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Invariant and Preserving Transforms for Cross Ratio of 4-Points in a line on Desargues Affine Plane

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Abstract

This paper introduces advances in the geometry of the transforms for cross ratio of four points in a line in the Desargues affine plane. The results given here have a clean, based Desargues affine plan axiomatics and definitions of addition and multiplication of points on a line in this plane, and for skew field properties. In this paper are studied, properties and results related to the some transforms for cross ratio for 4-points, in a line, which we divide into two categories, *Invariant* and *Preserving* transforms for cross ratio. The results in this paper are (1) the cross-ratio of four points is *Invariant* under transforms: Inversion, Natural Translation, Natural Dilation, Mobiüs Transform, in a line of Desargues affine plane. (2) the cross-ratio of four points is *Preserved* under transforms: parallel projection, translations and dilation's in the Desargues affine plane.

Keywords: Cross Ratio, Skew-Field, Desargues Affine Plane.
2010 MSC: 51-XX, 51Axx, 51A30, 51E15, 51N25, 30C20, 30F40.

1. Introduction and Preliminaries

Influenced by the recently achieved results, related to the ratio of 2 and 3 points (see, [16], [17], [19], [23]), but mainly on the results presented in the paper [18] for cross-ratio of four collinear points in a line ℓ^{OI}, in Desargues affine planes, in this paper we study some transforms regrading to cross-ratio of four collinear points (four point in a line ℓ^{OI} on Desargues affine plane). We divide this transforms in two categories *Invariant-Transforms* and *Preserving-Transforms*.

Earlier, we defined addition and multiplication of points in a line on Desarges affine plane, and we have proved that on each line on Desargues affine plane, we can construct a skew-field related to these two actions, so $(\ell^{OI}, +, \cdot)$-is a skew-field, this construction has been achieved, simply and constructively, using simple

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elements of elementary geometry, and only the basic axioms of Desargues affine plane (see [9], [10], [14], [20]). In this paper, we consider dilations and translations entirely in the Desargues affine plane (see [8], [15], [14], [20]).

The foundations for the study of the connections between axiomatic geometry and algebraic structures were set forth by D. Hilbert [4]. And some classic examples for this are, E. Artin [25], D.R. Huges and F.C. Piper [12], H. S. M Coxeter [13]. Marcel Berger in [5], Robin Hartshorne in [6], etc. In our earlier works [8, 9, 10, 15, 11, 14, 20, 21, 22, 24] we have brought up quite a few interesting facts about the association of algebraic structures with affine planes and with 'Desargues affine planes', and vice versa.

In this paper, all results are based on geometric intuition, on axiomatic approaches of Desargues affine plane and skew field properties, utilize a method that is naive and direct, without requiring the concept of coordinates.

1.1. Desargues Affine Plane

Let $\mathcal{P}$ be a nonempty space, $\mathcal{L}$ is a family of subsets of $\mathcal{P}$. The elements P of $\mathcal{P}$ are points and an element ℓ of $\mathcal{L}$ is a line.

Definition 1.1. The incidence structure $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, called affine plane, where satisfies the above axioms:

1^o For each points $P, Q \in \mathcal{P}$, there is exactly one line $\ell \in \mathcal{L}$ such that $P, Q \in \ell$.

2^o For each point $P \in \mathcal{P}, \ell \in \mathcal{L}, P \notin \ell$, there is exactly one line $\ell' \in \mathcal{L}$ such that $P \in \ell'$ and $\ell \cap \ell' = \emptyset$ (Playfair Parallel Axiom [1]). Put another way, if the point $P \notin \ell$, then there is a unique line ℓ' on P missing ℓ [2].

3^o There is a 3-subset of points $\{P, Q, R\} \subset \mathcal{P}$, which is not a subset of any ℓ in the plane. Put another way, there exist three non-collinear points $\mathcal{P}$ [2].

Desargues' Axiom, circa 1630 [7, §3.9, pp. 60-61] [3]. Let $A, B, C, A', B', C' \in \mathcal{P}$ and let pairwise distinct lines $\ell^{AA'}, \ell^{BB'}, \ell^{CC'}, \ell^{AC}, \ell^{A'C'} \in \mathcal{L}$ such that

$$\ell^{AA'} \parallel \ell^{BB'} \parallel \ell^{CC'} \text{ (Fig. 1(a)) } \textbf{or } \ell^{AA'} \cap \ell^{BB'} \cap \ell^{CC'} = P. \text{(Fig. 1(b))}$$
$$\text{and } \ell^{AB} \parallel \ell^{A'B'} \text{ and } \ell^{BC} \parallel \ell^{B'C'}.$$
$$A, B \in \ell^{AB}, A'B' \in \ell^{A'B'}, A, C \in \ell^{AC}, \text{ and } A'C' \in \ell^{A'C'}, B, C \in \ell^{BC}, B'C' \in \ell^{B'C'}.$$
$$A \neq C, A' \neq C', \text{ and } \ell^{AB} \neq \ell^{A'B'}, \ell^{BC} \neq \ell^{B'C'}.$$

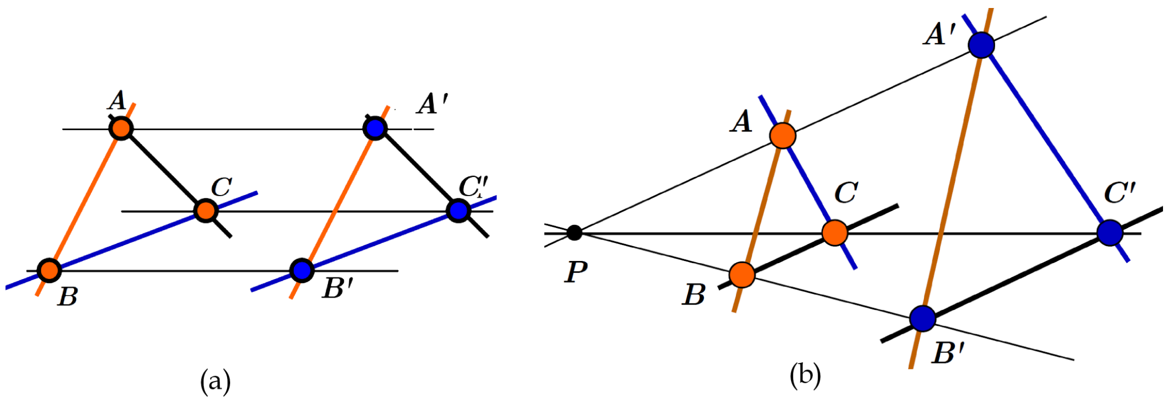

Figure 1: Desargues Axioms: (a) For parallel lines $\ell^{AA_1} \parallel \ell^{BB'} \parallel \ell^{CC'}$; (b) For lines which are cutting in a single point P, $\ell^{AA'} \cap \ell^{BB'} \cap \ell^{CC'} = P$.

Then $\boldsymbol{\ell^{AC} \parallel \ell^{A'C'}}$. ■

A **Desargues affine plane** is an affine plane that satisfies Desargues' Axiom.

Notation 1. Three vertexes ABC and $A'B'C'$, which, fulfilling the conditions of the Desargues Axiom, we call *'Desarguesian'*.

1.2. Addition and Multiplication of points in a line of Desargues affine plane

The process of constructing the points C for addition (Figure 2 (a)) and multiplication (Figure 2 (b)) of points in $\ell^{OI}-$line in affine plane, is presented in the two algorithm form

ADDITION ALGORITHM

Step.1 $B_1 \notin \ell^{OI}$

Step.2 $\ell_{OI}^{B_1} \cap \ell_{OB_1}^{A} = P_1$

Step.3 $\ell_{BB_1}^{P_1} \cap \ell^{OI} = C(= A + B)$

MULTIPLICATION ALGORITHM

Step.1 $B_1 \notin \ell^{OI}$

Step.2 $\ell_{IB_1}^{A} \cap \ell^{OB_1} = P_1$

Step.3 $\ell_{BB_1}^{P_1} \cap \ell^{OI} = C(= A \cdot B)$

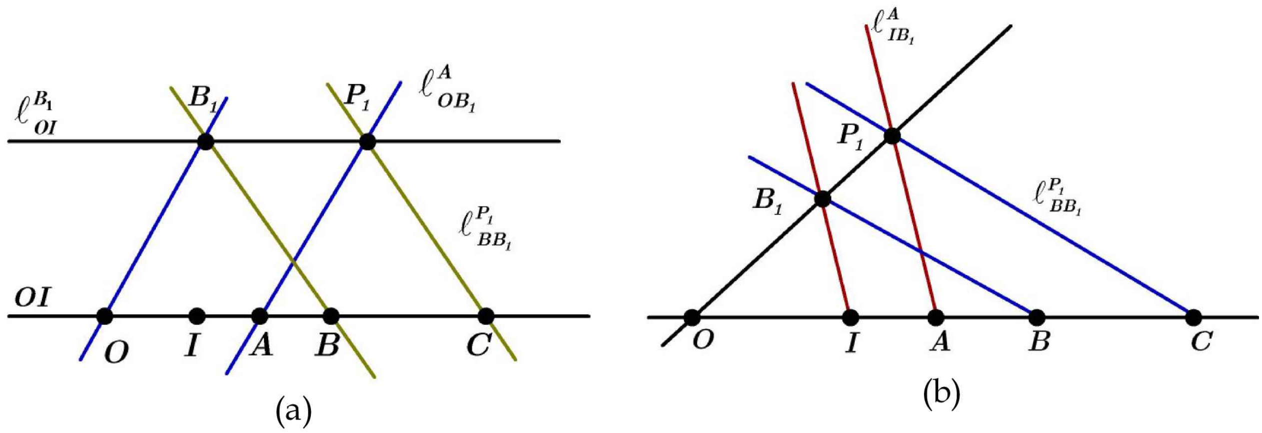

Figure 2: (a) Addition of points in a line in affine plane, (b) Multiplication of points in a line in affine plane

In [14] and [10], we have prove that $(\ell^{OI}, +, \cdot)$ is a skew field in Desargues affine plane, and is field (commutative skew field) in the Papus affine plane.

Definition 1.2. The parallel projection between the two lines in the Desargues affine plane, will be called, a function,

$$P_P : \ell_1 \to \ell_2, \quad \forall A, B \in \ell_1, \quad AP_P(A) || BP_P(B)$$

It is clear that this function is a bijection between any two lines in Desargues affine planes, for this reason, it can also be thought of as isomorphism between two lines.

Definition 1.3. [15] Dilatation of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, called a its collineation δ such that: $\forall P \neq Q \in \mathcal{P}, \delta(PQ) || PQ$.

Definition 1.4. [15] Translation of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, called identical dilatation $id_{\mathcal{P}}$ his and every other of its dilatation, about which he affine plane has not fixed points.

Some well-known results related to translations and dilation's in Desargues affine planes.

- The dilatation set $\mathbf{Dil}_{\mathcal{A}}$ of affine plane $\mathcal{A}$ forms a **group** with respect to composition $\circ$ ([15]).
- The translations set $\mathbf{Tr}_{\mathcal{A}}$ of affine plane $\mathcal{A}$ forms a **group** with respect to composition $\circ$; which is a sub-group of the dilation group $(\mathbf{Dil}_{\mathcal{A}}, \circ)$ ([15]).
- In a affine plane: the group $(\mathbf{Tr}_{\mathcal{A}}, \circ)$ of translations is **normal sub-group** of the group of dilatations $(\mathbf{Dil}_{\mathcal{A}}, \circ)$ ([15]).

- Every dilatation in Desargues affine plane $\mathcal{A}_{\mathcal{D}} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ which leads a line in itself ($\delta : \ell \to \ell$), is an automorphism of skew-fields constructed on the same line $\ell \in \mathcal{L}$, of the plane $\mathcal{A}_{\mathcal{D}}$ ([8]).

- Every translations in Desargues affine plane $\mathcal{A}_{\mathcal{D}} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ which leads a line in itself ($\varphi : \ell \to \ell$), is an automorphism of skew-fields constructed on the same line $\ell \in \mathcal{L}$, of the plane $\mathcal{A}_{\mathcal{D}}$ ([8]).

- Each dilatation in a Desargues affine plane, $\mathcal{A}_{\mathcal{D}} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is an isomorphism between skew-fields constructed over isomorphic lines $\ell_1, \ell_2 \in \mathcal{L}$ of that plane ([20]).

- Each translations in a Desargues affine plane, $\mathcal{A}_{\mathcal{D}} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is an isomorphism between skew-fields constructed over isomorphic lines $\ell_1, \ell_2 \in \mathcal{L}$ of that plane ([20]).

1.3. Some algebraic properties of Skew Fields

In this section K will denote a skew field [26] and $z[K]$ its center, where is the set K such that

$$z[K] = \{k \in K \qquad ak = ka, \quad \forall a \in K\}.$$

Proposition 1.5. *$z[K]$ is a commutative subfield of a skew field K.*

Let now $p \in K$ be a fixed element of the skew field K, we will denote by $z_K(p)$ the centralizer in K of the element p, where is the set,

$$z_K(p) = \{k \in K, pk = kp\}.$$

where $z_K(p)$ is sub skew field of K, but, in general, it is not commutative.

Let K be a skew field, $p \in K$, and let us denote by $[p_K]$ the conjugacy class of p:

$$[p_K] = \left\{q^{-1}pq \quad , \quad q \in K \setminus \{0\}\right\}$$

If, $p \in z[K]$, for all $q \in K$ we have that $q^{-1}pq = p$.

1.4. Ratio of two and three points

In the papers [16], [23] we have done a detailed study, related to the ratio of two and three points in a line of Desargues affine plane. Below we are listing some of the results for ratio of two and three points.

Definition 1.6. [16], [23] Let us have two different points $A, B \in \ell^{OI}-$line, and $B \neq O$, in Desargues affine plane. We define as ratio of this two points, a point $R \in \ell^{OI}$, such that,

$$R = B^{-1}A, \qquad \text{we mark this, with,} \qquad R = r(A : B) = B^{-1}A$$

For a *ratio-point* $R \in \ell^{OI}$, and for point $B \neq O$ in line ℓ^{OI}, is a unique defined point, $A \in \ell^{OI}$, such that $R = B^{-1}A = r(A : B)$.

Some results for Ratio of 2-points in Desargues affine plane (see [16], [23]).

- For two different points $A, B \in \ell^{OI}-$line, and $B \neq O$, in Desargues affine plane, then, $r^{-1}(A : B) = r(B : A)$.

- For three collinear points A, B, C and $C \neq O$, in $\ell^{OI}-$line, we have,

$$r(A + B : C) = r(A : C) + r(B : C).$$

- For three collinear points A, B, C and $C \neq O$, in $\ell^{OI}-$line, we have,
  1. $r(A \cdot B : C) = r(A : C) \cdot B$.
  2. $r(A : B \cdot C) = C^{-1}r(A : C)$.

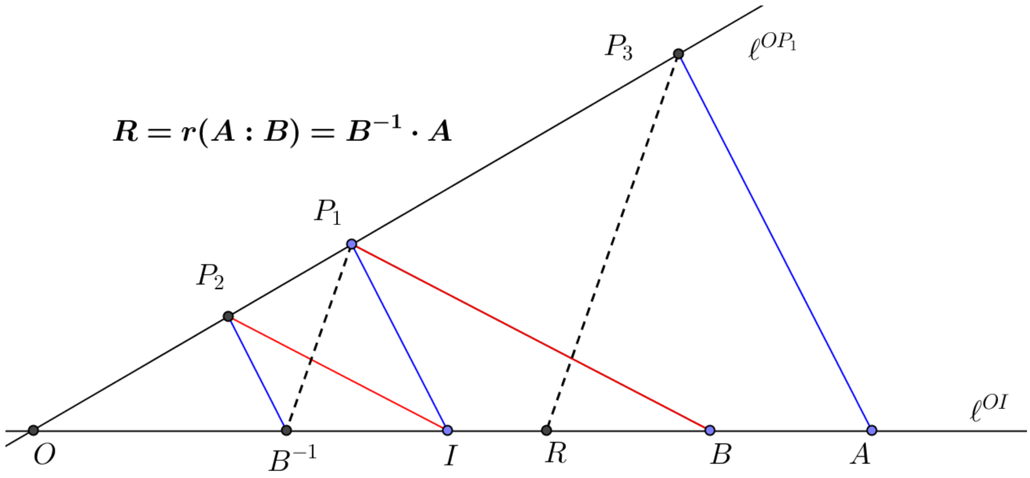

Figure 3: Ilustrate the Ratio-Point, of 2-Points in a line of Desargues affine plane $R = r(A : B) = B^{-1}A$.

- Let us have the points A, B in the line ℓ^{OI} where $B \neq O$. Then we have that,

$$r(A : B) = r(B : A) \Leftrightarrow A = B.$$

- This ratio-map, $r_B : \ell^{OI} \to \ell^{OI}$ is a bijection in $\ell^{OI}-$line in Desargues affine plane.
- The ratio-maps-set $\mathcal{R}_2 = \{r_B(X), \forall X \in \ell^{OI}\}$, for a fixed point B in the line ℓ^{OI}, forms a skew-field with 'addition and multiplication' of points. This, skew field $(\mathcal{R}_2, +, \cdot)$ is sub-skew field of the skew field $(\ell^{OI}, +, \cdot)$.

Definition 1.7. If A, B, C are three points on a line ℓ^{OI} (collinear) in Desargues affine plane, then we define their **ratio** to be a point $R \in \ell^{OI}$, such that:

$$(B - C) \cdot R = A - C, \quad \text{concisely} \quad R = (B - C)^{-1}(A - C),$$

and we mark this with $r(A, B; C) = (B - C)^{-1}(A - C)$.

Some Results for Ratio of 3-points in Desargues affine plane ([16], [23]).

- For 3-points A, B, C in a line ℓ^{OI} of Desargues affine plane, we have that,

$$r(-A, -B; -C) = r(A, B; C).$$

- For 3-points A, B, C in a line ℓ^{OI} in the Desargues affine plane, we have that

$$r^{-1}(A, B; C) = r(B, A; C).$$

- If A, B, C, are three different points, and different from point O, in a line ℓ^{OI} on Desargues affine plane, then

$$r(A^{-1}, B^{-1}; C^{-1}) = B[r(A, B; C)]A^{-1}.$$

- In the Pappus affine plane, for three point different from point O, in $\ell^{OI}-$line, we have $r(A^{-1}, B^{-1}; C^{-1}) = r(A, B; C) \cdot r(B, A; O)$.
- This ratio-map, $r_{BC} : \ell^{OI} \to \ell^{OI}$ is a bijection in $\ell^{OI}-$line in Desargues affine plane.
- The ratio-maps-set $\mathcal{R}_3 = \{r_{BC}(X), \forall X \in \ell^{OI}\}$, for a different fixed points B, C in $\ell^{OI}-$line, forms a skew-field with 'addition and multiplication' of points in $\ell^{OI}-$line. This, skew field $(\mathcal{R}_3, +, \cdot)$ is sub-skew field of the skew field $(\ell^{OI}, +, \cdot)$.

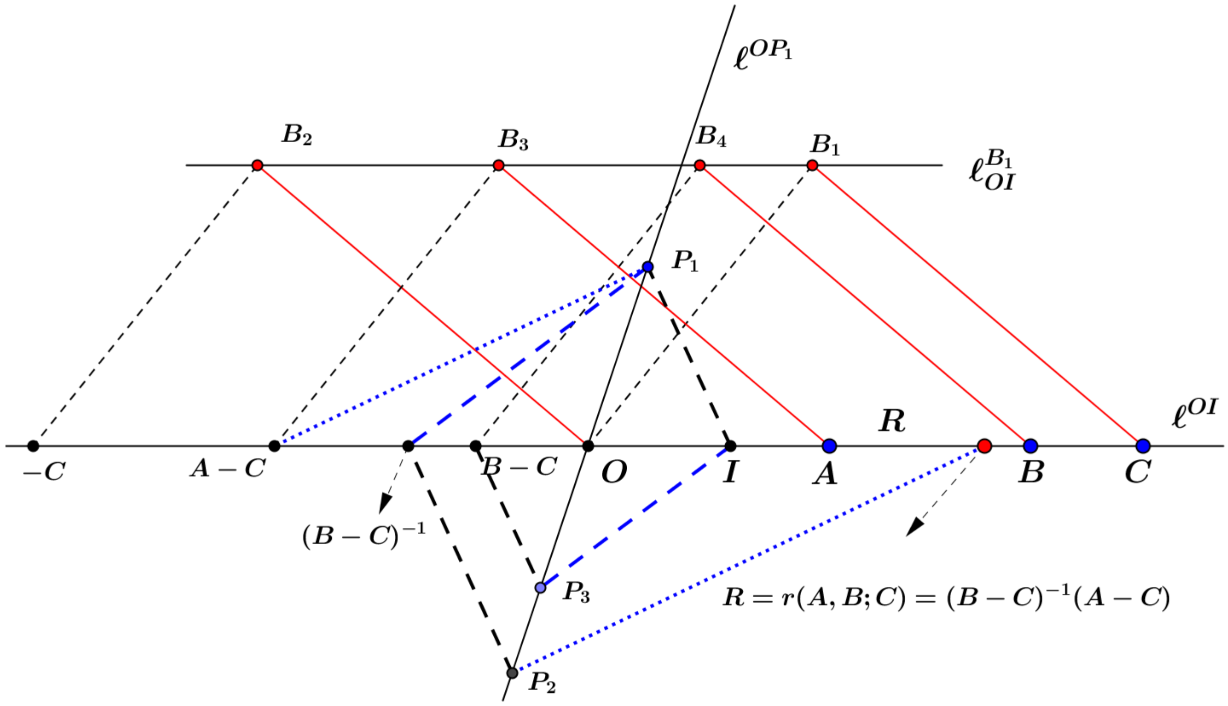

Figure 4: Ratio of 3-Points in a line of Desargues affine plane $R = r(A, B; C)$.

1.5. Cross-Ratio in a line of Desargues affine plane

Let us have the line ℓ^{OI} in Desarges affine plane $\mathcal{A}_{\mathcal{D}}$, and four points, $A, B, C, D \in \ell^{OI}$

Definition 1.8. If A, B, C, D are four points on a line ℓ^{OI} in Desargues affine plane $\mathcal{A}_{\mathcal{D}}$, no three of them equal, then we define their cross ratio to be a point:

$$c_r(A, B; C, D) = \left[(A-D)^{-1}(B-D)\right]\left[(B-C)^{-1}(A-C)\right]$$

Definition 1.9. If the line ℓ^{OI} in Desargues affine plane, is a infinite line (number of points in this line is $+\infty$), we define as follows:

$$c_r(\infty, B; C, D) = (B-D)(B-C)^{-1}$$
$$c_r(A, \infty; C, D) = (A-D)^{-1}(A-C)$$
$$c_r(A, B; \infty, D) = (A-D)^{-1}(B-D)$$
$$c_r(A, B; C, \infty) = (B-C)^{-1}(A-C)$$

From this definition and from ratio definition 1.7 we have that,

- $c_r(A, B; C, D) = \left[(A-D)^{-1}(B-D)\right]\left[(B-C)^{-1}(A-C)\right] = r(B, A; D) \cdot r(A, B; C)$.
- $c_r(\infty, B; C, D) = (B-D)(B-C)^{-1} = [(D-B)^{-1}(C-B)]^{-1} = r^{-1}(C, D; B)$.
- $c_r(A, \infty; C, D) = (A-D)^{-1}(A-C) = (D-A)^{-1}(C-A) = r(C, D; A)$.
- $c_r(A, B; \infty, D) = (A-D)^{-1}(B-D) = r(A, B; D)$.
- $c_r(A, B; C, \infty) = (B-C)^{-1}(A-C) = r(A, B; C)$.

Some results for Cross-Ratio of 4-collinear points in Desargues affine plane (see [18]).

- If A, B, C, D are distinct points in a $\ell^{OI}-$line, in Desargues affine plane, then

$$c_r(-A, -B; -C, -D) = c_r(A, B; D, C) \quad \text{and} \quad c_r^{-1}(A, B; C, D) = c_r(A, B; D, C).$$

- If A, B, C, D are distinct points in a line, in Desargues affine plane and I is unitary point for multiplications of points in same line, then,

  (a) $I - c_r(A, B; C, D) = c_r(A, C; B, D)$
  (b) $c_r(A, D; B, C) = I - c_r^{-1}(A, B; C, D)$
  (c) $c_r(A, C; D, B) = [I - c_r(A, B; C, D)]^{-1}$
  (d) $c_r(A, D; C, B) = [c_r(A, B; C, D) - I]^{-1} c_r(A, B; C, D)$

- If A, B, C, D are distinct points, and different from zero-point O, in a line, in Desargues affine plane and I is unitary point for multiplications of points in same line, have,

$$c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1}) = A \cdot c_r(A, B; C, D) \cdot A^{-1}$$

- If the point $A \in z[K]$ (center of skew field $K = (\ell^{OI}, +, \cdot)$), then,

$$c_r(A, C; B, D) = c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1}).$$

- If $A, B, C, D \in \ell^{OI}$ are distinct points in a line, in Desargues affine plane and I is unital point for multiplications of points in same line, then equation

$$c_r(A, B; C, D) = c_r(B, A; D, C)$$

  it's true, if

  (a) points A, B, C, D are in 'center of skew-field' $z[K]$;
  (b) ratio-points $r(A, B; C)$ are in 'center of skew-field';
  (c) ratio-point $r(B, A; D)$ are in 'center of skew-field';
  (d) ratio-point $r(A, B; D)$ is in centaralizer of point $r(A, B; C)$, or vice versa.

2. Some Invariant Transforms for Cross-Ratio of 4-points in a line of Desargues affine plane

In this section we will see some transformations, for which the cross-ratio are invariant under their action. First, we are defining these transforms and illustrating them with the corresponding figures, respectively: Inversion with Fig.5, Reflection with Fig.6, Natural Translation with Fig.7 and Natural Dilation with Fig.8.

Definition 2.1. Inversion of points in $\ell^{OI}-$line, called the map

$$j_P : \ell^{OI} \to \ell^{OI},$$

where $P \in \ell^{OI}$ is fix-point, and which satisfies the condition,

$$\forall A \in \ell^{OI} \quad j_P(A) = P \cdot A.$$

Notation 2. Inversion of points in $\ell^{OI}-$line,

$$j_P : \ell^{OI} \to \ell^{OI},$$

where $P = -I \in \ell^{OI}$, called *Involution* or *Reflection about the point O in $\ell^{OI}-$line* and we have,

$$\forall A \in \ell^{OI} \quad j_P(A) = -I \cdot A = -A,$$

where $-A$ is opposite point of point A, regrading to addition of points in $\ell^{OI}-$line.

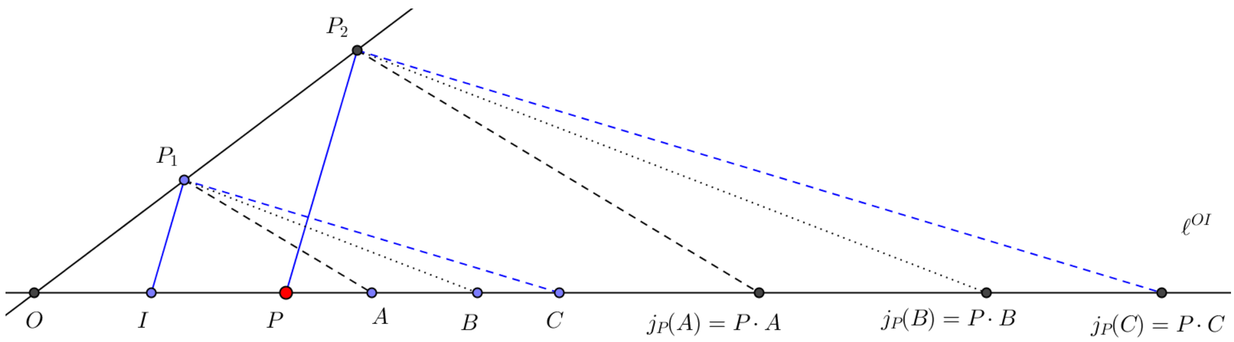

Figure 5: Ilustrate the Inversion of Points, in a line of Desargues affine plane $J_P(A) = P \cdot A$.

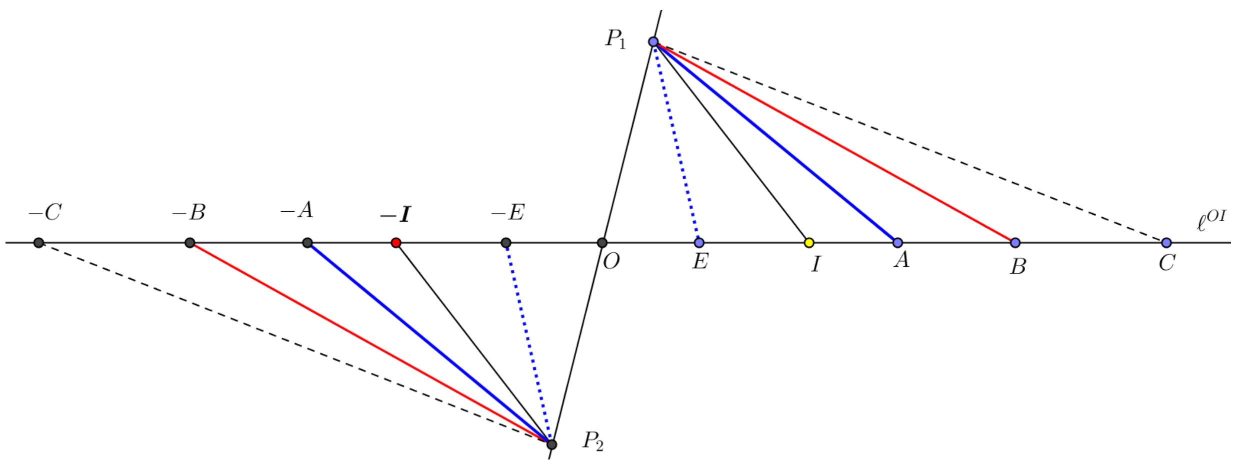

Figure 6: Ilustrate the Reflection of Points, in a line of Desargues affine plane $j_{-I}(A) = -I \cdot A = -A$.

Definition 2.2. A natural translation with point P, of points in $\ell^{OI}-$line, called the map

$$\varphi_P : \ell^{OI} \to \ell^{OI},$$

for a fixed $P \in \ell^{OI}$ which satisfies the condition,

$$\forall A \in \ell^{OI} \quad \varphi_P(A) = P + A.$$

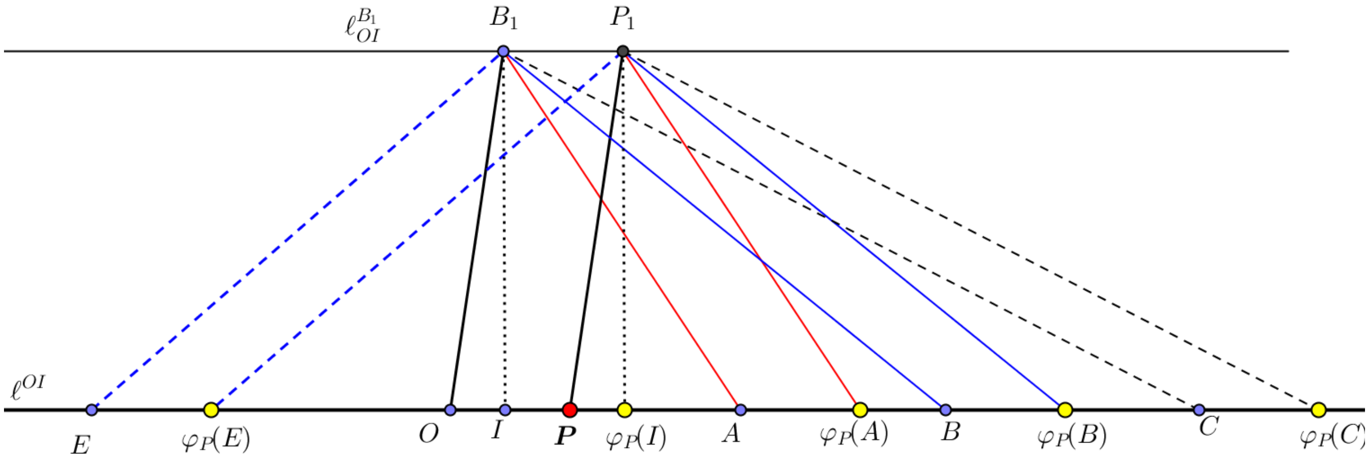

Figure 7: Ilustrate the Natural Translation of Points, in a line of Desargues affine plane $\varphi_P(A) = P + A$.

Definition 2.3. A natural Dilation of points in $\ell^{OI}-$line, called the map

$$\delta_n : \ell^{OI} \to \ell^{OI},$$

for a fixed natural number $n \in \mathbb{Z}$ which satisfies the condition,
if, $n > 0$, have,

$$\forall A \in \ell^{OI} \quad \delta_n(A) = nA = \underbrace{A + A + \cdots + A}_{n-times},$$

and if, $n < 0$, we have

$$\forall A \in \ell^{OI} \quad \delta_n(A) = nA = \underbrace{[-A] + [-A] + \cdots + [-A]}_{(-n)-times},$$

where $-A = (-I) \cdot A$ is the oposite point of point A, regarding to addition of points in $\ell^{OI}-$line.

Definition 2.4. Lets have three fixed points $B, C, D \in \ell^{OI}$. Mobiüs transform for cross-ratio, we called the map,

$$\mu : \ell^{OI} \to \ell^{OI},$$

which satisfies the condition,

$$\forall X \in \ell^{OI}, \quad \mu(X) = c_r(X, B; C, D).$$

Theorem 2.5. *The cross-ratios are invariant under the natural translation with a point P.*

Proof. From cross-ratio definition 1.8, have,

$$c_r(A, B; C, D) = [(A - D)^{-1}(B - D)][(B - C)^{-1}(A - C)]$$

so for cross-ratio we have that,

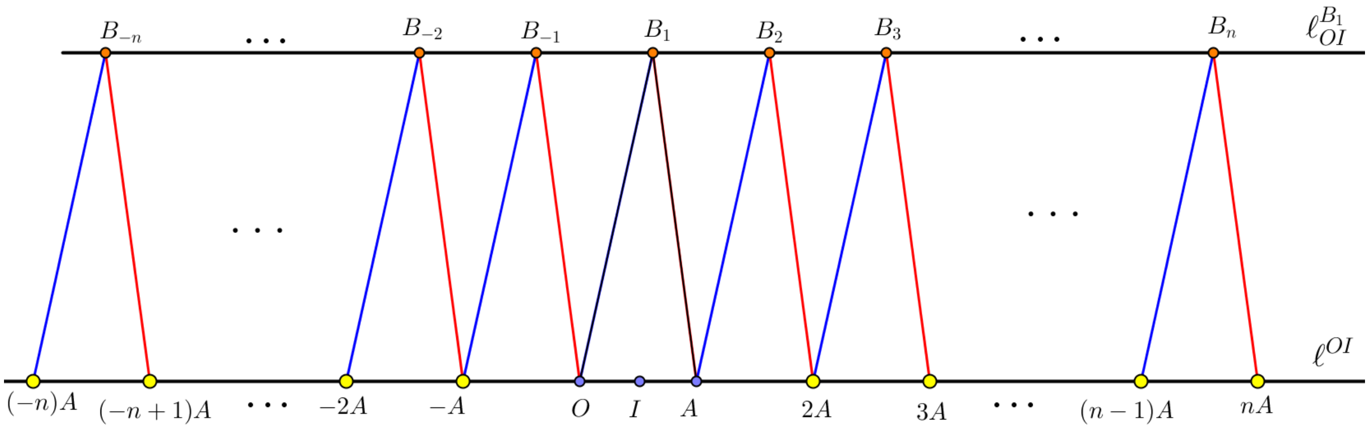

Figure 8: Ilustrate the Natural Translation of Points, in a line of Desargues affine plane $\delta_n(A) = nA$.

$$\begin{aligned} c_r[\varphi_P(A), \varphi_P(B); \varphi_P(C), \varphi_P(D)] &= c_r(A+P, B+P; C+P, D+P) \\ &= [((A+P)-(D+P))^{-1}((B+P)-(D+P))] \\ &\cdot [((B+P)-(C+P))^{-1}((A+P)-(C+P))] \\ &= [(A+P-D-P)^{-1}(B+P-D-P)] \\ &\cdot [(B+P-C-P)^{-1}(A+P-C-P)] \\ &= [(A-D)^{-1}(B-D)][(B-C)^{-1}(A-C)] \\ &= c_r(A, B; C, D) \end{aligned}$$

Theorem 2.6. *The Cross-Ratios are invariant under the natural dilation with a fixet $n \in \mathbb{Z}$.*

Proof. From cross-ratio definition 1.8, we have

$$c_r(A, B; C, D) = [(A-D)^{-1}(B-D)][(B-C)^{-1}(A-C)]$$

so for cross-ratio of *natural dilation-points* we have that,
Case.1 For $n > 0$, we have,

$$\begin{aligned} c_r[\delta_n(A), \delta_n(B); \delta_n(C), \delta_n(D)] &= c_r(nA, nB; nC, nD) \\ &= [(nA-nD)^{-1}(nB-nD)][(nB-nC)^{-1}(nA-nC)] \\ &= [(n(A-D))^{-1}n(B-D)][(n(B-C))^{-1}n(A-C)] \\ &= [(A-D)^{-1}n^{-1}n(B-D)][(B-C)^{-1}n^{-1}n(A-C)] \\ &= [(A-D)^{-1}(B-D)][(B-C)^{-1}(A-C)] \\ &= c_r(A, B; C, D) \end{aligned}$$

Case.2 For $n < 0$, we mark $m = -n > 0$ or $-m = n$ where $m > 0$, and we have that,

$$\begin{aligned}
c_r[\delta_n(A), \delta_n(B); \delta_n(C), \delta_n(D)] &= c_r(nA, nB; nC, nD) \\
&= c_r([-m]A, [-m]B; [-m]C, [-m]D) \\
&= [([-m]A - [-m]D)^{-1}([-m]B - [-m]D)] \\
&\quad \cdot [([-m]B - [-m]C)^{-1}([-m]A - [-m]C)] \\
&= [(m[-A] - m[-D])^{-1}(m[-B] - m[-D])] \\
&\quad \cdot [(m[-B] - m[-C])^{-1}(m[-A] - m[-C])] \\
&= [(m([-A] - [-D]))^{-1}m([-B] - [-D])] \\
&\quad \cdot [(m([-B] - [-C]))^{-1}m([-A] - [-C])] \\
&= [([-A] - [-D])^{-1}m^{-1}m([-B] - [-D])] \\
&\quad \cdot [([-B] - [-C])^{-1}m^{-1}m([-A] - [-C])] \\
&= [([-A] - [-D])^{-1}([-B] - [-D])] \\
&\quad \cdot [([-B] - [-C])^{-1}([-A] - [-C])] \\
&= [([-I][A - D])^{-1}([-I][B - D])] \\
&\quad \cdot [([-I][B - C])^{-1}([-I][A - C])] \\
&= [(A - D)^{-1}[-I]^{-1}[-I](B - D)] \\
&\quad \cdot [(B - C)^{-1}[-I]^{-1}[-I](A - C)] \\
&= [(A - D)^{-1}([-I][-I])(B - D)] \\
&\quad \cdot [(B - C)^{-1}([-I][-I])(A - C)] \\
&= [(A - D)^{-1}(I)(B - D)][(B - C)^{-1}(I)(A - C)] \\
&= [(A - D)^{-1}(B - D)][(B - C)^{-1}(A - C)] \\
&= c_r(A, B; C, D)
\end{aligned}$$

remember that, $[-I]^{-1} = -I$ and $(-I) \cdot (-I) = I$.

Another Proof of Case.2, for $n < 0$, we mark $m = -n > 0$ or $-m = n$ where $m > 0$, and we have that,

$$c_r[\delta_n(A), \delta_n(B); \delta_n(C), \delta_n(D)] = c_r(nA, nB; nC, nD) = c_r([-m]A, [-m]B; [-m]C, [-m]D)$$

so,

$$c_r([-m]A, [-m]B; [-m]C, [-m]D) = c_r(-[mA], -[mB]; -[mC], -[mD])$$

but from the results listed in section 1, for cross-ratio, we have that,

$$c_r(-A, -B; -C, -D) = c_r(A, B; C, D), \quad \text{for all different points A,B,C,D} \in \ell^{OI}$$

therefore

$$c_r(-[mA], -[mB]; -[mC], -[mD]) = c_r(mA, mB; mC, mD)$$

and from Case.1 (since $m > 0$), have

$$c_r(mA, mB; mC, mD) = c_r(A, B; C, D).$$

Hence,

$$c_r[\delta_n(A), \delta_n(B); \delta_n(C), \delta_n(D)] = c_r(A, B; C, D).$$

□

Theorem 2.7. *The Cross-Ratios are invariant under Inversion with a given point $P \in \ell^{OI}$.*

Proof. From cross-ratio definition 1.8, we have

$$c_r(A,B;C,D) = [(A-D)^{-1}(B-D)][(B-C)^{-1}(A-C)]$$

so, for cross-ratio of the points $j_P(A), j_P(B); j_P(C), j_P(D) \in \ell^{OI}$ (cross-ratio of Inversion-points) we have that,

$$\begin{aligned} c_r[j_P(A), j_P(B); j_P(C), j_P(D)] &= c_r(PA, PB; PC, PD) \\ &= [(PA-PD)^{-1}(PB-PD)][(PB-PC)^{-1}(PA-PC)] \\ &= [(P(A-D))^{-1}P(B-D)][(P(B-C))^{-1}P(A-C)] \\ &= [(A-D)^{-1}P^{-1}P(B-D)][(B-C)^{-1}P^{-1}P(A-C)] \\ &= [(A-D)^{-1}(P^{-1}P)(B-D)][(B-C)^{-1}(P^{-1}P)(A-C)] \\ &= [(A-D)^{-1}I(B-D)][(B-C)^{-1}I(A-C)] \\ &= [(A-D)^{-1}(B-D)][(B-C)^{-1}(A-C)] \\ &= c_r(A,B;C,D) \end{aligned}$$

□

Corollary 2.8. *The Cross-Ratios are invariant under* reflection about the point O in $\ell^{OI}-$line *in Desargues affine plane.*

Proof. We have the Inversion with point $(-I) \in \ell^{OI}-$line, and from cross-ratio definition 1.8, we have

$$c_r(A,B;C,D) = [(A-D)^{-1}(B-D)][(B-C)^{-1}(A-C)]$$

so, for cross-ratio of the points $j_{[-I]}(A), j_{[-I]}(B); j_{[-I]}(C), j_{[-I]}(D) \in \ell^{OI}$ (cross-ratio of Inversion-points) we have that,

$$\begin{aligned} c_r[j_{[-I]}(A), j_{[-I]}(B); j_{[-I]}(C), j_{[-I]}(D)] &= c_r([-I]A, [-I]B; [-I]C, [-I]D) \\ &= [([-I]A-[-I]D)^{-1}([-I]B-[-I]D)] \\ &\quad \cdot [([-I]B-[-I]C)^{-1}([-I]A-[-I]C)] \\ &= [([-I](A-D))^{-1}[-I](B-D)] \\ &\quad \cdot [([-I](B-C))^{-1}[-I](A-C)] \\ &= [(A-D)^{-1}[-I]^{-1}[-I](B-D)] \\ &\quad \cdot [(B-C)^{-1}[-I]^{-1}[-I](A-C)] \\ &= [(A-D)^{-1}([-I]^{-1}[-I])(B-D)] \\ &\quad \cdot [(B-C)^{-1}([-I]^{-1}[-I])(A-C)] \\ &= [(A-D)^{-1}([-I][-I])(B-D)] \\ &\quad \cdot [(B-C)^{-1}([-I][-I])(A-C)] \\ &= [(A-D)^{-1}I(B-D)][(B-C)^{-1}I(A-C)] \\ &= [(A-D)^{-1}(B-D)][(B-C)^{-1}(A-C)] \\ &= c_r(A,B;C,D) \end{aligned}$$

we used the fact that, in the skew field we have true $[-I]^{-1} = -I$ and $[-I][-I] = I$.

Hence

$$c_r[j_{[-I]}(A), j_{[-I]}(B); j_{[-I]}(C), j_{[-I]}(D)] = c_r(A,B;C,D)$$

Theorem 2.9. *Cross-Ratios are invariant under Mobiüs transform.*

Proof. From Mobiüs transform definition 2.4 we have

$$\mu(X) = c_r(X, B; C, D) = [(X - D)^{-1}(B - D)][(B - C)^{-1}(X - C)]$$

so, for cross-ratio of the points $\mu(A), \mu(B), \mu(C), \mu(D) \in \ell^{OI}$ first, we calculate, this point, according to following the definition of $\mu-$map, and we have

- $\mu(A) = c_r(A, B; C, D)$,
- $\mu(B) = c_r(B, B; C, D)$, so

$$\begin{aligned}\mu(B) &= [(B - D)^{-1}(B - D)][(B - C)^{-1}(B - C)] \\ &= [I][I] \\ &= I.\end{aligned}$$

  Thus

$$\mu(B) = I$$

- $\mu(C) = c_r(C, B; C, D)$, so,

$$\begin{aligned}\mu(C) &= [(C - D)^{-1}(B - D)][(B - C)^{-1}(C - C)] \\ &= [(C - D)^{-1}(B - D)][(B - C)^{-1}O] = O \\ &= [(C - D)^{-1}(B - D)][O] \\ &= O\end{aligned}$$

  Thus

$$\mu(C) = O.$$

- $\mu(D) = c_r(D, B; C, D)$, so,

$$\begin{aligned}\mu(D) &= [(D - D)^{-1}(B - D)][(B - C)^{-1}(D - C)] \\ &= [O^{-1}(B - D)][(B - C)^{-1}(D - C)] \\ &\text{(and } O^{-1} = \infty - \text{(point in infinity))} \\ &= [\infty][(B - C)^{-1}(D - C)] \\ &= \infty\end{aligned}$$

  Thus

$$\mu(D) = \infty$$

Now, calculate, cross-ratio of points $\mu(A), \mu(B), \mu(C), \mu(D)$, and have

$$\begin{aligned}c_r[\mu(A), \mu(B); \mu(C), \mu(D)] &= c_r(\mu(A), I; O, \infty) \\ &= r(\mu(A), I; O) \\ &= (I - O)^{-1}(\mu(A) - O) \\ &= (I)^{-1}(\mu(A)) \\ &= \mu(A) \\ &= c_r(A, B; C, D)\end{aligned}$$

□

3. Transforms which Preserving Cross-Ratios of 4-points in a line of Desargues affine plane

In this section we prove that the parallel projection, Translations and Dilation of a line in itself or in isomorphic line in Desargues affine plane, *preserving:* the cross-ratios for 4-points. The geometrical interpretations, even in the Euclidean view, are quite beautiful in the above theorems, regardless of the rather rendered figures. This is also the reason that we are giving the proofs in the algebraic sense. So we will always have in mind the close connection of skew field and a line in Desargues affine plane, and the properties of parallel projection, translations and dilations.

Theorem 3.1. *Translations φ with trace the same line ℓ^{OI} of points A, B, C, D, preserve the cross-ratio of this points,*

$$\varphi(c_r(A, B; C, D)) = c_r(\varphi(A), \varphi(B); \varphi(C), \varphi(D))$$

Proof. For the proof of that theorem, we refer to the translation properties, which are studied in the papers (see [14], [15], [8], [20]). We will follow algebraic properties for these proofs. Whatever the translation φ, automorphism or isomorphism, $\varphi : \ell^{OI} \to \ell^{OI}$ or $\varphi : \ell^{OI} \to \ell^{O'I'}$, where $\ell^{OI} \neq \ell^{O'I'}$ and $\ell^{OI} || \ell^{O'I'}$, we have true the above equals,

$$\begin{aligned}
\varphi(c_r(A, B; C, D)) &= \varphi\left\{\left[(A - D)^{-1}(B - D)\right]\left[(B - C)^{-1}(A - C)\right]\right\} \\
&\text{(For this equation, we use the fact that } \varphi \text{ is homomorphisms)} \\
&= \varphi\left[(A - D)^{-1}(B - D)\right] \cdot \varphi\left[(B - C)^{-1}(A - C)\right] \\
&= \varphi[(A - D)^{-1}(B - D)] \cdot \varphi[(B - C)^{-1}(A - C)] \\
&\text{(again, from the fact that } \varphi \text{ is a homomorphism,)} \\
&= \left\{\varphi[(A - D)^{-1}] \cdot \varphi(B - D)\right\} \cdot \left\{\varphi[(B - C)^{-1}] \cdot \varphi(A - C)\right\} \\
&\text{(translation } \varphi \text{ is bijective,)} \\
&= \left\{[\varphi(A - D)]^{-1} \cdot \varphi(B - D)\right\} \cdot \left\{[\varphi(B - C)]^{-1} \cdot \varphi(A - C)\right\} \\
&\text{(from the fact that } \varphi \text{ is a homomorphism)} \\
&= \left\{[\varphi(A) - \varphi(D)]^{-1} \cdot [\varphi(B) - \varphi(D)]\right\} \cdot \left\{[\varphi(B) - \varphi(C)]^{-1} \cdot [\varphi(A) - \varphi(C)]\right\} \\
&\text{(from cross-ratio definition 1.8)} \\
&= c_r(\varphi(A), \varphi(B); \varphi(C), \varphi(D))
\end{aligned}$$

□

Theorem 3.2. *The parallel projection between the two lines ℓ_1 and ℓ_2 in Desargues affine plane,* ***preserving the cross-ratio*** *of 4−points,*

$$P_P(c_r(A, B; C, D)) = c_r(P_P(A), P_P(B); P_P(C), P_P(D))$$

Proof. If $\ell_1 || \ell_2$, we have that the parallel projection is a translation, and have true this theorem.
If lines ℓ_1 and ℓ_2 they are not parallel (so, they are cut at a single point), we have $A, B, C \in \ell_1$ and $P_P(A), P_P(B), P_P(C), P_P(D) \in \ell_2$. Also since P_P is a bijection we have that,

$$\begin{aligned}
P_P(c_r(A, B; C, D)) &= P_P\left\{\left[(A - D)^{-1}(B - D)\right]\left[(B - C)^{-1}(A - C)\right]\right\} \\
&= P_P[(A - D)^{-1}(B - D)] \cdot P_P[(B - C)^{-1}(A - C)] \\
&= \left\{P_P[(A - D)^{-1}] \cdot P_P(B - D)\right\} \cdot \left\{P_P[(B - C)^{-1}] \cdot P_P(A - C)\right\} \\
&= \left\{[P_P(A - D)]^{-1} \cdot P_P(B - D)\right\} \cdot \left\{[P_P(B - C)]^{-1} \cdot P_P(A - C)\right\} \\
&= \left\{[P_P(A) - P_P(D)]^{-1} \cdot [P_P(B) - P_P(D)]\right\} \\
&\cdot \left\{[P_P(B) - P_P(C)]^{-1} \cdot [P_P(A) - P_P(C)]\right\} \\
&= c_r(\varphi(A), P_P(B); P_P(C), P_P(D))
\end{aligned}$$

Theorem 3.3. *Dilation δ with fixed point in the same line ℓ^{OI} of points A, B, C, D, preserve the cross-ratio of this points,*

$$\delta(c_r(A, B; C, D)) = c_r(\delta(A), \delta(B); \delta(C), \delta(D))$$

Proof. For the proof of that theorem, we refer to the dilation properties, which are studied in the papers (see [14], [15], [8], [20]). We will follow algebraic properties for these proofs. Whatever the dilation δ, is a automorphism or a isomorphism, $\delta : \ell^{OI} \to \ell^{OI}$ or $\delta : \ell^{OI} \to \ell^{O'I'}$, where $\ell^{OI} \neq \ell^{O'I'}$ and $\ell^{OI} || \ell^{O'I'}$, relying on isomorphism (or automorphism) properties, we have

$$\begin{aligned}
\delta(c_r(A, B; C, D)) &= \delta\left\{\left[(A - D)^{-1}(B - D)\right]\left[(B - C)^{-1}(A - C)\right]\right\} \\
&= \delta[(A - D)^{-1}(B - D)] \cdot \delta[(B - C)^{-1}(A - C)] \\
&= \left\{\delta[(A - D)^{-1}] \cdot \delta(B - D)\right\} \cdot \left\{\delta[(B - C)^{-1}] \cdot \delta(A - C)\right\} \\
&= \left\{[\delta(A - D)]^{-1} \cdot \delta(B - D)\right\} \cdot \left\{[\delta(B - C)]^{-1} \cdot \delta(A - C)\right\} \\
&= \left\{[\delta(A) - \delta(D)]^{-1} \cdot [\delta(B) - \delta(D)]\right\} \cdot \left\{[\delta(B) - \delta(C)]^{-1} \cdot [\delta(A) - \delta(C)]\right\} \\
&= c_r(\delta(A), \delta(B); \delta(C), \delta(D))
\end{aligned}$$

□

4. Declarations

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