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Generalized Graph Energies of a Regular Graph under Vertex Duplication Operation

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Abstract

We provide a thorough examination of the graph energies in regular graphs that arise from the vertex duplication process in this paper. Understanding the numerous structural components of graphs requires understanding the thought of graph energy, known as a measurement obtained by computing eigenvalues the adjacency matrix of a graph. We derived generalized closed-form expressions for a number of important energy metrics, such as minimum degree, energy maximum degree energy, first Zagreb energy, second Zagreb energy and degree square sum energy, utilizing proficient algebraic graph theory techniques and eigenvalue spectrum analysis. Our work emphasizes on vertex duplication techniques and the impact they have on these energy metrics, primarily on regular graphs, a basic class of graphs where every vertex has the same degree. The resulting formulations offer further explanations for the behavior and attributes of these energy functions within the framework of regular graphs, providing a more comprehensive knowledge of how these operations affect the structural complexity of the graph. These findings greatly expand the conceptual model of graph energy and have potential uses in fields like combinatorics, chemistry, and network analysis where the energy models of graphs are extensively employed.

Keywords: Vertex duplication of graph, matrices, regular graph, minimum degree energy, maximum degree energy, first Zagreb energy, second Zagreb energy, degree square sum energy, graph energy, eigenvalues.
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1. Introduction

Spectral graph theory provides a powerful framework for analyzing the structure and properties of graphs through matrix representations and their eigenvalues. Concepts such as graph energy, the characteristic equation, the spectrum, and graph operations are central tools in this theory, offering deep insights into various graph properties and applications in fields ranging from network science to quantum chemistry.

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In the context of molecular graphs, graph energy was initially developed in 1978 [1] and was used to approximate the total π-electron energy of a molecule. Graph energy, derived from the eigenvalues of the adjacency matrix, provides a useful measure of a graph's structure. Through graph operations, the energy of more complex graphs can be calculated, offering insights into their properties and applications in various fields. A key idea in linear algebra, especially when studying matrices and linear transformations, is the characteristic equation.The characteristic equation is derived from a square matrix

$$\det(\eta I - A) = 0,$$

where η represents an eigenvalue of $A(G)$ and I is the identity matrix. A method for determining a matrix's eigenvalues is provided by the characteristic equation. Understanding the behavior of differential equation systems and linear transformations requires an understanding of eigenvalues. The eigenvalues of the graph change when a vertex is duplicated, and this, in turn, affects the energy of the graph. Since graph energy $E(G)$ is the sum of the absolute values of the eigenvalues, duplicating a vertex is typically used to increase the graph energy. The exact impact on the spectrum and energy depends on the specific structure of the graph and the vertex being duplicated. For instance, duplicating a highly connected vertex (i.e., a vertex with many neighbors) tends to have a larger effect on the spectrum than duplicating an isolated or less connected vertex.The spectrum can provide crucial information about the properties of the matrix, such as stability, invertibility, and the nature of its associated dynamical systems [2]. Eigenvalues play a vital role in various applications across mathematics, physics, and engineering, including stability analysis, vibrations analysis, and quantum mechanics, where they represent measurable quantities like energy levels. The study of eigenvalues and spectra extends to infinite-dimensional spaces and operators in functional analysis, leading to rich theoretical developments and applications in fields such as quantum mechanics and differential equations.Understanding the concepts of eigenvalues and spectra is essential for further exploration of linear algebra, differential equations, and their numerous applications in both theoretical and applied contexts.

In network design, duplicating critical nodes (vertices) can provide fault tolerance. If one node fails, its duplicate can maintain network connectivity. Vertex duplication in communication or transportation networks can improve reliability by creating alternative routes for data or traffic in case of node failures. Vertex duplication can help identify overlapping communities. By duplicating nodes that connect different communities, researchers can analyze connections between groups more effectively. In biological networks, vertex duplication is used to model gene duplication events in evolutionary biology. Duplicating a gene vertex and its interactions allows researchers to study how duplicate genes evolve different functions over time. In biological networks, vertex duplication is used to model gene duplication events in evolutionary biology. Duplicating a gene vertex and its interactions allows researchers to study how duplicate genes evolve different functions over time. The connection between energy and Sombor energy of the m-splitting graph and the m-shadow graph of the k-regular graph was discovered by the author in [3]. The author presented the partition energy of an m-splitting graph and their generalized complements in [4]. A statistical overview of graph energies research and applications may be found in [5]. The author of [6] computed the graph energy of particular graph operations using block matrix partition. Results for m-splitting and m-shadow of a graph energy of a graph are provided by the author in [7]. In [8], Kousar and Nazeer presented findings for multiple graph energies of both complete graph and regular subdivision graphs. Maximum and minimum energy of p-splitting and p-shadow graphs as a multiple of maximum and minimum energy of graph G were reported by Rao and Srinivasa in [9]. The Randić energy of an m-splitting graph, m-shadow graph, and m-duplicate graph of a given graph are computed in [10]. For notations and fundamental definitions, we refer to the readers [11, 12, 13].

The structure of this article is as follows: Basic definitions of various graph energies are provided in Section 2. Section 3 gives the result of the maximum degree energy, minimum degree energy, first Zagreb energy, second Zagreb energy and degree square sum energy of regular graph for vertex duplication.

2. Preliminaries

The adjacency matrix $A(G)$ of a graph G with n vertices $w_1, w_2, \ldots, w_n$ is an $n \times n$ matrix where each entry a_{ij} is given by [14]

$$a_{ij} = \begin{cases} 1, & \text{if } w_i w_j \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

The energy $E(G)$ of the graph G is defined as [15]:

$$E(G) = \sum_{i=1}^{n} |\eta_i|,$$

where the adjacency matrix $A(G)$'s eigenvalues are $\eta_1, \eta_2, \ldots, \eta_n$.

The characteristic polynomial of a matrix $A(G)$ is given by [1]

$$\phi_A(\eta) = \det(A - \eta I)$$

where η is a scalar and I is the identity matrix of the same dimension as $A(G)$. The roots of $\phi_A(\eta) = 0$ are the eigenvalues of A(G).

Given a graph G, its spectrum is given as

$$Spec(G) = \begin{pmatrix} \eta_1 & \eta_2 & \ldots & \eta_n \\ m_1 & m_2 & \ldots & m_n \end{pmatrix}.$$

We define the vertex duplication of a graph G, denoted by G_1, as the graph obtained by duplicating each vertex of G [16]. Duplication of a vertex w_k *by a new edge* $e = w'w''$ in a graph G produces a new graph G_1 such that $N(w') = w_k, w''$ and $N(w'') = \{w_k, w'\}$.

The matrix of maximum degree $M_e(G) = [M_{ij}]$ Of G, is described as [17]

$$[M_{ij}] = \begin{cases} max(d_{w_i}, d_{w_j}), & \text{if } w_i w_j \in E(G) \\ 0, & \text{otherwise} \end{cases}$$

For a simple connected graph G, the author defined the maximum degree energy $M(G)$ as the sum of the absolute values of the eigenvalues of the maximum degree matrix $M_{[ij]}$ of G. The minimum degree matrix $m_e(G) = [m_{ij}]$ of G is defined as [18]

$$[m_{ij}] = \begin{cases} min(d_{w_i}, d_{w_j}), & \text{if } w_i w_j \in E(G) \\ 0, & \text{otherwise} \end{cases}$$

The sum of the absolute values of the eigenvalues of the minimum degree matrix $m_{[ij]}$ of a simple connected graph G is the minimal degree energy $m(G)$ of G. In [19] using the first and second Zagreb topological indices, Gutman introduced the First Zagreb Energy and Second Zagreb Energy. A simple connected graph G's first Zagreb energy ZE_1 is defined as the sum of the absolute values of the first Zagreb matrix's eigenvalues. $Z^{(1)}(G)$ of G where $Z^{(1)}(G) = [z_{ij}^{(1)}]$ and

$$z_{ij}^{(1)} = \begin{cases} d_{w_i} + d_{w_j}, & \text{if } w_i w_j \in E(G) \\ 0, & \text{otherwise} \end{cases}$$

A simple connected graph G's second Zagreb energy ZE_2 is equal to the sum of the absolute values of its second Zagreb matrix's $Z^{(2)}(G)$ eigenvalues of G where $Z^{(2)}(G) = [z_{ij}^{(2)}]$ and

$$z_{ij}^{(2)} = \begin{cases} d_{w_i} . d_{w_j}, & \text{if } w_i w_j \in E(G) \\ 0, & \text{otherwise} \end{cases}$$

According to [20], the degree square sum energy $E_{DSS}(G)$ of a simple connected graph G is defined as the sum of the absolute values of the eigenvalues of the degree square sum matrix $DSS(G)$. Here $DSS(G) = [dss_{ij}]$ where

$$dss_{ij} = \begin{cases} d_{w_i}^2 + d_{w_j}^2, & \text{if } w_iw_j \in E(G) \\ 0, & \text{otherwise} \end{cases}$$

Let $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{p\times q}$. Then the *Kronecker product* (or tensor product) of C and D is defined as the matrix

$$C \otimes D = \begin{bmatrix} c_{11}D & \cdots & c_{1n}D \\ \vdots & \ddots & \vdots \\ c_{m1}D & \cdots & c_{mn}D \end{bmatrix}$$

Proposition 2.1. *[21] If $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times n}$ be invertible matrices then*

$$(X \otimes Y)^{-1} = X^{-1} \otimes Y^{-1}$$

Proposition 2.2. *[22, 21] Let $M, N, P, Q \in$ be matrices, Q be invertible and*

$$S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$$

Then,

$$det(S) = det\left[Q\right].det\left[M - NQ^{-1}P\right].$$

3. Main Results

3.1. Maximum Degree Energy of Vertex Duplication of Regular Graph

The maximum degree energy of vertex duplication is shown in this section in terms of the adjacency matrix's eigenvalues for a regular graph G. An example of vertex duplication of C_4 is provided for explanation.

Theorem 3.1. *The maximum degree energy $E_M(G_1)$ of the vertex-duplicated graph G_1 of a given t-regular graph G, where η_i are the eigenvalues of $A(G)$, is given as:*

$$E_M(G_1) = 2n + \sum_{\eta_i \le 3} \sqrt{[(t+2)\eta_i + 2]^2 - 4[2(t+2)\eta_i - 2(t+2)^2]} + \sum_{\eta_i > 3} [(t+2)\eta_i + 2)].$$

Proof. Consider a connected, undirected, simple t-regular graph G with n vertices and m edges. Subsequently, the incidence matrix $B(G)$ and adjacency matrix $A(G)$ are provided by

$$A(G) = \begin{bmatrix} & \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \cdots & \mathbf{w}_n \\ \mathbf{w}_1 & 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ \mathbf{w}_2 & a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ \mathbf{w}_3 & a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{w}_n & a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix},$$

$$B(G) = \begin{bmatrix} & \mathbf{w}_1' & \mathbf{w}_1'' & \mathbf{w}_2' & \mathbf{w}_2'' & \cdots & \mathbf{w}_n' & \mathbf{w}_n'' \\ \mathbf{w}_1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \mathbf{w}_2 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{w}_n & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}.$$

We duplicate the vertices $w_1, w_2, \ldots, w_n$ all together by the edges $e_1, e_2, \ldots, e_n$, respectively, such that:

$$e_1 = w_1' w_1'', \quad e_2 = w_2' w_2'', \quad \ldots, \quad e_n = w_n' w_n''$$

to obtain graph G_1. The following is the maximum degree matrix of G_1 in terms of block matrices:

$$M(G_1) = \begin{bmatrix} & w_1 & w_2 & \cdots & w_n & w_1' & w_1'' & w_2' & w_2'' & \cdots & w_n' & w_n'' \\ w_1 & 0 & (t+2)a_{12} & \cdots & (t+2)a_{1n} & (t+2) & (t+2) & 0 & 0 & \cdots & 0 & 0 \\ w_2 & (t+2)a_{21} & 0 & \cdots & (t+2)a_{2n} & 0 & 0 & (t+2) & (t+2) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_n & (t+2)a_{n1} & (t+2)a_{n2} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & (t+2) & (t+2) \\ w_1' & (t+2) & 0 & \cdots & 0 & 0 & 2 & 0 & 0 & \cdots & 0 & 0 \\ w_1'' & (t+2) & 0 & \cdots & 0 & 2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ w_2' & 0 & (t+2) & \cdots & 0 & 0 & 0 & 0 & 2 & \cdots & 0 & 0 \\ w_2'' & 0 & (t+2) & \cdots & 0 & 0 & 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_n' & 0 & 0 & \cdots & (t+2) & 0 & 0 & 0 & 0 & \cdots & 0 & 2 \\ w_n'' & 0 & 0 & \cdots & (t+2) & 0 & 0 & 0 & 0 & \cdots & 2 & 0 \end{bmatrix}$$

That is

$$M(G_1) = \begin{bmatrix} (t+2)A(G) & (t+2)B(G) \\ (t+2)B^T(G) & I_n \otimes 2A(K_2) \end{bmatrix},$$

The characteristic polynomial $\phi(G_1, z)$, which is provided by, is first computed in order to find the eigenvalues of $M(G_1)$:

$$\phi(G_1, z) = \det(zI_{3n} - M(G_1)).$$

The block matrix determinant formula allows us to have:

$$\phi(G_1, z) = \begin{vmatrix} zI_n - (t+2)A(G) & (t+2)B(G) \\ (t+2)B^T(G) & I_n \otimes (zI_2 - 2A(K_2)) \end{vmatrix}.$$

By taking into account the block matrix's structure, this can be made simpler

$$\begin{aligned} \phi(G_1, z) &= \left| I_n \otimes (zI_2 - 2A(K_2)) \right| \left| zI_n - (t+2)A(G) - (t+2)B(G)(I_n \otimes zI_2 - 2A(K_2))^{-1}(t+2)B^T(G) \right| \\ &= (z^2-4)^n \left| zI_n - (t+2)A(G) - \tfrac{(t+2)^2 B(G)(zI_2 + 2A(K_2) \otimes I_n)B^T(G)}{z^2-4} \right| \\ &= \left| (z^2-4)(zI_n - (t+2)A(G)) - (t+2)^2 B(G)(zI_2 + 2A(K_2) \otimes I_n)B^T(G) \right|. \end{aligned}$$

Now,

$$B\left((zI_2 + 2A(K_2)) \otimes I_n\right)B^T = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} z & 2 & 0 & 0 & \cdots & 0 & 0 \\ 2 & z & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & z & 2 & \cdots & 0 & 0 \\ 0 & 0 & 2 & z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & z & 2 \\ 0 & 0 & 0 & 0 & \cdots & 2 & z \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} z+2 & z+2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & z+2 & z+2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & z+2 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$
= \begin{pmatrix} 2z+4 & 0 & 0 & \cdots & 0 \\ 0 & 2z+4 & 0 & \cdots & 0 \\ 0 & 0 & 2z+4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2z+4 \end{pmatrix}
$$
$$
= (2z+4)I_n.
$$

Continuing the theorem's proof

$$
\begin{aligned}
\phi(G_1, z) &= \left|(z^2-4)(zI_n - (t+2)A(G)) - (t+2)^2 B(G)(zI_2 + 2A(K_2) \otimes I_n)B^T(G)\right|. \\
&= \left|(z^2-4)(zI_n - (t+2)A(G)) - (t+2)^2(2z+4)I_n\right|.
\end{aligned}
$$

In the situation η_i represents the eigenvalues of $A(G)$, with $i = 1, 2, ..., n$

$$
\begin{aligned}
\phi(G_1, z) &= \prod_{i=1}^{n} \left[(z^2-4)(z-(t+2)\eta_i) - (t+2)^2(2z+4)\right] \\
&= (z+2)^n \prod_{i=1}^{n} \left[(z-2)(z-(t+2)\eta_i) - 2(t+2)^2\right] \\
&= (z+2)^n \prod_{i=1}^{n} [(z-2)(z-(t+2)\eta_i) - 2(t+2)^2 \\
&= (z+2)^n \prod_{i=1}^{n} [(z^2 - z(t+2)\eta_i - 2z + 2(t+2)\eta_i - 2(t+2)^2].
\end{aligned}
$$

The characteristic polynomial above has the following roots:

$$
z = -2(\text{n–times}), \quad z = \frac{(t+2)\eta_i + 2 \pm \sqrt{[(t+2)\eta_i + 2]^2 - 4[2(t+2)\eta_i - 2(t+2)^2]}}{2}
$$

For each $i = 1, 2, ..., n$.

$$
Spec(M(G_1)) = \begin{pmatrix} -2 & \alpha_1 ... & \alpha_2 \\ n & 1 & 1 \\ \alpha_3 & \alpha_4 ... & \alpha_5 \\ 1 & 1 & 1 \end{pmatrix}.
$$

where

$$
\begin{aligned}
\alpha_1 &= \frac{(r+2)\eta_1 + 2 + \sqrt{[(t+2)\eta_1 + 2]^2 - 4[2(t+2)\eta_1 - 2(t+2)^2]}}{2}, \\
\alpha_2 &= \frac{(t+2)\eta_n + 2 + \sqrt{[(t+2)\eta_n + 2]^2 - 4[2(t+2)\eta_n - 2(t+2)^2]}}{2}, \\
\alpha_3 &= \frac{(t+2)\eta_1 + 2 - \sqrt{[(t+2)\eta_1 + 2]^2 - 4[2(t+2)\eta_1 - 2(t+2)^2]}}{2}, \\
\alpha_4 &= \frac{(t+2)\eta_+ 22 - \sqrt{[(t+2)\eta_2 + 2]^2 - 4[2(t+2)\eta_2 - 2(t+2)^2]}}{2},
\end{aligned}
$$

and

$$
\alpha_5 = \frac{(t+2)\eta_n + 2 - \sqrt{[(t+2)\eta_n + 2]^2 - 4[2(t+2)\eta_n - 2(t+2)^2]}}{2}.
$$

The calculation of energy considers only positive eigenvalues. For the graph under consideration, the positivity of eigenvalues depends upon the value $((t+2)\eta_1+2)-\sqrt{[(t+2)\eta_1+2]^2-4[2(t+2)\eta_1-2(t+2)^2]}$. This gives rise to the following two possibilities:

$$((r+2)\eta_i+2) < \sqrt{[(t+2)\eta_i+2]^2-4[2(t+2)\eta_i-2(t+2)^2]} \quad \text{if}\, \eta_i \leq 3$$

$$((t+2)\eta_i+2) \geq \sqrt{[(t+2)\eta_i+2]^2-4[2(t+2)\eta_i-2(t+2)^2]} \quad \text{if}\, \eta_i > 3$$

Here,

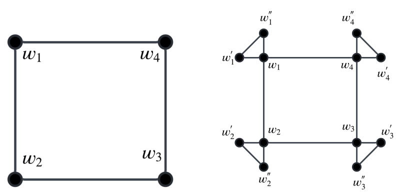
$$\begin{aligned} E_M(G_1) &= \sum_{i=1}^{3n} |\eta_i| \\ &= \sum_{i=1}^{n} |-2| + \sum_{i=1}^{n} \left| \frac{((t+2)\eta_i+2)+\sqrt{[(t+2)\eta_i+2]^2-4[2(t+2)\eta_i-2(t+2)^2]}}{2} \right| \\ &\quad + \sum_{i=1}^{n} \left| \frac{((t+2)\eta_i+2)-\sqrt{[(t+2)\eta_i+2]^2-4[2(t+2)\eta_i-2(t+2)^2]}}{2} \right| \end{aligned}$$

$$\begin{aligned} E_M(G_1) &= \sum_{i=1}^{n} |-2| + \sum_{\eta_i \leq 3} \left| \frac{((t+2)\eta_i+2)+\sqrt{[(t+2)\eta_i+2]^2-4[2(t+2)\eta_i-2(t+2)^2]}}{2} \right. \\ &\quad \left. + \frac{\sqrt{[(t+2)\eta_i+2]^2-4[2(t+2)\eta_i-2(t+2)^2]}-((t+2)\eta_i+2)}{2} \right| \\ &\quad + \sum_{\eta_i > 3} \left| \frac{((t+2)\eta_i+2)+\sqrt{[(t+2)\eta_i+2]^2-4[2(t+2)\eta_i-2(t+2)^2]}}{2} \right. \\ &\quad \left. + \frac{((t+2)\eta_i+2)-\sqrt{[(t+2)\eta_i+2]^2-4[2(t+2)\eta_i-2(t+2)^2]}((t+2)\eta_i+2)}{2} \right| \\ &= 2n + \sum_{\eta_i \leq 3} \sqrt{[(t+2)\eta_i+2]^2-4[2(t+2)\eta_i-2(t+2)^2]} + \sum_{\eta_i > 3} [(t+2)\eta_i+2)]. \end{aligned}$$

This concludes the theorem's proof □

Illustration 3.1. Let us consider cycle C_4 and a graph (let's say G_1) produced by duplicating every vertex in each edge of C_4.

$$Spec(C_4) = \begin{pmatrix} -2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$



(a) C_4 (b) G_1

Figure 1: Vertex Duplication of C_4

$$M(G_1) = \begin{pmatrix} & w_1 & w_2 & w_3 & w_4 & w_1' & w_1'' & w_2' & w_2'' & w_3' & w_3'' & w_4' & w_4'' \\ w_1 & 0 & 4 & 0 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_2 & 4 & 0 & 4 & 0 & 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \\ w_3 & 0 & 4 & 0 & 4 & 0 & 0 & 0 & 0 & 4 & 4 & 0 & 0 \\ w_4 & 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 \\ w_1' & 4 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_1'' & 4 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_2' & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ w_2'' & 0 & 4 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ w_3' & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ w_3'' & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ w_4' & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ w_4'' & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

$$Spec(M(G_1)) = \begin{pmatrix} -10.5489 & -4.7446 & -2 & -1.4031 & 4.5498 & 6.7446 & 11.4031 \\ 1 & 2 & 4 & 1 & 1 & 2 & 1 \end{pmatrix}.$$

Table 1:

spectrum of C_4	**spectrum of $M(G') = \frac{(t+2)\eta_i+2\pm\sqrt{[(t+2)\eta_i+2]^2-4[2(t+2)\eta_i-2(t+2)^2]}}{2}$**
$\eta_1 = -2$	$\sqrt{228}$
$\eta_2 = 2$	$\sqrt{164}$
$\eta_3 = 0$	$\sqrt{132}, \sqrt{132}$

$$E_M(G_1) = 58.8842.$$

3.2. Minimum Degree Energy of Vertex Duplication of Regular Graph

In terms of the adjacency matrix's eigenvalues, the minimal degree energy of vertex duplication of a regular graph G is presented in this section. Vertex duplication of C_4 is given as an example.

Theorem 3.2. *The minimum degree energy $E_n(G_1)$ of the vertex-duplicated graph G_1 of a given t-regular graph G, where ζ_i are the eigenvalues of $A(G)$, is given as:*

$$E_m(G_1) = 2n + \sum_{\zeta_i<1} \sqrt{[(t+2)\zeta_i + 2]^2 - 4[2(t+2)\zeta_i - -8]} + \sum_{\zeta_i \geq 1} [(t+2)\zeta_i + 2)].$$

Proof. Let G be a simple, undirected, and connected t-regular graph with n vertices, and m edges. Then, the adjacency matrix $A(G)$ and incidence matrix $B(G)$ are given by

$$A(G) = \begin{bmatrix} & \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \cdots & \mathbf{w}_n \\ \mathbf{w}_1 & 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ \mathbf{w}_2 & a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ \mathbf{w}_3 & a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{w}_n & a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix},$$

$$B(G) = \begin{bmatrix} & \mathbf{w}_1' & \mathbf{w}_1'' & \mathbf{w}_2' & \mathbf{w}_2'' & \cdots & \mathbf{w}_n' & \mathbf{w}_n'' \\ \mathbf{w}_1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \mathbf{w}_2 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{w}_n & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}.$$

We duplicate the vertices $w_1, w_2, \ldots, w_n$ all together by the edges $e_1, e_2, \ldots, e_n$, respectively, such that:

$$e_1 = w_1'w_1'', \quad e_2 = w_2'w_2'', \quad \ldots, \quad e_n = w_n'w_n''$$

to obtain graph G_1. The minimum degree matrix of G_1 is given in terms of block matrices as follows:

$$m(G_1) = \begin{bmatrix} & w_1 & w_2 & \cdots & w_n & w_1' & w_1'' & w_2' & w_2'' & \cdots & w_n' & w_n'' \\ w_1 & 0 & (t+2)a_{12} & \cdots & (t+2)a_{1n} & 2 & 2 & 0 & 0 & \cdots & 0 & 0 \\ w_2 & (t+2)a_{21} & 0 & \cdots & (t+2)a_{2n} & 0 & 0 & 2 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_n & (t+2)a_{n1} & (t+2)a_{n2} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 2 \\ w_1' & 2 & 0 & \cdots & 0 & 0 & 2 & 0 & 0 & \cdots & 0 & 0 \\ w_1'' & 2 & 0 & \cdots & 0 & 2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ w_2' & 0 & 2 & \cdots & 0 & 0 & 0 & 0 & 2 & \cdots & 0 & 0 \\ w_2'' & 0 & 2 & \cdots & 0 & 0 & 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_n' & 0 & 0 & \cdots & 2 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 \\ w_n'' & 0 & 0 & \cdots & 2 & 0 & 0 & 0 & 0 & \cdots & 2 & 0 \end{bmatrix}$$

That is

$$m(G_1) = \begin{bmatrix} (t+2)A(G) & 2B(G) \\ 2B^T(G) & I_n \otimes 2A(K_2) \end{bmatrix},$$

Calculating the characteristic polynomial $\phi(G_1, z)$ is the first step towards determining the eigenvalues of $m(G_1)$

$$\phi(G_1, z) = \det(zI_{3n} - m(G_1)).$$

Applying the formula for the block matrix determinant, we have:

$$\phi(G_1, z) = \begin{vmatrix} zI_n - (t+2)A(G) & 2B(G) \\ 2B^T(G) & I_n \otimes (zI_2 - 2A(K_2)) \end{vmatrix}.$$

By taking into account the block matrix's structure, this can be made simpler:

$$\begin{aligned} \phi(G_1, z) &= \left| I_n \otimes (zI_2 - 2A(K_2)) \right| \left| zI_n - (t+2)A(G) - 2B(G)(I_n \otimes zI_2 - 2A(K_2))^{-1} 2B^T(G) \right| \\ &= (z^2-4)^n \left| zI_n - (t+2)A(G) - \frac{4B(G)(zI_2 + 2A(K_2) \otimes I_n)B^T(G)}{z^2-4} \right| \\ &= \left| (z^2-4)(zI_n - (t+2)A(G)) - 4B(G)(zI_2 + 2A(K_2) \otimes I_n)B^T(G) \right|. \end{aligned}$$

Now,

$$B\left((zI_2 + 2A(K_2)) \otimes I_n\right) B^T = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} z & 2 & 0 & 0 & \cdots & 0 & 0 \\ 2 & z & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & z & 2 & \cdots & 0 & 0 \\ 0 & 0 & 2 & z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & z & 2 \\ 0 & 0 & 0 & 0 & \cdots & 2 & z \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} z+2 & z+2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & z+2 & z+2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & z+2 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2z+4 & 0 & 0 & \cdots & 0 \\ 0 & 2z+4 & 0 & \cdots & 0 \\ 0 & 0 & 2z+4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2z+4 \end{pmatrix}$$
$$=(2z+4)I_n.$$

Continuing proof of theorem

$$\begin{aligned} \phi(G_1, z) &= \left|(z^2-4)(zI_n-(t+2)A(G)) - 4B(G)(zI_2+2A(K_2)\otimes I_n)B^T(G)\right| \\ &= \left|(z^2-4)(zI_n-(t+2)A(G)) - 4(2z+4)I_n\right|. \end{aligned}$$

Given that ζ_i, where $i = 1, 2, ..., n$, is the eigenvalue of $A(G)$

$$\begin{aligned} \phi(G_1, z) = &\prod_{i=1}^{n} \left[(z^2-4)(z-(t+2)\zeta_i) - 4(2z+4)\right] \\ =&(z+2)^n \prod_{i=1}^{n} [(z-2)(z-(t+2)\zeta_i) - 8] \\ =&(z+2)^n \prod_{i=1}^{n} [(z-2)(z-(t+2)\zeta_i) - 8 \\ =&(z+2)^n \prod_{i=1}^{n} [(z^2 - z(t+2)\zeta_i - 2z + 2(t+2)\zeta_i - 8]. \end{aligned}$$

The characteristic polynomial mentioned above has the roots

$$z = -2(n--times), \quad z = \frac{(t+2)\zeta_i + 2 \pm \sqrt{[(t+2)\zeta_i+2]^2 - 4[2(t+2)\zeta_i - 8]}}{2}.$$

For each $i = 1, 2, ..., n$.

$$Spec(m(G_1)) = \begin{pmatrix} -2 & \alpha_1... & \alpha_2 \\ n & 1 & 1 \\ \alpha_3 & \alpha_4... & \alpha_5 \\ 1 & 1 & 1 \end{pmatrix}.$$

where

$$\begin{aligned} \alpha_1 &= \frac{(t+2)\zeta_1 + 2 + \sqrt{[(t+2)\zeta_1+2]^2 - 4[2(t+2)\zeta_1 - 8]}}{2}, \\ \alpha_2 &= \frac{(t+2)\zeta_n + 2 + \sqrt{[(t+2)\zeta_n+2]^2 - 4[2(t+2)\zeta_n - 8]}}{2}, \\ \alpha_3 &= \frac{(t+2)\zeta_1 + 2 - \sqrt{[(t+2)\zeta_1+2]^2 - 4[2(t+2)\zeta_1 - 8]}}{2}, \\ \alpha_4 &= \frac{(t+2)\zeta_2 + 2 - \sqrt{[(t+2)\zeta_2+2]^2 - 4[2(t+2)\zeta_2 - 8]}}{2}, \end{aligned}$$

and

$$\alpha_5 = \frac{(t+2)\zeta_n + 2 - \sqrt{[(t+2)\zeta_n+2]^2 - 4[2(t+2)\zeta_n - 8]}}{2}.$$

Energy is calculated with only positive eigenvalues taken into account. The positivity of the eigenvalues for the graph in question is dependent upon the value $((t+2)\zeta_i+2)-\sqrt{[(t+2)\zeta_i+2]^2-4[2(t+2)\zeta_i-8]}$. This gives rise to the following two possibilities:

$$((r+2)\zeta_i+2)\geq\sqrt{[(t+2)\zeta_i+2]^2-4[2(t+2)\zeta_i-8]}\quad \text{if}\,\zeta_i\geq 1,$$

$$((t+2)\zeta_i+2)<\sqrt{[(t+2)\zeta_i+2]^2-4[2(t+2)\zeta_i-8]}\quad \text{if}\,\zeta_i< 1.$$

Here,

$$\begin{aligned}E_m(G_1)&=\sum_{i=1}^{3n}|\zeta_i|\\&=\sum_{i=1}^{n}|-2|+\sum_{i=1}^{n}\left|\frac{((t+2)\zeta_i+2)+\sqrt{[(t+2)\zeta_i+2]^2-4[2(t+2)\zeta_i-8]}}{2}\right|\\&\quad+\sum_{i=1}^{n}\left|\frac{((t+2)\zeta_i+2)-\sqrt{[(t+2)\zeta_i+2]^2-4[2(t+2)\zeta_i-8]}}{2}\right|\end{aligned}$$

$$\begin{aligned}E_m(G_1)&=\sum_{i=1}^{n}|-2|+\sum_{\zeta_i<1}^{n}\left|\frac{((t+2)\zeta_i+2)+\sqrt{[(t+2)\zeta_i+2]^2-4[2(t+2)\zeta_i-8]}}{2}+\frac{\sqrt{[(t+2)\zeta_i+2]^2-4[2(t+2)\zeta_i-8]}-((t+2)\zeta_i+2)}{2}\right|\\&\quad+\sum_{\zeta_i\geq 1}^{n}\left|\frac{((t+2)\zeta_i+2)+\sqrt{[(t+2)\zeta_i+2]^2-4[2(t+2)\zeta_i-2(t+2)^2]}}{2}\right.\\&\quad\left.+\frac{((t+2)\zeta_i+2)-\sqrt{[(t+2)\zeta_i+2]^2-4[2(t+2)\zeta_i-2(t+2)^2]}((t+2)\zeta_i+2)}{2}\right|\\&=2n+\sum_{\zeta_i<1}\sqrt{[(t+2)\zeta_i+2]^2-4[2(t+2)\zeta_i-8]}+\sum_{\zeta_i\geq 1}[(t+2)\zeta_i+2)].\end{aligned}$$

The theorem's proof is now complete. □

Illustration 3.2. Examine cycle C_4 and a graph (let's say G_1) created by duplicating every vertex by edge in C_4, as illustrated in Figure 1.

$$Spec(C_4)=\begin{pmatrix}-2&0&2\\1&2&1\end{pmatrix}$$

$$m(G_1)=\begin{pmatrix} & w_1 & w_2 & w_3 & w_4 & w_1' & w_1'' & w_2' & w_2'' & w_3' & w_3'' & w_4' & w_4''\\ w_1 & 0&4&0&4&2&2&0&0&0&0&0&0\\ w_2 & 4&0&4&0&0&0&2&2&0&0&0&0\\ w_3 & 0&4&0&4&0&0&0&0&2&2&0&0\\ w_4 & 4&0&4&0&0&0&0&0&0&0&2&2\\ w_1' & 2&0&0&0&0&2&0&0&0&0&0&0\\ w_1'' & 2&0&0&0&2&0&0&0&0&0&0&0\\ w_2' & 0&2&0&0&0&0&0&2&0&0&0&0\\ w_2'' & 0&2&0&0&0&0&2&0&0&0&0&0\\ w_3' & 0&0&2&0&0&0&0&0&0&2&0&0\\ w_3'' & 0&0&2&0&0&0&0&0&2&0&0&0\\ w_4' & 0&0&0&2&0&0&0&0&0&0&0&2\\ w_4'' & 0&0&0&2&0&0&0&0&0&0&2&0\end{pmatrix}$$

$$Spec(m(G_1)) = \begin{pmatrix} -8.7446 & -2 & 0.8769 & 2.7446 & 4 & 9.1231 \\ 1 & 6 & 1 & 1 & 2 & 1 \end{pmatrix}.$$
$$E_m(G_1) = 41.4891.$$

Table 2:

spectrum of C_4	**spectrum of** $m(G_1) = \frac{(t+2)\zeta_i+2\pm\sqrt{[(t+2)\zeta_i+2]^2-4[2(t+2)\zeta_i-8]}}{2}$
$\zeta_1 = -2$	$-8.7446, 2.7446$
$\zeta_2 = 0$	$-2, 4$
$\zeta_3 = 2$	$0.8769, 9.1231$

3.3. First Zagreb Energy of Vertex Duplication of Regular Graph

The first Zagreb energy of vertex duplication of a regular graph G is presented in this section using the eigenvalues of the graph's adjacency matrix. An example of vertex duplication of C_4 is provided for understanding.

Theorem 3.3. *Given a t-regular graph G, its vertex-duplicated graph G_1, where η_i are the eigenvalues of $A(G)$, the first Zagreb energy $ZE_1(G_1)$ is given as:*

$$ZE_1(G_1) = 4n + \sum_{\eta_i \leq 2} \sqrt{[(2t+4)\eta_i + 4]^2 - 8[2(2t+4)\eta_i - (t+4)^2]} + \sum_{\eta_i \geq 3} [(2t+4)\eta_i + 4].$$

Proof. Let G be a simple, undirected, and connected t-regular graph with n vertices, and m edges. Then, the adjacency matrix $A(G)$ and incidence matrix $B(G)$ are given by

$$A(G) = \begin{bmatrix} & \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \cdots & \mathbf{w}_n \\ \mathbf{w}_1 & 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ \mathbf{w}_2 & a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ \mathbf{w}_3 & a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{w}_n & a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix},$$

$$B(G) = \begin{bmatrix} & \mathbf{w}'_1 & \mathbf{w}''_1 & \mathbf{w}'_2 & \mathbf{w}''_2 & \cdots & \mathbf{w}'_n & \mathbf{w}''_n \\ \mathbf{w}_1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \mathbf{w}_2 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{w}_n & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}.$$

We duplicate the vertices $w_1, w_2, \ldots, w_n$ all together by the edges $e_1, e_2, \ldots, e_n$, respectively, such that:

$$e_1 = w'_1 w''_1, \quad e_2 = w'_2 w''_2, \quad \ldots, \quad e_n = w'_n w''_n$$

to obtain graph G_1. n terms of block matrices, the first Zagreb matrix of G_1 is provided as follows:

$$Z^1(G_1) = \begin{bmatrix} w_1 & w_2 & \cdots & w_n & w'_1 & w''_1 & w'_2 & w''_2 & \cdots & w'_n & w''_n & \\ w_1 & 0 & (2t+4)a_{12} & \cdots & (2t+4)a_{1n} & (t+4) & (t+4) & 0 & 0 & \cdots & 0 & 0 \\ w_2 & (2t+4)a_{21} & 0 & \cdots & (2t+4)a_{2n} & 0 & 0 & (t+4) & (t+4) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_n & (2t+4)a_{n1} & (2t+4)a_{n2} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & (t+4) & (t+4) \\ w'_1 & (t+4) & 0 & \cdots & 0 & 0 & 4 & 0 & 0 & \cdots & 0 & 0 \\ w''_1 & (t+4) & 0 & \cdots & 0 & 4 & 0 & 0 & 0 & \cdots & 0 & 0 \\ w'_2 & 0 & (t+4) & \cdots & 0 & 0 & 0 & 0 & 4 & \cdots & 0 & 0 \\ w''_2 & 0 & (t+4) & \cdots & 0 & 0 & 0 & 4 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w'_n & 0 & 0 & \cdots & (t+4) & 0 & 0 & 0 & 0 & \cdots & 0 & 4 \\ w''_n & 0 & 0 & \cdots & (t+4) & 0 & 0 & 0 & 0 & \cdots & 4 & 0 \end{bmatrix}$$

That is

$$Z^1(G_1) = \begin{bmatrix} (2t+4)A(G) & (t+4)B(G) \\ (t+4)B^T(G) & I_n \otimes 4A(K_2) \end{bmatrix},$$

The characteristic polynomial $\phi(G_1, z)$ is first computed in order to obtain the eigenvalues of $Z^1(G_1)$:

$$\phi(G_1, z) = \det(zI_{3n} - Z^1(G_1)).$$

Applying the formula for the block matrix determinant, we have:

$$\phi(G_1, z) = \begin{vmatrix} zI_n - (2t+4)A(G) & (t+4)B(G) \\ (t+4)B^T(G) & I_n \otimes (zI_2 - 4A(K_2)) \end{vmatrix}.$$

Taking into account the block matrix's structure will simplify this:

$$\begin{aligned}
\phi(G_1, z) =& \left|I_n \otimes (zI_2 - 4A(K_2))\right| \left|zI_n - (2t+4)A(G) - (t+4)B(G)(I_n \otimes zI_2 - 4A(K_2))^{-1}(t+4)B^T(G)\right| \\
=&(z^2-16)^n \left|zI_n - (2t+4)A(G) - \tfrac{(t+4)^2 B(G)(zI_2+4A(K_2)\otimes I_n)B^T(G)}{z^2-16}\right| \\
=& \left|(z^2-16)(zI_n - (2t+4)A(G)) - (t+4)^2 B(G)(zI_2 + 4A(K_2) \otimes I_n)B^T(G)\right|.
\end{aligned}$$

Now,

$$B\left((zI_2 + 4A(K_2)) \otimes I_n\right) B^T = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} z & 4 & 0 & 0 & \cdots & 0 & 0 \\ 4 & z & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & z & 4 & \cdots & 0 & 0 \\ 0 & 0 & 4 & z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & z & 4 \\ 0 & 0 & 0 & 0 & \cdots & 4 & z \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} z+4 & z+4 & 0 & 0 & \cdots & 0 \\ 0 & 0 & z+4 & z+4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & z+4 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2z+8 & 0 & 0 & \cdots & 0 \\ 0 & 2z+8 & 0 & \cdots & 0 \\ 0 & 0 & 2z+8 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2z+8 \end{pmatrix}$$

$$= (2z+8)I_n.$$

Continuing proof of theorem

$$\begin{aligned}
\phi(G_1, z) &= \left|(z^2-16)(zI_n - (2t+4)A(G)) - (t+4)^2 B(G)(zI_2 + 4A(K_2) \otimes I_n)B^T(G)\right| \\
&= \left|(z^2-16)(zI_n - (2t+4)A(G)) - (t+4)^2(2z+8)I_n\right|.
\end{aligned}$$

Given that the eigenvalues of $A(G)$ are η_i, where $i = 1, 2, ..., n$

$$\phi(G_1, z) = \prod_{i=1}^{n} \left[(z^2-16)(z - (2t+4)\eta_i) - (t+4)^2(2z+8)\right]$$

$$
\begin{aligned}
&= (z+4)^n \prod_{i=1}^{n} \left[(z-4)(z-(2t+4)\eta_i) - 2(t+4)^2\right] \\
&= (z+4)^n \prod_{i=1}^{n} [(z-4)(z-(2t+4)\eta_i) - 2(t+4)^2] \\
&= (z+4)^n \prod_{i=1}^{n} [z^2 - z(2t+4)\eta_i - 4z + 4(2t+4)\eta_i - 2(t+4)^2].
\end{aligned}
$$

The characteristic polynomial mentioned above has the following roots

$$z = -4(n--times), \quad z = \frac{(2t+4)\eta_i + 4 \pm \sqrt{[(2t+4)\eta_i+4]^2 - 8[2(2t+4)\eta_i - (t+4)^2]}}{2}.$$

For each $i = 1, 2, ..., n$.

$$Spec(m(G_1)) = \begin{pmatrix} -4 & \alpha_1 ... & \alpha_2 \\ n & 1 & 1 \\ \alpha_3 & \alpha_4 ... & \alpha_5 \\ 1 & 1 & 1 \end{pmatrix}.$$

where

$$
\begin{aligned}
\alpha_1 &= \frac{(2t+4)\eta_1 + 4 + \sqrt{[(2t+4)\eta_1+4]^2 - 8[2(2t+4)\eta_1 - (t+4)^2]}}{2}, \\
\alpha_2 &= \frac{(2t+4)\eta_n + 4 + \sqrt{[(2t+4)\eta_n+4]^2 - 8[2(2t+4)\eta_n - (t+4)^2]}}{2}, \\
\alpha_3 &= \frac{(2t+4)\eta_1 + 4 - \sqrt{[(2t+4)\eta_1+4]^2 - 8[2(2t+4)\eta_1 - (t+4)^2]}}{2}, \\
\alpha_4 &= \frac{(2t+4)\eta_2 + 4 - \sqrt{[(2t+4)\eta_2+4]^2 - 8[2(2t+4)\eta_2 - (t+4)^2]}}{2},
\end{aligned}
$$

and

$$\alpha_5 = \frac{(2t+4)\eta_2 + 4 - \sqrt{[(2t+4)\eta_2+4]^2 - 8[2(2t+4)\eta_2 - (t+4)^2]}}{2}.$$

Energy is calculated with only positive eigenvalues taken into account. The positivity of the eigenvalues for the graph in question is dependent upon the value $[(2t+4)\eta_i+4] - \sqrt{[(2t+4)\eta_i+4]^2 - 8[2(2t+4)\eta_i - (t+4)^2]}$. This gives rise to the following two possibilities:

$$[(2t+4)\eta_i + 4] < \sqrt{[(2t+4)\eta_i+4]^2 - 8[2(2t+4)\eta_i - (t+4)^2]} \quad \text{if } \eta_i \leq 2.$$

$$[(2t+4)\eta_i + 4] > \sqrt{[(2t+4)\eta_i+4]^2 - 8[2(2t+4)\eta_i - (t+4)^2]} \quad \text{if } \eta_i \geq 3.$$

Here,

$$
\begin{aligned}
ZE_1(G_1) &= \sum_{i=1}^{3n} |\eta_i| \\
&= \sum_{i=1}^{n} |-4| + \sum_{i=1}^{n} \left| \frac{[(2t+4)\eta_i+4] + \sqrt{[(2t+4)\eta_i+4]^2 - 8[2(2t+4)\eta_i - (t+4)^2]}}{2} \right| \\
&\quad + \sum_{i=1}^{n} \left| \frac{[(2t+4)\eta_i+4] - \sqrt{[(2t+4)\eta_i+4]^2 - 8[2(2t+4)\eta_i - (t+4)^2]}}{2} \right|
\end{aligned}
$$

$$\begin{aligned} ZE_1(G_1) &= \sum_{i=1}^{n} |-4| \\ &+ \sum_{\eta_i \le 2}^{n} \left| \frac{[(2t+4)\eta_i+4]+\sqrt{[(2t+4)\eta_i+4]^2-8[2(2t+4)\eta_i-(t+4)^2]}}{2} + \frac{\sqrt{[(2t+4)\eta_i+4]^2-8[2(2t+4)\eta_i-(t+4)^2]}-[(2t+4)\eta_i+4]}{2} \right| \\ &+ \sum_{\eta_i \ge 3}^{n} \left| \frac{[(2t+4)\eta_i+4]+\sqrt{[(2t+4)\eta_i+4]^2-8[2(2t+4)\eta_i-(t+4)^2]}}{2} + \frac{[(2t+4)\eta_i+4]-\sqrt{[(2t+4)\eta_i+4]^2-8[2(2t+4)\eta_i-(t+4)^2]}}{2} \right| \\ &= 4n + \sum_{\eta_i \le 2} \sqrt{[(2t+4)\eta_i+4]^2-8[2(2t+4)\eta_i-(t+4)^2]} + \sum_{\eta_i \ge 3} [(2t+4)\eta_i+4]. \end{aligned}$$

Now the theorem is fully proved. □

Illustration 3.3. Let us examine cycle C_4 and a graph (let's say G_1) that is derived from C_4 by duplicating every vertex by edge, as illustrated in Figure 1.

$$Spec(C_4) = \begin{pmatrix} -2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$Z^1(G_1) = \begin{pmatrix} & w_1 & w_2 & w_3 & w_4 & w_1' & w_1'' & w_2' & w_2'' & w_3' & w_3'' & w_4' & w_4'' \\ w_1 & 0 & 8 & 0 & 8 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_2 & 8 & 0 & 8 & 0 & 0 & 0 & 6 & 6 & 0 & 0 & 0 & 0 \\ w_3 & 0 & 8 & 0 & 8 & 0 & 0 & 0 & 0 & 6 & 6 & 0 & 0 \\ w_4 & 8 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 6 \\ w_1' & 6 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_1'' & 6 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_2' & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ w_2'' & 0 & 6 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ w_3' & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ w_3'' & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ w_4' & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ w_4'' & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \end{pmatrix}$$

$$Spec(Z^1(G_1)) = \begin{pmatrix} -19.1149 & -6.7178 & -4 & -0.3923 & 7.1149 & 10.7178 & 20.3923 \\ 1 & 2 & 4 & 1 & 1 & 2 & 1 \end{pmatrix}.$$

$$ZE_1(G_1) = 97.8856.$$

Table 3:

spectrum of C_4	**spectrum of $Z^1(G_1) = \frac{(2t+4)\eta_i+4\pm\sqrt{[(2t+4)\eta_i+4]^2-8[2(2t+4)\eta_i-(t+4)^2]}}{2}$**
$\eta_1 = -2$	$-7.1149, -19.1149$
$\eta_2 = 0$	$-0.3923, 20.3923$
$\eta_3 = 2$	$-6.7178, 10.7178$

3.4. Second Zagreb Energy of Vertex Duplication of Regular Graph

The second Zagreb energy of vertex duplication of a regular graph G is presented in this section using the eigenvalues of the graph's adjacency matrix. An example of vertex duplication of C_4 is provided for understanding.

Theorem 3.4. *The second Zagreb energy $ZE_2(G_1)$ of the vertex-duplicated graph G_1 of a given t-regular graph G, where ζ_i are the eigenvalues of $A(G)$, is given as:*

$$ZE_2(G_1) = 4n + \sum_{\eta_i<2} \sqrt{[(t+2)^2\eta_i + 4]^2 - 8[2(t+2)^2\zeta_i - (2t+4)^2]} + \sum_{\zeta_i \geq 2} [(t+2)^2\zeta_i + 4].$$

Proof. Let G be a connected, simple, undirected t-regular graph with n vertices and m edges. Subsequently, the incidence matrix $B(G)$ and adjacency matrix $A(G)$ are obtained by

$$A(G) = \begin{bmatrix} & \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \cdots & \mathbf{w}_n \\ \mathbf{w}_1 & 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ \mathbf{w}_2 & a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ \mathbf{w}_3 & a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{w}_n & a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix},$$

$$B(G) = \begin{bmatrix} & \mathbf{w}'_1 & \mathbf{w}''_1 & \mathbf{w}'_2 & \mathbf{w}''_2 & \cdots & \mathbf{w}'_n & \mathbf{w}''_n \\ \mathbf{w}_1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \mathbf{w}_2 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{w}_n & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}.$$

We duplicate the vertices $w_1, w_2, \ldots, w_n$ all together by the edges $e_1, e_2, \ldots, e_n$, respectively, such that:

$$e_1 = w'_1 w''_1, \quad e_2 = w'_2 w''_2, \quad \ldots, \quad e_n = w'_n w''_n$$

to obtain graph G_1. In terms of block matrices, the second Zagreb matrix of G_1 is provided as follows:

$$Z^2(G_1) = \begin{bmatrix} w_1 & w_2 & \cdots & w_n & w'_1 & w''_1 & w'_2 & w''_2 & \cdots & w'_n & w''_n & \\ w_1 & 0 & (t+2)^2 a_{12} & \cdots & (t+2)^2 a_{1n} & (2t+4) & (2t+4) & 0 & 0 & \cdots & 0 & 0 \\ w_2 & (t+2)^2 a_{21} & 0 & \cdots & (t+2)^2 a_{2n} & 0 & 0 & (2t+4) & (2t+4) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_n & (t+2)^2 a_{n1} & (t+2)^2 a_{n2} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & (2t+4) & (2t+4) \\ w'_1 & (2t+4) & 0 & \cdots & 0 & 0 & 4 & 0 & 0 & \cdots & 0 & 0 \\ w''_1 & (2t+4) & 0 & \cdots & 0 & 4 & 0 & 0 & 0 & \cdots & 0 & 0 \\ w'_2 & 0 & (2t+4) & \cdots & 0 & 0 & 0 & 0 & 4 & \cdots & 0 & 0 \\ w''_2 & 0 & (2t+4) & \cdots & 0 & 0 & 0 & 4 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w'_n & 0 & 0 & \cdots & (2t+4) & 0 & 0 & 0 & 0 & \cdots & 0 & 4 \\ w''_n & 0 & 0 & \cdots & (2t+4) & 0 & 0 & 0 & 0 & \cdots & 4 & 0 \end{bmatrix}$$

That is

$$Z^2(G_1) = \begin{bmatrix} (t+2)^2 A(G) & (2t+4)B(G) \\ (2t+4)B^T(G) & I_n \otimes 4A(K_2) \end{bmatrix},$$

The characteristic polynomial $\phi(G_1, z)$ is first computed in order to obtain the eigenvalues of $Z^2(G_1)$. This is provided by:

$$\phi(G_1, z) = \det(zI_{3n} - Z^2(G_1)).$$

Using the block matrix determinant formula, we have:

$$\phi(G_1, z) = \begin{vmatrix} zI_n - (t+2)^2 A(G) & (2t+4)B(G) \\ (2t+4)B^T(G) & I_n \otimes (zI_2 - 4A(K_2)) \end{vmatrix}.$$

Taking into account the block matrix's structure will simplify this:

$$\begin{aligned}\phi(G_1,z) =& \left|I_n\otimes(zI_2-4A(K_2))\right|\left|zI_n-(t+2)^2A(G)-(2t+4)B(G)(I_n\otimes zI_2-4A(K_2))^{-1}(2t+4)B^T(G)\right|\\ =&(z^2-16)^n\left|zI_n-(t+2)^2A(G)-\tfrac{(2t+4)^2B(G)(zI_2+4A(K_2)\otimes I_n)B^T(G)}{z^2-16}\right|\\ =&\left|(z^2-16)(zI_n-(t+2)^2A(G))-(2t+4)^2B(G)(zI_2+4A(K_2)\otimes I_n)B^T(G)\right|.\end{aligned}$$

Now,

$$B\left((zI_2+4A(K_2))\otimes I_n\right)B^T=\begin{pmatrix}1&1&0&0&\cdots&0&0\\0&1&1&0&\cdots&0&0\\0&0&1&1&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&0&\cdots&1&1\\0&0&0&0&\cdots&1&1\end{pmatrix}\begin{pmatrix}z&4&0&0&\cdots&0&0\\4&z&0&0&\cdots&0&0\\0&0&z&4&\cdots&0&0\\0&0&4&z&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&0&\cdots&z&4\\0&0&0&0&\cdots&4&z\end{pmatrix}\begin{pmatrix}1&0&\cdots&0\\1&0&\cdots&0\\0&1&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\\0&0&\cdots&1\end{pmatrix}$$

$$=\begin{pmatrix}z+4&z+4&0&0&\cdots&0\\0&0&z+4&z+4&\cdots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&\cdots&z+4\end{pmatrix}\begin{pmatrix}1&0&\cdots&0\\1&0&\cdots&0\\0&1&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\\0&0&\cdots&1\end{pmatrix}$$

$$=\begin{pmatrix}2z+8&0&0&\cdots&0\\0&2z+8&0&\cdots&0\\0&0&2z+8&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&2z+8\end{pmatrix}$$

$$=(2z+8)I_n.$$

Continuing the theorem's proof

$$\begin{aligned}\phi(G_1,z)&=\left|(z^2-16)(zI_n-(t+2)^2A(G))-(t+4)^2B(G)(zI_2+4A(K_2)\otimes I_n)B^T(G)\right|\\&=\left|(z^2-16)(zI_n-(t+2)^2A(G))-(2t+4)^2(2z+8)I_n\right|.\end{aligned}$$

Given that the eigenvalue of $A(G)$ is ζ_i, where $i=1,2,...,n$

$$\begin{aligned}\phi(G_1,z)&=\prod_{i=1}^{n}\left[(z^2-16)(z-(t+2)^2\zeta_i)-(2t+4)^2(2z+8)\right]\\&=(z+4)^n\prod_{i=1}^{n}\left[(z-4)(z-(t+2)^2\zeta_i)-2(2t+4)^2\right]\\&=(z+4)^n\prod_{i=1}^{n}[(z-4)(z-(t+2)^2\zeta_i)-2(2t+4)^2]\\&=(z+4)^n\prod_{i=1}^{n}[(z^2-z(t+2)^2\zeta_i-4z+4(t+2)^2\zeta_i-2(2t+4)^2)].\end{aligned}$$

The roots of the previously mentioned characteristic polynomial are as follows

$$z=-4(n--times),\quad z=\frac{(t+2)^2\zeta_i+4\pm\sqrt{[(t+2)^2\zeta_i+4]^2-8[2(t+2)^2\zeta_i-(2t+4)^2]}}{2}$$

For each $i = 1, 2, ..., n$.

$$Spec(m(G_1)) = \begin{pmatrix} -4 & \alpha_1... & \alpha_2 \\ n & 1 & 1 \\ \alpha_3 & \alpha_4... & \alpha_5 \\ 1 & 1 & 1 \end{pmatrix},$$

where

$$\alpha_1 = \frac{(t+2)^2\zeta_1 + 4 + \sqrt{[(t+2)^2\zeta_1 + 4]^2 - 8[2(t+2)^2\zeta_1 - (2t+4)^2]}}{2},$$
$$\alpha_2 = \frac{(t+2)^2\zeta_n + 4 + \sqrt{[(t+2)^2\zeta_n + 4]^2 - 8[2(t+2)^2\zeta_n - (2t+4)^2]}}{2},$$
$$\alpha_3 = \frac{(t+2)^2\zeta_1 + 4 - \sqrt{[(t+2)^2\zeta_1 + 4]^2 - 8[2(t+2)^2\zeta_1 - (2t+4)^2]}}{2},$$
$$\alpha_4 = \frac{(t+2)^2\zeta_2 + 4 - \sqrt{[(t+2)^2\zeta_2 + 4]^2 - 8[2(t+2)^2\zeta_2 - (2t+4)^2]}}{2},$$

and

$$\alpha_5 = \frac{(t+2)^2\zeta_n + 4 - \sqrt{[(t+2)^2\zeta_n + 4]^2 - 8[2(t+2)^2\zeta_n - (2t+4)^2]}}{2}.$$

Only positive eigenvalues are taken into account when calculating energy. The value of the given graph determines the positivity of the eigenvalues. $(t+2)^2\zeta_1 + 4 - \sqrt{[(t+2)^2\zeta_1 + 4]^2 - 8[2(t+2)^2\zeta_1 - (2t+4)^2]}$. This gives rise to the following two possibilities:

$$[(t+2)^2\zeta_1 + 4] < \sqrt{[(t+2)^2\zeta_1 + 4]^2 - 8[2(t+2)^2\zeta_1 - (2t+4)^2]} \quad \text{if } \zeta_i < 2.$$

$$[(t+2)^2\zeta_1 + 4] \geq \sqrt{[(t+2)^2\zeta_1 + 4]^2 - 8[2(t+2)^2\zeta_1 - (2t+4)^2]} \quad \text{if } \zeta_i \geq 2.$$

Here,

$$\begin{aligned} ZE_2(G_1) &= \sum_{i=1}^{3n} |\zeta_i| \\ &= \sum_{i=1}^{n} |-4| + \sum_{i=1}^{n} \left| \frac{[(t+2)^2\zeta_i + 4] + \sqrt{[(t+2)^2\zeta_i + 4]^2 - 8[2(t+2)^2\zeta_i - (2t+4)^2]}}{2} \right| \\ &+ \sum_{i=1}^{n} \left| \frac{[(t+2)^2\zeta_i + 4] - \sqrt{[(t+2)^2\zeta_i + 4]^2 - 8[2(t+2)^2\zeta_i - (2t+4)^2]}}{2} \right| \end{aligned}$$

$$\begin{aligned} ZE_2(G_1) &= \sum_{i=1}^{n} |-4| \\ &+ \sum_{\zeta_i<2}^{n} \left| \frac{[(t+2)^2\zeta_i+4] + \sqrt{[(t+2)^2\zeta_i+4]^2 - 8[2(t+2)^2\zeta_i - (2t+4)^2]}}{2} + \frac{\sqrt{[(t+2)^2\zeta_i+4]^2 - 8[2(t+2)^2\zeta_i - (2t+4)^2]} - [(t+2)^2\zeta_1 + 4}{2} \right| \\ &+ \sum_{\zeta_i\geq 2}^{n} \left| \frac{[(t+2)^2\zeta_i+4] + \sqrt{[(t+2)^2\zeta_i+4]^2 - 8[2(t+2)^2\zeta_i - (2t+4)^2]}}{2} + \frac{[(t+2)^2\zeta_i+4] - \sqrt{[(t+2)^2\zeta_i+4]^2 - 8[2(t+2)^2\zeta_i - (2t+4)^2]}}{2} \right| \end{aligned}$$

$$ZE_2(G_1) = 4n + \sum_{\zeta_i<2} \sqrt{[(t+2)^2\zeta_i + 4]^2 - 8[2(t+2)^2\zeta_i - (2t+4)^2]} + \sum_{\zeta_i\geq 2} [(t+2)^2\zeta_i + 4].$$

This concludes the theorem's proof. □

Illustration 3.4. Let us examine cycle C_4 and a graph (let's say G_1) that is derived from C_4 by duplicating every vertex by edge, as illustrated in figure 1.

$$Spec(C_4) = \begin{pmatrix} -2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$Z^2(G_1) = \begin{pmatrix} & w_1 & w_2 & w_3 & w_4 & w_1' & w_1'' & w_2' & w_2'' & w_3' & w_3'' & w_4' & w_4'' \\ w_1 & 0 & 16 & 0 & 16 & 8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_2 & 16 & 0 & 16 & 0 & 0 & 0 & 8 & 8 & 0 & 0 & 0 & 0 \\ w_3 & 0 & 16 & 0 & 16 & 0 & 0 & 0 & 0 & 8 & 8 & 0 & 0 \\ w_4 & 16 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 \\ w_1' & 8 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_1'' & 8 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_2' & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ w_2'' & 0 & 8 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ w_3' & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ w_3'' & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ w_4' & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ w_4'' & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \end{pmatrix}$$

$$Spec(Z^2(G_1)) = \begin{pmatrix} -35.2603 & -9.4891 & -4 & 0 & 7.2603 & 13.4891 & 36 \\ 1 & 2 & 4 & 1 & 1 & 2 & 1 \end{pmatrix}.$$

$$ZE_2(G_1) = 140.4771.$$

Table 4:

spectrum of C_4	**spectrum of** $Z^2(G_1) = \frac{(t+2)^2\zeta_i+4\pm\sqrt{[(t+2)^2\zeta_i+4]^2-8[2(t+2)^2\zeta_i-(2t+4)^2]}}{2}$
$\zeta_1 = -2$	$-35.2603, 7.2603$
$\zeta_2 = 0$	$-9.4891, 13.4891$
$\zeta_3 = 2$	$0, 36$

3.5. Degree Square Sum Energy of Vertex Duplication of Regular Graph

In terms of the adjacency matrix eigenvalues of the regular graph G, this section defines the degree square sum energy of vertex duplication of a regular graph G. As an example, a particular case of vertex duplication of C_4 is shown.

Theorem 3.5. *The degree square sum energy $E_{DSS}(G_1)$ of the vertex-duplicated graph G_1, for a given t-regular graph G, where η_i are the eigenvalues of $A(G)$, is given as:*

$$E_{DSS}(G_1) = 8n + \sum_{\eta_i \le 3} \sqrt{[(t+2)^2\eta_i + 4]^2 - 16(t+2)^2\eta_i + 2[(t+2)^2 + 4]^2} + \sum_{\eta_i \ge 4} [(t+2)^2\eta_i + 4].$$

Proof. A simple, connected, undirected t-regular graph with n vertices and m edges is denoted by G. Next, given by are the incidence matrix $B(G)$ and adjacency matrix $A(G)$

$$A(G) = \begin{bmatrix} & \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \cdots & \mathbf{w}_n \\ \mathbf{w}_1 & 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ \mathbf{w}_2 & a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ \mathbf{w}_3 & a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{w}_n & a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix},$$

$$B(G)=\begin{bmatrix} & \mathbf{w}'_1 & \mathbf{w}''_1 & \mathbf{w}'_2 & \mathbf{w}''_2 & \cdots & \mathbf{w}'_n & \mathbf{w}''_n \\ \mathbf{w}_1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \mathbf{w}_2 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{w}_n & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}.$$

We duplicate the vertices $w_1, w_2, \dots, w_n$ all together by the edges $e_1, e_2, \dots, e_n$, respectiwely, such that:

$$e_1 = w'_1 w''_1, \quad e_2 = w'_2 w''_2, \quad \dots, \quad e_n = w'_n w''_n$$

to obtain graph G_1. The degree square sum matrix of G_1 is given in terms of block matrices as follows:

$$DSS(G_1)=\begin{bmatrix} & w_1 & w_2 & \cdots & w_n & w'_1 & w''_1 & w'_2 & w''_2 & \cdots & w'_n & w''_n \\ w_1 & 0 & 2(t+2)^2 a_{12} & \cdots & 2(t+2)^2 a_{1n} & (t+2)^2 & (t+2)^2 & 0 & 0 & \cdots & 0 & 0 \\ w_2 & 2(t+2)^2 a_{21} & 0 & \cdots & 2(t+2)^2 a_{2n} & 0 & 0 & (t+2)^2 & (t+2)^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_n & 2(t+2)^2 a_{n1} & 2(t+2)^2 a_{n2} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & (t+2)^2 & (t+2)^2 \\ w'_1 & (t+2)^2 & 0 & \cdots & 0 & 0 & 8 & 0 & 0 & \cdots & 0 & 0 \\ w''_1 & (t+2)^2 & 0 & \cdots & 0 & 8 & 0 & 0 & 0 & \cdots & 0 & 0 \\ w'_2 & 0 & (t+2)^2 & \cdots & 0 & 0 & 0 & 0 & 8 & \cdots & 0 & 0 \\ w''_2 & 0 & (t+2)^2 & \cdots & 0 & 0 & 0 & 8 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w'_n & 0 & 0 & \cdots & (t+2)^2 & 0 & 0 & 0 & 0 & \cdots & 0 & 8 \\ w''_n & 0 & 0 & \cdots & (t+2)^2 & 0 & 0 & 0 & 0 & \cdots & 8 & 0 \end{bmatrix}$$

That is

$$DSS(G_1)=\begin{bmatrix} 2(t+2)^2 A(G) & [(t+2)^2+4]B(G) \\ [(t+2)^2+4]B^T(G) & I_n \otimes 8A(K_2) \end{bmatrix},$$

The characteristic polynomial $\phi(G_1, z)$ is first computed in order to obtain the eigenvalues of $DSS(G_1)$. This is provided by:

$$\phi(G_1, z) = \det(zI_{3n} - DSS(G_1)).$$

Applying the formula for the block matrix determinant, we have:

$$\phi(G_1, z) = \begin{vmatrix} zI_n - 2(t+2)^2 A(G) & [(t+2)^2+4]B(G) \\ [(t+2)^2+4]B^T(G) & I_n \otimes (zI_2 - 8A(K_2)) \end{vmatrix}.$$

Taking into account the block matrix's structure helps simplify this:

$$\begin{aligned}\phi(G_1, z) &= \left|I_n \otimes (zI_2 - 8A(K_2))\right| \left|zI_n - 2(t+2)^2 A(G) - [(t+2)^2+4]^2 B(G)(I_n \otimes zI_2 - 8A(K_2))^{-1} B^T(G)\right| \\ &= (z^2-64)^n \left|zI_n - 2(t+2)^2 A(G) - \tfrac{[(t+2)+4]^2 B(G)(zI_2 + 8A(K_2)\otimes I_n)B^T(G)}{z^2-64}\right| \\ &= \left|(z^2-64)(zI_n - 2(t+2)^2 A(G)) - [(t+2)^2+4]^2 B(G)(zI_2 + 8A(K_2)\otimes I_n)B^T(G)\right|.\end{aligned}$$

Now,

$$B\left((zI_2 + 8A(K_2)) \otimes I_n\right) B^T = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} z & 8 & 0 & 0 & \cdots & 0 & 0 \\ 8 & z & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & z & 8 & \cdots & 0 & 0 \\ 0 & 0 & 8 & z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & z & 8 \\ 0 & 0 & 0 & 0 & \cdots & 8 & z \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} z+8 & z+8 & 0 & 0 & \cdots & 0 \\ 0 & 0 & z+8 & z+8 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & z+8 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2z+16 & 0 & 0 & \cdots & 0 \\ 0 & 2z+16 & 0 & \cdots & 0 \\ 0 & 0 & 2z+16 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2z+16 \end{pmatrix}$$
$$= (2z+16)I_n.$$

Continuing proof of theorem

$$\begin{aligned}\phi(G_1, z) &= \left|(z^2-64)(zI_n - 2(t+2)^2A(G)) - [(t+2)^2+4]^2B(G)(zI_2 + 8A(K_2)\otimes I_n)B^T(G)\right| \\ &= \left|(z^2-64)(zI_n - 2(t+2)^2A(G)) - [(t+2)^2+4]^2(2z+16)I_n\right|.\end{aligned}$$

Given that η_i, where $i = 1, 2, ..., n$, is the eigenvalue of $A(G)$

$$\begin{aligned}\phi(G_1, z) &= \prod_{i=1}^{n}\left[(z^2-64)(z-2(t+2)^2\eta_i) - [(t+2)^2+4]^2(2z+16)\right] \\ &= (z+8)^n\prod_{i=1}^{n}\left[(z-8)(z-2(t+2)^2\eta_i) - 2[(t+2)^2+4]^2\right] \\ &= (z+8)^n\prod_{i=1}^{n}[(z-8)(z-2(t+2)^2\eta_i) - 2[(t+2)^2+4]^2 \\ &= (z+8)^n\prod_{i=1}^{n}[(z^2-2z(t+2)^2\eta_i - 8z + 16(t+2)^2\eta_i - 2[(t+2)^2+4]^2].\end{aligned}$$

The characteristic polynomial above has the following roots

$$z = -8(n--times), \quad z = (t+2)^2\eta_i + 4 \pm \sqrt{[(t+2)^2\eta_i+4]^2 - 16(t+2)^2\eta_i + 2[(t+2)^2+4]^2}$$

For each $i = 1, 2, ..., n$.

$$Spec(m(G_1)) = \begin{pmatrix} -8 & \alpha_1... & \alpha_2 \\ n & 1 & 1 \\ \alpha_3 & \alpha_4... & \alpha_5 \\ 1 & 1 & 1 \end{pmatrix},$$

where

$$\begin{aligned}\alpha_1 &= (t+2)^2\eta_1 + 4 + \sqrt{[(t+2)^2\eta_1+4]^2 - 16(t+2)^2\eta_1 + 2[(t+2)^2+4]^2}, \\ \alpha_2 &= (t+2)^2\eta_n + 4 + \sqrt{[(t+2)^2\eta_n+4]^2 - 16(t+2)^2\eta_n + 2[(t+2)^2+4]^2}, \\ \alpha_3 &= (t+2)^2\eta_1 + 4 - \sqrt{[(t+2)^2\eta_1+4]^2 - 16(t+2)^2\eta_1 + 2[(t+2)^2+4]^2}, \\ \alpha_4 &= (t+2)^2\eta_2 + 4 - \sqrt{[(t+2)^2\eta_2+4]^2 - 16(t+2)^2\eta_2 + 2[(t+2)^2+4]^2},\end{aligned}$$

and

$$\alpha_5 = (t+2)^2\eta_n + 4 - \sqrt{[(t+2)^2\eta_n+4]^2 - 16(t+2)^2\eta_n + 2[(t+2)^2+4]^2}.$$

Only positive eigenvalues are used in the calculation of energy. The value of the eigenvalues for the given graph determines their positivity. $[(t+2)^2\eta_i+4] - \sqrt{[(t+2)^2\eta_i+4]^2 - 16(t+2)^2\eta_i + 2[(t+2)^2+4]^2}$. This gives rise to the following two possibilities:

$$[(t+2)^2\eta_i+4] < \sqrt{[(t+2)^2\eta_i+4]^2 - 16(t+2)^2\eta_i + 2[(t+2)^2+4]^2} \quad \text{if}\, \eta_i \leq 3.$$

$$[(t+2)^2\eta_i + 4] > \sqrt{[(t+2)^2\eta_i + 4]^2 - 16(t+2)^2\eta_i + 2[(t+2)^2+4]^2} \quad \text{if } \eta_i \geq 4.$$

Here,

$$\begin{aligned} E_{DSS}(G_1) &= \sum_{i=1}^{3n} |\eta_i| \\ &= \sum_{i=1}^{n} |-8| + \sum_{i=1}^{n} \left| [(t+2)^2\eta_i + 4] + \sqrt{[(t+2)^2\eta_i + 4]^2 - 16(t+2)^2\eta_i + 2[(t+2)^2+4]^2} \right| \\ &+ \sum_{i=1}^{n} \left| [(t+2)^2\eta_i + 4] - \sqrt{[(t+2)^2\eta_i + 4]^2 - 16(t+2)^2\eta_i + 2[(t+2)^2+4]^2} \right| \end{aligned}$$

$$\begin{aligned} E_{DSS}(G_1) =& \sum_{i=1}^{n} |-8| \\ &+ \sum_{\eta_i \leq 3}^{n} \Big| (t+2)^2\eta_i + 4 + \sqrt{[(t+2)^2\eta_i + 4]^2 - 16(t+2)^2\eta_i + 2[(t+2)^2+4]^2} \\ &+ \sqrt{[(t+2)^2\eta_i + 4]^2 - 16(t+2)^2\eta_i + 2[(t+2)^2+4]^2} - [(t+2)^2\eta_1 + 4] \Big| \\ &+ \sum_{\eta_i \geq 4}^{n} \Big| [(t+2)^2\eta_i + 4] + \sqrt{[(t+2)^2\eta_i + 4]^2 - 16(t+2)^2\eta_i + 2[(t+2)^2+4]^2} + [(t+2)^2\eta_i + 4] \\ &- \sqrt{[(t+2)^2\eta_i + 4]^2 - 16(t+2)^2\eta_i + 2[(t+2)^2+4]^2} \Big| \end{aligned}$$

$$E_{DSS}(G_1) = 8n + \sum_{\eta_i \leq 3} \sqrt{[(t+2)^2\eta_i + 4]^2 - 16(t+2)^2\eta_i + 2[(t+2)^2+4]^2} + \sum_{\eta_i \geq 4} [(t+2)^2\eta_i + 4].$$

Now the theorem is fully proved. □

Illustration 3.5. Let us examine cycle C_4 and a graph (let's say G_1) that is derived from C_4 by duplicating every vertex by edge, as illustrated in figure 1.

$$Spec(C_4) = \begin{pmatrix} -2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$DSS(G_1) = \begin{pmatrix} & w_1 & w_2 & w_3 & w_4 & w_1' & w_1'' & w_2' & w_2'' & w_3' & w_3'' & w_4' & w_4'' \\ w_1 & 0 & 32 & 0 & 32 & 20 & 20 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_2 & 32 & 0 & 32 & 0 & 0 & 0 & 20 & 20 & 0 & 0 & 0 & 0 \\ w_3 & 0 & 32 & 0 & 32 & 0 & 0 & 0 & 0 & 20 & 20 & 0 & 0 \\ w_4 & 32 & 0 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 20 & 20 \\ w_1' & 20 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_1'' & 20 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w_2' & 0 & 20 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\ w_2'' & 0 & 20 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\ w_3' & 0 & 0 & 20 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ w_3'' & 0 & 0 & 20 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\ w_4' & 0 & 0 & 0 & 20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ w_4'' & 0 & 0 & 0 & 20 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \end{pmatrix}$$

$$Spec(DSS(G_1)) = \begin{pmatrix} -73.7821 & -24.5657 & -8 & -3.7995 & 17.7821 & 32.5657 & 75.7995 \\ 1 & 2 & 4 & 1 & 1 & 2 & 1 \end{pmatrix}.$$

$$E_{DSS}(G_1) = 371.4260.$$

Table 5:

spectrum of C_4	**spectrum of** $DSS(G_1) = (t+2)^2\eta_i + 4 \pm \sqrt{[(t+2)^2\eta_i+4]^2 - 16(t+2)^2\eta_i + 2[(t+2)^2+4]^2}$
$\eta_1 = -2$	$-73.7821, 17.7821$
$\eta_2 = 2$	$-3.7995, 75.7995$
$\eta_3 = 0$	$-24.5657, 32.5657$

4. Application

In social networks, ties or interactions are represented by edges, and individuals are represented by vertices. The graph energy can give insights into the overall connectedness of the network, the influence of certain nodes (people), and the resilience of the network [23]. Higher graph energy might correspond to more highly connected and active networks [24]. In biology, [25] metabolic and protein interaction networks can be analyzed using graph energy. These graphs help to understand how different proteins or metabolites interact with each other and how perturbations (like a disease) affect the overall network's function. Lower energy in such graphs could imply robustness, while higher energy might indicate fragility or dysfunction, useful in analyzing network behavior in diseases, see also [26, 27, 28]. The spectrum of a network representing financial markets or trade networks can be used to study market stability and resilience. Large eigenvalues may indicate fragility, while smaller eigenvalues can signal a stable and robust system. Vertex duplication has applications in various areas of network theory, such as network robustness, redundancy analysis, and molecular graph theory [29]. For example, in communication networks, duplicating key nodes can increase redundancy, making the network more resilient to failures.

5. Conclusion

In this work, we have obtained a generalized equation for the energy following vertex duplication in a regular graph. The findings deepen our understanding of the energy implications of such graph operations and lay the groundwork for future research into various other kinds of graph alterations.

References

[1] Gutman, I. (1978). The energy of a graph. *Statist. Sekt. Forschungsz. Graz.* 1, 2

[2] Akram, M., and Naz, S. (2018). Energy of Pythagorean fuzzy graphs with applications. *Mathematics, 6*(8), 136. MDPI. 1

[3] Singh, R., and Patekar, S. C. (2022). On the Sombor index and Sombor energy of m-splitting graph and m-shadow graph of regular graphs. *arXiv preprint arXiv:2205.09480.* 1

[4] Shwetha, B. C., Sriraj, M. A., Roopa, S. V., and Veena, C. R. (2024). Partition energy of m-splitting graph and their generalized complements. *Proyecciones (Antofagasta, Online), 43*(5), 1113–1139. 1

[5] Gutman, I., and Furtula, B. (2019). Graph energies and their applications. *Bulletin of the Serbian Academy of Sciences and Arts: Mathematical Sciences, 44*, 29–45. JSTOR. 1

[6] Chu, Z.-Q., Nazeer, S., Zia, T. J., Ahmed, I., and Shahid, S. (2019). Some new results on various graph energies of the splitting graph. *Journal of Chemistry, 2019*(1), 7214047. Wiley Online Library. 1

[7] Vaidya, S. K., and Popat, K. M. (2017). Energy of m-splitting and m-shadow graphs. *Far East Journal of Mathematical Sciences, 102*(8), 1571–1578. 1

[8] Kousar, I., Nazeer, S., Mahboob, A., Shahid, S., and Lv, Y.-P. (2021). Numerous graph energies of regular subdivision graph and complete graph. *AIMS Mathematics, 6*(8), 8466–8476. AIMS Press. 1

[9] Rao, K. S., Saravanan, K., Prakasha, K. N., and Cangul, I. N. (2022). Maximum and minimum degree energies of p-splitting and p-shadow graphs. *TWMS Journal of Applied and Engineering Mathematics, 12*(1), 1. Elman Hasanoglu. 1

[10] Chithra, A. V. (2021). Energy and Randić energy of special graphs. *arXiv preprint arXiv:2105.07195.* 1

[11] Agrahari, G., and Froncek, D. (2024). On Some Classes of Cycles-Related γ-Harmonious Graphs. *Utilitas Mathematica, 120*, 75–91. 1

[12] Saurabh, S. (2024). Key Pre-distribution Schemes via Certain Resolvable Block Designs. *Utilitas Mathematica, 120*, 19–25. 1

[13] Kathiresan, K. M., and Jeyagermani, S. (2024). Multiplicative Distance-Location Number of Graphs. *Utilitas Mathematica, 120*, 27–36. 1

[14] Bondy, J. A. (1976). *Graph theory with applications.* North-Holland. 2

[15] Gutman, I. (2001). The energy of a graph: Old and new results. In *Algebraic Combinatorics and Applications* (pp. 196–211). Springer. 2

[16] Vaidya, S. K., and Popat, K. M. (2020). L-spectra of graphs obtained by duplicating graph elements. In *Recent Advancements in Graph Theory* (pp. 381–390). CRC Press. 2

[17] Adiga, C., and Smitha, M. (2009). On maximum degree energy of a graph. *International Journal of Contemporary Mathematics and Sciences, 4*(8), 385–396. 2

[18] Basavanagoud, B., and Jakkannavar, P. (2019). Minimum degree energy of graphs. *Electronic Journal of Mathematical Analysis and Applications, 7*(2), 230–243. 2

[19] Rad, N. J., Jahanbani, A., and Gutman, I. (2018). Zagreb energy and Zagreb Estrada index of graphs. *MATCH-Communications in Mathematical and Computer Chemistry, 79*(2018). MATCH-Communications. 2

[20] Basavanagoud, B., and Chitra, E. (2018). Degree square sum energy of graphs. *International Journal of Mathematics and Its Applications, 6*(2-B), 193–205. 2

[21] Horn, R. A., and Johnson, C. R. (1991). Topics in matrix analysis. Cambridge University Press. 2.1, 2.2

[22] Zhang, F. (2006). *The Schur complement and its applications* (Vol. 4). Springer Science and Business Media. 2.2

[23] Hubert, L. J. (1974). Some applications of graph theory to clustering. *Psychometrika, 39*(3), 283–309. Springer. 4

[24] Singh, R. P. (2014). Application of graph theory in computer science and engineering. *International Journal of Computer Applications, 104*(1). Citeseer. 4

[25] Pavlopoulos, G. A., Secrier, M., Moschopoulos, C. N., Soldatos, T. G., Kossida, S., Aerts, J., Schneider, R., and Bagos, P. G. (2011). Using graph theory to analyze biological networks. *BioData Mining, 4*, 1–27. Springer. 4

[26] Adams, P., Bunge, R. C., Dulowski, J., El-Zanati, S. I., Kenney, M., Odabası, U., and Zale, K. (2024). The λ-fold spectrum problem for the orientations of the 6-cycle. *Journal of Combinatorial Mathematics and Combinatorial Computing, 122*, 361–370. 4

[27] Sun, X., and Du, J. (2024). The multiplicative sum Zagreb indices of graphs with given clique number. *Journal of Combinatorial Mathematics and Combinatorial Computing, 122*, 343–350. 4

[28] Hutson, K. R., and Hedetniemi, S. T. (2024). Efficient domination in grid graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing, 122*, 325–342. 4

[29] Balaban, A. T. (1985). Applications of graph theory in chemistry. *Journal of Chemical Information and Computer Sciences, 25*(3), 334–343. ACS Publications. 4