_n(x)\Big)^m + \Big(L_n(x) - \sqrt{x^2+4}F_n(x)\Big)^m = 2^m L_{nm}(x).$$

As a direct consequence of the previous corollary, we obtain the following interesting identities.

Corollary 4.23. *For every $n \geq 0$ and $m \geq 0$, the Fibonacci numbers F_n and the Lucas numbers L_n satisfy*

1)

$$(L_n + \sqrt{5}F_n)^m = 2^{m-1}(L_{nm} + \sqrt{5}F_{nm}).$$

2)

$$(L_n - \sqrt{5}F_n)^m = 2^{m-1}(L_{nm} - \sqrt{5}F_{nm}).$$

3)

$$(L_n + \sqrt{5}F_n)^m + (L_n - \sqrt{5}F_n)^m = 2^m L_{nm}.$$

5. Consequences on the Dickson polynomials of the first and the second kind

Dickson polynomials, named after the American mathematician Leonard Eugene Dickson, were first introduced in the early 20th century. Leonard E. Dickson, known for his work in algebra and number theory, developed these polynomials as part of his broader investigations into algebraic structures and polynomial functions. The initial work on Dickson polynomials aimed to explore their properties and applications within the context of algebraic number theory.

Dickson's foundational work laid the groundwork for subsequent research into these polynomials. In his seminal papers, Dickson analyzed the structure and behavior of these polynomials, deriving important properties and relations. Over the decades, mathematicians expanded on Dickson's work, exploring various aspects of these polynomials, including their recurrence relations, algebraic properties, and applications in different mathematical fields.

In modern mathematics, Dickson polynomials have found applications in areas such as cryptography, pseudoprimality testing, coding theory, and combinatorial design theory, and related topics. Their properties, including their relation to other polynomial sequences and their role in various mathematical structures, have been extensively studied. Researchers have developed numerous results and generalizations, including connections with classical polynomials like the Fibonacci and Lucas polynomials.

The influence of Dickson polynomials extends beyond their immediate applications. They have served as a bridge between different areas of mathematics, providing insights into polynomial theory and its applications. Ongoing research continues to explore new properties and applications of Dickson polynomials, contributing to a deeper understanding of their role in mathematical theory and practice.

In this section, we investigate this important family of sequence polynomials. Notably, we apply the results of the last section to produce a number of interesting statements. We refer the interested reader to the very nice book on Dickson polynomials [16], that presents a comprehensive collection of results of these polynomials and provides a number of applications. Dickson polynomials as one of the timeless mathematical topics attracted the attention of many researchers since their appearance. Actually, Dickson polynomials are an essential research topic open to further progress.

The Dickson polynomials of the first kind $D_n(x,a)$ can be generated by

$$D_0(x,a) = 2, \;\; D_1(x,a) = x, \;\;\; D_{n+2}(x,a) = xD_{n+1}(x,a) - aD_n(x,a), \;\; n \geq 0,$$

The first few examples of the Dickson polynomials of the first kind are

$$\begin{aligned} D_1(x,a) &= x, \\ D_2(x,a) &= x^2 - 2a, \\ D_3(x,a) &= x^3 - 3xa, \\ D_4(x,a) &= x^4 - 4x^2a + 2a^2, \\ D_5(x,a) &= x^5 - 5x^3a + 5xa^2. \end{aligned}$$

The Dickson polynomials of the second kind $E_n(x,a)$ can be defined by

$$E_0(x,a) = 1, \;\; E_1(x,a) = x, \;\;\; E_{n+2}(x,a) = xE_{n+1}(x,a) - aE_n(x,a), \;\; n \geq 0,$$

These polynomials have not studied much in the literature. The first few examples of the Dickson polynomials of the second kind are

$$\begin{aligned}
E_1(x,a) &= x,\\
E_2(x,a) &= x^2 - a,\\
E_3(x,a) &= x^3 - 2xa,\\
E_4(x,a) &= x^4 - 3x^2a + a^2,\\
E_5(x,a) &= x^5 - 4x^3a + 3xa^2.
\end{aligned}$$

It is clear that the Dickson polynomials of the first kind are particular cases of the Lucas polynomials of order 2, that is when $q_1(x,a) = x, q_2(x,a) = -a$. In the rest of the paper, we assume that $q_1(x,a) = x, q_2(x,a) = -a$.

In the following result, we establish a relationship between the Dickson polynomials of the second kind and the Fibonacci polynomials of order 2.

Theorem 5.1. *The Dickson polynomials of the second kind satisfy the following*

$$E_n(x,a) = F_{n+1}(x,a), \;\; n \geq 0.$$

Proof. By (2.3), it is clear that for every $n \geq 1$ we have

$$\begin{aligned}
E_n(x,a) &= P_n^{(0)}(x,a) + xP_n^{(1)}(x,a)\\
&= -aP_{n-1}^{(1)}(x,a) + xP_n^{(1)}(x,a)\\
&= -aF_{n-1}(x,a) + xF_n(x,a)\\
&= F_{n+1}(x,a).
\end{aligned}$$

which gives the desired formula. □

In determinant form, the Dickson polynomials of the second kind are given by

Theorem 5.2. *For every $n \geq 1$*

$$E_n(x,a) = \begin{vmatrix}
x & a & 0 & \cdots & \cdots & 0\\
1 & \ddots & \ddots & \ddots & & \vdots\\
0 & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & 0\\
\vdots & & \ddots & \ddots & \ddots & a\\
0 & \cdots & \cdots & 0 & 1 & x
\end{vmatrix}_{n\times n}.$$

Proof. According to the above Theorem, we have $E_n(x,a) = F_{n+1}(x,a), n \geq 1$, and by using Theorem 4.13, we deduce that

$$E_n(x,a) = \begin{vmatrix}
x & -a & 0 & \cdots & \cdots & 0\\
-1 & \ddots & \ddots & \ddots & & \vdots\\
0 & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & 0\\
\vdots & & \ddots & \ddots & \ddots & -a\\
0 & \cdots & \cdots & 0 & -1 & x
\end{vmatrix}_{n\times n}.$$

On the other hand, it is not hard to prove that

$$\begin{vmatrix} b & c & 0 & \cdots & \cdots & 0 \\ a & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & c \\ 0 & \cdots & \cdots & 0 & a & b \end{vmatrix}_{n\times n} = \begin{cases} \dfrac{(b+\sqrt{b^2-4ac})^{n+1}-(b-\sqrt{b^2-4ac})^{n+1}}{2^{n+1}\sqrt{b^2-4ac}}, & b^2 \neq 4ac, \\ (n+1)\left(\dfrac{b}{2}\right)^n, \quad b^2 = 4ac. \end{cases}$$

Consequently, we obtain

$$\begin{vmatrix} x & -a & 0 & \cdots & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & -a \\ 0 & \cdots & \cdots & 0 & -1 & x \end{vmatrix} = \begin{vmatrix} x & a & 0 & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & a \\ 0 & \cdots & \cdots & 0 & 1 & x \end{vmatrix}.$$

This proves the desired determinantal formula. □

For $a = 1$, we get the following result stated in [16] and [22] without proof.

Corollary 5.3. *For every $n \geq 1$*

$$E_n(x,1) = \begin{vmatrix} x & 1 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & x \end{vmatrix}.$$

Remark 5.4. Note that the polynomial $E_n(x,1)$ appears as the numerator and the denominator of the approximant to some continued fractions, for more details see [16, p.15].

In the following result, we provide explicit formulas for the Dickson polynomials of the first and the second kind.

Theorem 5.5. *For $n \geq 0$, the following identities hold true*

$$D_n(x,a) = \sum_{i=0}^{[\frac{n}{2}]} \frac{n}{n-i}\binom{n-i}{i} x^{n-2i}(-a)^i,$$

and

$$E_n(x,a) = \sum_{i=0}^{[\frac{n}{2}]} \binom{n-i}{i} x^{n-2i}(-a)^i.$$

Proof. It is clear that $D_n(x,a) = L_n(x,a)$ when $q_1(x,a) = x$ and $q_2(x,a) = -a$, so the first identity follows form Theorem 4.19. The second identity can be proved easily with the aid of Theorem 4.17. □

Taking into account Theorem 4.21, we can easily derive the following interesting identities.

Theorem 5.6. *1) For every $n \geq 1$ and $m \geq 1$, we have*

$$\Big(D_n(x,a) + \sqrt{x^2-4a}E_{n-1}(x,a)\Big)^m = 2^{m-1}\Big(D_{nm}(x,a) + \sqrt{x^2-4a}E_{nm-1}(x,a)\Big).$$

2) For every $n \geq 1$ and $m \geq 1$, we have

$$\Big(D_n(x,a) - \sqrt{x^2-4a}E_{n-1}(x,a)\Big)^m = 2^{m-1}\Big(D_{nm}(x,a) - \sqrt{x^2-4a}E_{nm-1}(x,a)\Big).$$

3) For every $n \geq 1$ and $m \geq 0$, we have

$$\Big(D_n(x,a) + \sqrt{x^2-4a}E_{n-1}(x,a)\Big)^m + \Big(D_n(x,a) - \sqrt{x^2-4a}E_{n-1}(x,a)\Big)^m = 2^m D_{nm}(x,a).$$

Finally, using the results from the previous section, we derive the following theorem, which establishes relationships between the Dickson polynomials of the first and second kinds.

Theorem 5.7. *The following identities hold true:*

a)

$$D_n(x,a) = xE_{n-1}(x,a) - 2aE_{n-2}(x,a),\ n \geq 2,$$

b)

$$2D_{n+1}(x,a) - xD_n(x,a) = (x^2-4a)E_{n-1}(x,a),\ n \geq 1,$$

c)

$$D_{n+p}(x,a) = D_{p+1}(x,a)E_{n-1}(x,a) - aD_p(x,a)E_{n-2}(x,a),\ n \geq 2, p \geq 0,$$

d)

$$D_{2n}(x,a) = D_{n+1}(x,a)E_{n-1}(x,a) - aD_n(x,a)E_{n-2}(x,a),\ n \geq 2,$$

d)

$$D_{2n}(x,a) = E_{n+1}^2(x,a) - 2aE_{n-1}^2(x,a) + a^2E_{n-2}^2(x,a),\ n \geq 2,$$

e)

$$D_n(x,a) = 2E_n(x,a) - xE_{n-1}(x,a),\ n \geq 1.$$

6. Conclusion

This paper presents a comprehensive study of polynomial sequences in multiple variables, focusing on both general and specific cases. It introduces a broader family of polynomials that satisfy a linear recurrence relation and provides explicit formulas and determinantal expressions for these polynomials. The research highlights several significant identities and relationships involving Fibonacci and Lucas polynomials of order 2, classical Fibonacci and Lucas polynomials, as well as Dickson polynomials of both kinds.

By integrating results from high-order polynomial sequences and second-order recurrent polynomials, the paper offers a unified generalization of existing works. It not only recapitulates and generalizes established results but also presents new insights into the interrelationships among these polynomial sequences. This work enhances the understanding of some polynomial theory and establishes a solid basis for future investigation and advancement in this field.

Acknowledgements:

The authors are sincerely grateful to the anonymous referees for their careful reading, critical comments, and valuable suggestions, which significantly contributed to improving the manuscript during the revision process.

References

[1] H. Armstrong. *A treatment of cascaded active four-terminal networks, with application to transistor circuits*, IRE Transactions on Circuit Theory. **3** (1956), 138–140. 4

[2] H. Armstrong. *Note on the use of Tchebycheff functions in dealing with iterated networks* , IRE Trans. on Circuit Theory. **2** (1955), 169–70. 4

[3] R. Barakat, E. Baumann. *Mth Powers of an $N \times N$ matrix and its connection with the generalized Lucas polynomials* , J. Math. Phys. **10** (1969), 1474–1476. 2, 4

[4] P. Catarino. *Diagonal functions of k-Pell and k-Pell-Lucas polynomials and some identities*, Acta Math. Univ. Comenianae **87** (2018), 147–159. 1

[5] P. Catarino. *A note on certain matrices with h(x)-Fibonacci polynomials*, Acta Math. Univ. Comenianae. **86** (2017), 263–270. 1

[6] L. Chen, X. Wang. *The power sums involving Fibonacci polynomials and their applications*, Symmetry. **11** (2019), 635. 1

[7] G. Everest, A. Van Der Poorten, I. Shparlinski, and T. Ward. *Recurrence sequences*, American Mathematical Society, Mathematical Surveys and Monographs 104, 2003. 4

[8] H, Guan, W. A. Khan, C. Kızılateş and C. S. Ryoo. *On Certain Properties of Parametric Kinds of Apostol-Type Frobenius-Euler-Fibonacci Polynomials*, Axioms. **13** (2024), 348. 1

[9] T. Koshy. *Fibonacci and Lucas numbers with applications*, A Wiley-Interscience Publication, 2001. 4

[10] T. Koshy. *Pell and Pell-Lucas numbers with applications*, New York: Springer, 2014. 1

[11] B. Kuloglu, E. Özkan and M. Marin. *Fibonacci and Lucas polynomials in n-gon*, Ann. Stiint. Univ. Ovidius Constanta Ser. Math. **31** (2023), 127–140. 1

[12] C. Levesque. *On m-th order linear recurrences*, Fibonacci Q. **23** (1985), 290–293. 3

[13] E. Özkan, İ. Altun. *Generalized Lucas polynomials and relationships between the Fibonacci polynomials and Lucas polynomials*, Commun. Algebra. **47** (2019), 4020–4030. 2.2, 4, 4.3

[14] E. Özkan, B. Kuloğlu. *On the new Narayana polynomials, the Gauss Narayana numbers and their polynomials*, Asian-Eur. J. Math. **14** (2021), 2150100. 1

[15] V. Mowery. *On hypergeometric functions in iterated networks*, IEEE Trans. Circuit Theory. **11** (1964), 232–247. 4

[16] G. L. Mullen, R. Lidl and G. Turnwald. *Dickson Polynomials*, Vol.65, Chapman and Hall/CRC, 1993. 2.1, 5, 5, 5.4

[17] M. C. Pease. *The iterated network and its application to differentiators*, Proc. IRE. **40** (1952), 709–711. 4

[18] J. L. Ramírez. *On convolved generalized Fibonacci and Lucas polynomials*, Appl. Math. Comput. **229** (2014), 208–213. 1

[19] P. E. Ricci. *A note on Q-matrices and higher order Fibonacci polynomials*, Notes Number Theory Discret. Math. **27** (2021), 91–100. 1

[20] P. E. Ricci, R. Srivastava and D. Caratelli. *Laguerre-Type Bernoulli and Euler Numbers and Related Fractional Polynomials*, Mathematics. **12** (2024), 381. 2.4

[21] Y. Soykan. *Generalized Tribonacci Polynomials*, Earthline Journal of Mathematical Sciences. **13** (2023), 1–20. 1

[22] J. J. Sylvester. *Note on contiuants*, Messenger Maths. **8** (1879), 187–189. (Call. Math. Papers, Vol. Cambridge Univ. Press, Cambridge, 1909, 249-251. 5

[23] M. Taştan, E. Özkan. *On The Gauss K-Fibonacci Polynomials*, Electron. J. Math. Anal. Appl. **9** (2021), 124–130. 1

[24] N. Tuglu, E. G. Kocer and A. Stakhov. *Bivariate Fibonacci like p-polynomials*, Appl. Math. Comput. **217** (2011), 10239–10246. 1

[25] T. Wang, W. P. Zhang. *Some identities involving Fibonacci, Lucas polynomials and their applications*, Bull. Math. Soc. Sci. Math. Roumanie. **55** (2012), 95–103. 1

[26] X. Ye and Z. Zhang. *A common generalization of convolved generalized Fibonacci and Lucas polynomials and its applications*, Appl. Math. Comput. **306** (2017), 31–37. 1

[27] Y. Yi, W. P. Zhang. *Some identities involving the Fibonacci polynomials*, Fibonacci Q. **40** (2002), 314–318. 1

[28] S. Zriaa and M. Mouçouf. *Explicit, recurrent, determinantal expressions of the kth power of formal power series and applications to the generalized Bernoulli numbers*. arXiv:2305.06072[math.NT](2023).