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Deterministic and stochastic approaches for a fat receptor-breast cancer model with crossover effects

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Abstract

In this paper, the dynamics of a fat receptor-breast cancer model have been investigated by employing the deterministic and stochastic approaches. The existence of the endemic equilibrium, positivity of solutions and the calculation of the reproduction number are examined for the deterministic model and also the existence-uniqueness of the stochastic model is discussed. Then, we will examine the crossover tendencies of the deterministic-stochastic model with the help of piecewise differential operators that take into account stochastic and power law processes followed by generalized Mittag-Leffler functions have been investigated. We employ a numerical scheme based on Newton polynomial to solve the deterministic-stochastic tumor growth model with fractional differential operators numerically. The graphical representations are simulated for different values of fractional order and the crossover tendencies of the deterministic-stochastic model are observed during the simulations.

Keywords: Breast cancer model, numerical scheme, piecewise differential operators.

1. Introduction

Tumor, which occurs as a result of abnormal growth of some tissues due to excessive division of cells in the body, is one of the diseases that cause death worldwide and seriously threatens human life. When the literature is examined, chemotherapy, radiotherapy, gene therapy and virotherapy can be found among the treatment approaches. Treatment should be considered as a whole and the dose of the drug to be used and the time should be determined well. For this reason, cancer disease has attracted great attention not only in branches of science such as medicine and biology, but also in the field of mathematics. Mathematical models represent complex processes with different aspects and methods. Since some real world problems can have random behaviors, many models have been examined by using the stochastic approaches [1-8]. However,

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mathematical modeling, which is one of the important tools that helps in especially in interpreting the treatment process made significant contributions in analyzing tumor growth [9-16].

In the realm of cancer research, diverse mathematical models have emerged, each shedding light on distinct facets of tumor behavior and treatment responses. One study by McKenna M.T et al. extensively analyzes doxorubicin uptake through a parameterized model using experimental imaging data, showcasing variations in pharmacokinetic parameters among investigated cell lines. This work not only describes drug kinetics but also effectively predicts population dynamics following treatment [9].

Another model [10] dissects the complexities within treated tumors, delineating subsets with differing immune signaling and growth rates. It foresees scenarios of dormancy, elimination, and immune evasion, emphasizing the potential of dormant cells in controlling aggressive counterparts through immune responses. Furthermore, Mehdizadeh et. al. propose a set of differential equations probes interactions among cancer stem cells, tumor cells, and healthy cells, elucidating the impact of estrogen surplus and the immune response. This model showcases how immune cells contribute to cancer elimination while underlining the continuous existence of cancer in their absence [11]. Additionally, studies delve into population-specific models, such as one focused on breast cancer in the Saudi Arabian population, emphasizing the association between chemotherapy and cardiotoxicity. This research aims to mitigate cardiotoxicity while improving treatment outcomes [12].

Another model specifically targets triple negative breast cancer, integrating immunotherapy. It reveals equilibria indicating tumor-free and large-tumor states, highlighting the potential of immune checkpoint inhibitors when combined with immune booster therapy [13]. Moreover, a comprehensive model encompasses the interplay between immune responses, vaccination, and chemotherapy treatments. It emphasizes the synergy between these therapies in eradicating tumors, showcasing their potential when used in combination [14]. Additionally, an innovative approach utilizes piecewise differential operators to amalgamate models representing various stages of tumor growth. This technique offers a holistic view enabling comprehensive analysis and prediction of tumor progression, providing insights into diverse behaviors [15]. Lastly, an optimal control problem centered on a stochastic delay differential model delves into tumor-immune interactions. It identifies variables crucial for controlling tumor growth and showcases the impact of white noises and time delays on tumor dynamics [16].

Fractional differentiation has been incorporated and applied in various problem-solving scenarios as an extension of the classical differentiation concept. This idea has been widely recognized and utilized since before 1965. Their popularity primarily stems from the characteristics of their kernels, which were employed in defining these operators. For example, when the Riemann-Liouville and Caputo derivatives [17] reveal a power-law relationship in certain data, it could suggest specific mechanisms underlying the natural phenomenon in question. Power law phenomena are widespread across diverse fields, providing deep insights into the intricate workings of complex systems. These phenomena involve relationships where changes in one quantity lead to proportional changes in another, playing crucial roles in disciplines spanning physics, biology, economics, and the social sciences. Knowledge of power laws not only reveals fundamental principles governing natural and social phenomena but also uncovers hierarchical structures, scaling characteristics, and complex dynamics that shape our understanding of the world. The Caputo-Fabrizio derivative [18] is a mathematical tool that extends fractional calculus by integrating an exponential kernel. This enhancement enables the modeling of non-local and non-Markovian behaviors in diverse physical and engineering systems. Applications of this derivative include viscoelasticity, anomalous diffusion, and control theory, addressing complexities beyond the capabilities of traditional derivatives. The Atangana-Baleanu derivative [19] is an advanced concept in fractional calculus that introduces a novel kernel to extend classical fractional derivatives. Atangana and Baleanu proposed this derivative to deal with complex systems exhibiting non-local and non-singular behavior, which are commonly encountered in physics, engineering and various scientific fields. Its applications span fields such as viscoelasticity, fluid mechanics, electromagnetic theory and control systems, where conventional derivatives struggle to capture the full range of dynamical phenomena. Widely recognized for its ability to model systems with memory effects, the Atangana-Baleanu derivative has attracted considerable interest and research activity in recent years. There is no doubt that models modified

by the addition of processes with power law [17], fading memory [18] and generalized Mittag-Leffler [19] function present different scenarios. Moreover, it can be observed that the model related to piecewise differential operators[20], which makes it possible to handle deterministic and stochastic processes together, has crossover tendencies. Therefore, in this study, the fat receptor breast cancer model was discussed using deterministic and stochastic approaches [21].

Since it is vital to examine the duration and prevalence of the tumor in a particular region when evaluating the tumor spread model, we address the scenario by introducing the existence of an endemic equilibrium that has proven to be globally asymptotically stable in deterministic models. Then, the relevant model will be modified with piecewise differential operators that take these two processes into account together and will be solved with a numerical method based on Newton polynomials [22].

Now, we present the definitions of the fractional derivatives with power law, exponential decay and Mittag-Leffler kernels and piecewise derivative.

The Caputo-Fabrizio fractional derivative [18] of the function $\omega(t) \in H^1(0,T)$ is given by

$$ {}_0^{CF}D_t^{\alpha}\omega(t) = \frac{1}{1-\alpha}\int_0^t \omega'(\zeta)\exp\left[-\frac{\alpha}{1-\alpha}(t-\zeta)\right]d\zeta, \tag{1.1} $$

where $0 < \alpha < 1$ and $H^1(0,T)$ describes the Hilbert space. The associated integral is defined by

$$ {}_0^{CF}J_t^{\alpha}\omega(t) = (1-\alpha)\omega(t) + \alpha\int_0^t \omega(\zeta)\,d\zeta. \tag{1.2} $$

The Caputo fractional derivative [17] of the function $\omega(t) \in H^1(0,T)$ is defined by

$$ {}_0^{C}D_t^{\alpha}\omega(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \omega'(\zeta)(t-\zeta)^{-\alpha}\,d\zeta, \tag{1.3} $$

where $0 < \alpha \leq 1$ and the Riemann-Liouville fractional derivative of the function $\omega(t) \in C(0,T)$ is defined by

$$ {}_0^{RL}D_t^{\alpha}\omega(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \omega(\zeta)(t-\zeta)^{-\alpha}\,d\zeta. \tag{1.4} $$

The integral with power-law kernel [23]

$$ {}_0^{RL}J_t^{\alpha}\omega(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \omega(\zeta)(t-\zeta)^{\alpha-1}\,d\zeta. \tag{1.5} $$

The following formulas describe the Atangana-Baleanu fractional derivative [19] which has the crossover behavior from stretched exponential to power-law,

$$ {}_0^{ABC}D_t^{\alpha}\omega(t) = \frac{1}{1-\alpha}\int_0^t \omega'(\zeta)E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-\zeta)^{\alpha}\right]d\zeta, \tag{1.6} $$

and

$$ {}_0^{ABR}D_t^{\alpha}\omega(t) = \frac{1}{1-\alpha}\frac{d}{dt}\int_0^t \omega(\zeta)E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-\zeta)^{\alpha}\right]d\zeta. \tag{1.7} $$

The above operators are called as Atangana-Baleanu fractional derivative in Caputo sense and Atangana-Baleanu fractional derivative in Riemann-Liouville sense [19], respectively. The associated integral is given by

$$ {}_0^{AB}J_t^{\alpha}\omega(t) = (1-\alpha)\omega(t) + \frac{\alpha}{\Gamma(\alpha)}\int_0^t \omega(\zeta)(t-\zeta)^{\alpha-1}\,d\zeta. \tag{1.8} $$

We now present the definitions of the piecewise derivative and integral operators which made significant contribution to literature [20].

The piecewise derivative with classical and fractional derivative with power-law kernel such that can be taken as [20]

$$ {}_0^{PRL}D_t^{\alpha}\omega\ (t) = \begin{cases} \omega'(t) \text{ if } 0 \le t \le t_0 \\ {}_{t_0}^{RL}D_t^{\alpha}\omega(t) \text{ if } t_0 \le t \le T \end{cases}, \tag{1.9} $$

where ${}_0^{PRL}D_t^{\alpha}$ represents classical derivative within $0 \le t \le t_0$ and Riemann-Liouville fractional derivative within $t_0 \le t \le T$.

The piecewise with Caputo derivative is given as [20]

$$ {}_0^{PC}D_t^{\alpha}\omega\ (t) = \begin{cases} \omega'(t) \text{ if } 0 \le t \le t_0 \\ {}_{t_0}^{C}D_t^{\alpha}\omega(t) \text{ if } t_0 \le t \le T \end{cases}, \tag{1.10} $$

where the function $\omega(t)$ is differentiable in $[0,T]$. Here ${}_0^{PRL}D_t^{\alpha}$ represents classical derivative on $0 \le t \le t_0$ and Caputo fractional derivative [17] on $t_0 \le t \le T$. The associated piecewise integral of ω is given as [20]

$$ {}^{PPL}I_t\omega\ (t) = \begin{cases} \int_0^t \omega(\zeta)\, d\zeta \text{ if } 0 \le t \le t_0 \\ \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \omega(\zeta)(t-\zeta)^{\alpha-1}\, d\zeta \text{ if } t_0 \le t \le T \end{cases}, \tag{1.11} $$

where ${}_0^{PPL}I_t^{\alpha}$ represents classical integral on $0 \le t \le t_0$ and the integral with power-law kernel on $t_0 \le t \le T$.

The piecewise derivative with classical derivative and Mittag-Leffler kernel is defined by [20]

$$ {}_0^{PAB}D_t^{\alpha}\omega\ (t) = \begin{cases} \omega'(t) \text{ if } 0 \le t \le t_0 \\ {}_{t_0}^{ABC}D_t^{\alpha}\omega(t) \text{ if } t_0 \le t \le T \end{cases}, \tag{1.12} $$

where ${}_0^{PAB}D_t^{\alpha}$ represents classical derivative on $0 \le t \le t_0$ and Atangana-Baleanu fractional derivative [19] on $t_0 \le t \le T$. The associated piecewise integral is given as [20]

$$ {}^{PAB}I_t\omega\ (t) = \begin{cases} \int_0^t \omega(\zeta)\, d\zeta \text{ if } 0 \le t \le t_0 \\ (1-\alpha)\omega(t) + \frac{\alpha}{\Gamma(\alpha)} \int_{t_0}^t \omega(\zeta)(t-\zeta)^{\alpha-1}\, d\zeta \text{ if } t_0 \le t \le T \end{cases}. \tag{1.13} $$

In this section, we will investigate the behavior of a mathematical model that takes into account the effect of fat cells and estrogen on tumor growth[21] under deterministic and stochastic approaches. The deterministic model under investigation is represented by

$$ \begin{cases} \frac{dH}{dt} = H(\kappa_1 - \delta_1 H - \eta T) - \mu_1 HE \\ \frac{dT}{dt} = T(\kappa_2 - \delta_2 T - \phi_1 I) + \mu_2 HE + \varepsilon_1 TF \\ \frac{dI}{dt} = \Lambda + I\left(\frac{qT}{\theta_1 + T} - \frac{\mu_3 E}{\theta_2 + E} - p - \phi_2 T\right) \\ \frac{dE}{dt} = w_1 + \varepsilon_2 EF - w_2 E \\ \frac{dF}{dt} = \kappa_3 F(1 - \varepsilon_3 F) \end{cases}, \tag{1.14} $$

with the initial condition

$$ H(0) = H_0,\ T(0) = T_0,\ I(0) = I_0,\ E(0) = E_0,\ F(0) = F_0. \tag{1.15} $$

$H(t)$, $T(t)$, $I(t)$, $E(t)$ and $F(t)$ describe healthy cells, breast tumor cells, immune cells, estrogen, and fat cells, respectively.

The description of the parameters for the system (1.14) is presented in Table 1.

Table 1: The description of the parameters for the system (1.14)

Parameter	Description of parameters	Parameter value	Reference
κ_1	The growth rate of healthy breast cells	0.7	[21]
δ_1	The death rate of healthy breast cells	0.3	Estimated
η	The rate of suppression of healthy breast cells.	0.1	[21]
μ_1	The rate of increase in the growth factor of estrogen levels	0.0015	[21]
κ_2	The growth rate of tumor cells	0.98	[21]
δ_2	The death rate of tumor breast cells	0.94	[21]
ϕ_1	The rate of the immune response	2.1	Estimated
μ_2	The rate of overflowing estrogen free ONA	0.2	Estimated
ε_1	The growth rate of tumor cells from fat cells	0.5	[21]
Λ	Immune cells constant source	0.4	[21]
p	The natural death rate of immune cells	0.1	[21]
q	The growth rate of the immune response	0.4	Estimated
θ_1	The rate of immune cells	0.9	[21]
μ_3	The rate of immune suppression by estrogen	0.002	Estimated
θ_2	The threshold of estrogen	0.1	[21]
ϕ_2	The rate of tumor cells effect on immune cells	0.05	Estimated
w_1	The production of an increased estrogen rate	0.9	Estimated
ε_2	The contribution of excess estrogen by fat cells	0.2562	[21]
w_2	The rate of estrogen that affects breast cells	0.8788	[21]
κ_3	The rate of fat	0.93	Estimated
ε_3	The inverse rate of fat carrying capacity	0.5	[21]

1.1. Equilibrium Points:

In this subsection, the equilibrium points of system (1.14) are determined by setting the left-hand side of system (1.14) to zero. Considering both the mathematical and biological significance of system (1.14), we identify two types of equilibrium points: tumor-free equilibrium points and coexisting equilibrium points, using parameter value the assumptions presented in [21]. It is concluded that the system (1.14) has eight steady states. The tumor-free equilibrium point is

$$\left(\frac{\kappa_1-\mu_1E^*}{\delta_1},0,\frac{\Lambda\left(\theta_2+E^*\right)}{p\theta_2+\left(\mu_3+p\right)E^*},\frac{\varepsilon_3w_1}{\varepsilon_3w_2-\varepsilon_2},\frac{1}{\varepsilon_3}\right), \tag{1.16}$$

noting that $\varepsilon_3w_2>\varepsilon_2$. The co-existing equilibrium points are

$$\begin{aligned}
\widetilde{H}^* &= \frac{\kappa_1-\eta\widetilde{T}-\mu_1E^*}{\delta_1}, \\
\widetilde{T}^* &= \frac{-\Pi+\sqrt{\Pi^2-4\delta_1\delta_2\mu_2\left(\mu_1E^{*2}-\kappa_1E^*\right)}}{2\delta_1\delta_2}, \\
\widetilde{I}^* &= \frac{\Lambda}{p+\theta_2\widetilde{T}^*-\dfrac{q\widetilde{T}^*}{\theta_1+\widetilde{T}^*}+\dfrac{\mu_3E^*}{\theta_2+E^*}}, \\
\widetilde{E}^* &= \frac{\varepsilon_3w_1}{\varepsilon_3w_2-\varepsilon_2}, \\
\widetilde{F}^* &= \frac{1}{\varepsilon_3}.
\end{aligned} \tag{1.17}$$

1.2. Positivity of Solutions

In this subsection, we prove that the solutions of the system (1.14) are positive if the initial conditions are positive. To achieve our goal, we start our proof with the function $F(t)$

$$\frac{dF}{dt} - \kappa_3 F = \kappa_3 \varepsilon_3 F^2, \tag{1.18}$$

and solving above yields

$$F(t) \geq \frac{1}{\kappa_3 \varepsilon_3 + ce^{-\kappa_3 t}}, \tag{1.19}$$

where

$$c = \frac{1 - F_0 \kappa_3 \varepsilon_3}{F_0}. \tag{1.20}$$

For the function $E(t)$, we can have

$$\begin{aligned} \frac{dE}{dt} &= w_1 + \varepsilon_2 EF - w_2 E \\ &\geq (\varepsilon_2 |F| - w_2) E \\ &\geq \left(\varepsilon_2 \sup_{t\epsilon[0,T]} |F| - w_2 \right) E. \end{aligned} \tag{1.21}$$

To proceed, we define the following norm

$$\|F\|_\infty = \sup_{t\epsilon[0,T]} |F(t)|, \tag{1.22}$$

using the norm presented above yields

$$E(t) \geq E(0) \exp\left((\varepsilon_2 \|F\|_\infty - w_2) t \right), \tag{1.23}$$

where

$$\varepsilon_2 \|F\|_\infty < w_2. \tag{1.24}$$

We proceed with the function $I(t)$

$$\begin{aligned} \frac{dI}{dt} &\geq \left(\frac{q \|T\|_\infty}{\theta_1 + \|T\|_\infty} - \frac{\mu_3 \|E\|_\infty}{\theta_2 + \|E\|_\infty} - p - \phi_1 \|T\|_\infty \right) I \\ &\geq I(0) \exp(\kappa t), \end{aligned} \tag{1.25}$$

where

$$\frac{q \|T\|_\infty}{\theta_1 + \|T\|_\infty} < \frac{\mu_3 \|E\|_\infty}{\theta_2 + \|E\|_\infty} + p + \theta_2 \|T\|_\infty. \tag{1.26}$$

Doing the same routine presented earlier, we obtain

$$\begin{aligned} \frac{dT}{dt} &\geq (\kappa_2 - \phi_1 \|I\|_\infty) T - \delta_2 T^2 \\ &\geq \frac{1}{\delta_2 + c_1 e^{-(\kappa_2 - \phi_1 \|I\|_\infty) t}}, \end{aligned} \tag{1.27}$$

where $\kappa_2 > \phi_1 \|I\|_\infty$ and $c_1 = \dfrac{1 - T_0 \delta_2}{T_0}$.

Finally, the function $H(t)$ is evaluated as

$$\begin{aligned} \frac{dH}{dt} &\geq H(\kappa_1 - \eta \|T\|_\infty - \mu_1 \|E\|_\infty) - \delta_1 H^2 \\ &\geq \frac{1}{\delta_1 + c_2 e^{-(\kappa_1 - \eta \|T\|_\infty - \mu_1 \|E\|_\infty) t}}, \end{aligned} \tag{1.28}$$

where $\kappa_1 > \eta \|T\|_\infty + \mu_1 \|E\|_\infty$ and $c_2 = \dfrac{1 - H_0 \delta_1}{H_0}$.

1.3. Reproduction number for the considered

In this subsection, we evaluate the basic reproduction number for the system (1.14). By employing the advanced method developed by Driessche and Watmough [24], the reproduction number can be found. However, we will find the associated number in an easier way. To perform our aim, we will use the following;

$$\frac{dT}{dt} > 0, \tag{1.29}$$

which yields

$$T\left(k_2 - \delta_2 T - \phi_1 I\right) + \mu_2 HE + \varepsilon_1 TF > 0. \tag{1.30}$$

It is worth noting that the tumor-free equilibrium point exists when the estrogen level is normal, meaning $\mu_2 = 0$. Then, equation (1.30) can be arranged at the tumor-free equilibrium as follows:

$$\begin{aligned} T\kappa_2 - \phi_1 T I^* + \varepsilon_1 T F^* &> 0 \\ T\left(\kappa_2 + \varepsilon_1 F^*\right)\left(1 - \frac{\phi_1 I^*}{\kappa_2 + \varepsilon_1 F^*}\right) &> 0 \\ T\left(\kappa_2 + \varepsilon_1 F^*\right)\left(1 - \frac{1}{R_0}\right) &> 0. \end{aligned} \tag{1.31}$$

Then, the reproduction number can be obtained as:

$$\begin{aligned} R_0 &= \frac{\kappa_2 + \varepsilon_1 F^*}{\phi_1 I^*} \\ &= \frac{\kappa_2 + \varepsilon_1\left(\frac{1}{\varepsilon_3}\right)}{\phi_1\left(\frac{\Lambda\left(\theta_2\left(\varepsilon_3 w_3 - \varepsilon_2\right) + \varepsilon_3 w_1\right)}{p\theta_2\left(\varepsilon_3 w_3 - \varepsilon_2\right) + \left(p + \mu_3\right)\varepsilon_3 w_1}\right)} \\ &= \frac{\left(\kappa_2\varepsilon_3 + \varepsilon_1\right)\left(p\theta_2\left(\varepsilon_3 w_3 - \varepsilon_2\right) + \left(p + \mu_3\right)\varepsilon_3 w_1\right)}{\varepsilon_3\phi_1\Lambda\left(\theta_2\left(\varepsilon_3 w_3 - \varepsilon_2\right) + \varepsilon_3 w_1\right)}, \end{aligned} \tag{1.32}$$

knowing that $\varepsilon_3 w_2 > \varepsilon_2$.

2. Stochastic model for the considered model

In this section, we intend to establish a deep analysis of a fat-estrogen breast cancer model[21] with the addition of a stochastic component. The mathematical model under investigation is presented as follows :

$$\begin{cases} dH = \left[H\left(\kappa_1 - \delta_1 H - \eta T\right) - \mu_1 HE\right]dt + \sigma_1 H dB_1(t) \\ dT = \left[T\left(\kappa_2 - \delta_2 T - \phi_1 I\right) + \mu_2 HE + \varepsilon_1 TF\right]dt + \sigma_2 T dB_2(t) \\ dI = \left[\Lambda + I\left(\frac{qT}{\theta_1 + T} - \frac{\mu_3 E}{\theta_2 + E} - p - \phi_2 T\right)\right]dt + \sigma_3 I dB_3(t) \\ dE = \left[w_1 + \varepsilon_2 EF - w_2 E\right]dt + \sigma_4 E dB_4(t) \\ dF = \left[\kappa_3 F\left(1 - \varepsilon_3 F\right)\right]dt + \sigma_5 F dB_5(t) \end{cases}, \tag{2.1}$$

where $B_i(t)$ and σ_i $(i = 1, ..., 5)$ are Brownian motion and stochastic constants.

We will begin our analysis by providing a theorem that will demonstrate the conditions under which the system (1.14) has a unique solution[6].

Theorem 2.1. *Assuming that there are two positive constants, m_i and $\widetilde{m_i}$, that satisfy the conditions listed below:*

i) *Lipschitz condition :* $\forall\ x, \widetilde{x} \in \mathbb{R}^5, i \in \{1, .., 5\}$

$$max\left\{\left|f_i(x,t) - f_i(\widetilde{x},t)\right|^2,\ \left|g_i(x,t) - g_i(\widetilde{x},t)\right|^2\right\} \leq m_i|x - \widetilde{x}|^2, \tag{2.2}$$

ii) *Linear growth condition :* $\forall\ (x,t) \in \mathbb{R}^5 \times [t_0, T]$

$$max\left\{|f\left(x,t\right)|^2,\ |g\left(x,t\right)|^2\right\} \leq \widetilde{m_i}\left(1+|x|^2\right). \tag{2.3}$$

The initial condition x_0 is a random variable that is independent of the σ_i-algebra $F_{\infty}^{(m)}$ generated by w_i^s, $s \geq 0$ and such that $E|x_0|^2 < \infty$. In this case, the stochastic model under investigation has a unique solution in $M^2\left([t_0,T],\mathbb{R}^5\right)$.

Proof. In order to start our proof, we need to introduce some notations to avoid complexity

$$x = \begin{pmatrix} H \\ T \\ I \\ E \\ F \end{pmatrix},\ f\left(x,t\right) = \begin{pmatrix} f_1\left(t,H\right) \\ f_2\left(t,T\right) \\ f_3\left(t,I\right) \\ f_4\left(t,E\right) \\ f_5\left(t,F\right) \end{pmatrix} = \begin{pmatrix} H\left(\kappa_1 - \delta_1 H - \eta T\right) - \mu_1 HE \\ T\left(\kappa_2 - \delta_2 T - \phi_1 I\right) + \mu_2 HE + \varepsilon_1 TF \\ \Lambda + I\left(\dfrac{qT}{\theta_1+T} - \dfrac{\mu_3 E}{\theta_2+E} - p - \phi_2 T\right) \\ w_1 + \varepsilon_2 EF - w_2 E \\ \kappa_3 F\left(1-\varepsilon_3 F\right) \end{pmatrix}. \tag{2.4}$$

Our objective is to prove that the Lipschitz and linear growth conditions for each function on the right-hand side of the model are satisfied. To achieve this, we will start with the function $f_1\left(t,H\right)$ and then proceed to prove the same for the other functions. We verify the first condition for the function $f_1\left(t,H\right)$ by following:

$$\begin{aligned}
\left|f_1\left(t,H\right) - f_1\left(t,\widetilde{H}\right)\right|^2 &= \left|\left(H-\widetilde{H}\right)\left(\kappa_1 - \delta_1\left(H-\widetilde{H}\right) - \eta T\right) - \mu_1 E\left(H-\widetilde{H}\right)\right|^2 \\
&\leq 2\left|\left(H-\widetilde{H}\right)\right|^2\left|\left(\kappa_1 - \delta_1\left(H-\widetilde{H}\right) - \eta T\right)^2\right| + 2\mu_1^2|E|^2\left|H-\widetilde{H}\right|^2 \\
&\leq 2\left|\left(H-\widetilde{H}\right)\right|^2\left[\left(3\kappa_1^2 + 3\delta_1^2\left|H-\widetilde{H}\right|^2 + 3\eta^2|T|\right) + \mu_1^2|E|^2\right] \\
&\leq 2\left|\left(H-\widetilde{H}\right)\right|^2\left[3\kappa_1^2 + 3\delta_1^2\sup_{t\in[0,t]}\left|H-\widetilde{H}\right|^2 + 3\eta^2\sup_{t\in[0,t]}|T|^2 + \mu_1^2\sup_{t\in[0,t]}|E|^2\right] \\
&\leq 2\left(3\kappa_1^2 + 3\delta_1^2\left\|H-\widetilde{H}\right\|_{\infty}^2 + 3\eta^2\|T\|_{\infty}^2 + \mu_1^2\|E\|_{\infty}^2\right)\left|H-\widetilde{H}\right|^2,
\end{aligned} \tag{2.5}$$

by employing the norm defined before. Thus, we have

$$\left|f_1\left(t,H\right) - f_1\left(t,\widetilde{H}\right)\right|^2 \leq m_1\left|H-\widetilde{H}\right|, \tag{2.6}$$

where

$$m_1 = 2\left(3\kappa_1^2 + 3\delta_1^2\left\|H-\widetilde{H}\right\|_{\infty}^2 + 3\eta^2\|T\|_{\infty}^2 + \mu_1^2\|E\|_{\infty}^2\right). \tag{2.7}$$

We will continue our proof with the second function, $f_2\left(t,T\right)$ Then, we obtain

$$\begin{aligned}
\left|f_2\left(t,T\right) - f_2\left(t,\widetilde{T}\right)\right|^2 &= \left|\left(T-\widetilde{T}\right)\left(\kappa_2 - \delta_2\left(T-\widetilde{T}\right) - \phi_1 I\right) + \varepsilon_1\left(T-\widetilde{T}\right)F\right|^2 \\
&\leq 2\left|T-\widetilde{T}\right|^2\left(3\kappa_2^2 + 3\delta_2^2\left|T-\widetilde{T}\right|^2 + 3\phi_1^2|I|^2\right) + 2\varepsilon_1^2|F|^2\left|T-\widetilde{T}\right|^2 \\
&\leq 2\left|T-\widetilde{T}\right|^2\left(3\kappa_2^2 + 3\delta_2^2\sup_{t\in[0,t]}\left|T-\widetilde{T}\right|^2 + 3\phi_1^2\sup_{t\in[0,t]}|I|^2 + \varepsilon_1^2\sup_{t\in[0,t]}|F|^2\right) \\
&\leq 2\left|T-\widetilde{T}\right|^2\left(3\kappa_2^2 + 3\delta_2^2\left\|T-\widetilde{T}\right\|_{\infty}^2 + 3\phi_1^2\left\|TI-\widetilde{T}\right\|_{\infty}^2 + \varepsilon_1^2\|F\|_{\infty}^2\right) \\
&\leq m_2\left|T-\widetilde{T}\right|^2,
\end{aligned} \tag{2.8}$$

where

$$m_2 = \left(6\kappa_2^2 + 6\delta_2^2 \left\|T - \widetilde{T}\right\|_\infty^2 + 6\phi_1^2 \left\|TI - \widetilde{T}\right\|_\infty^2 + 2\varepsilon_1^2 \|F\|_\infty^2\right). \tag{2.9}$$

Doing same routine for other function, we obtain

$$\begin{aligned}
\left|f_3(t,I) - f_3\left(t,\widetilde{I}\right)\right|^2 &= \left|\frac{qT}{\theta_1+T} - \frac{\mu_3 E}{\theta_2+E} - p - \phi_2 T\right|^2 \left|I-\widetilde{I}\right|^2 \qquad (2.10)\\
&\leq \left|\frac{2q^2 \sup\limits_{t\in[0,t]}|T|^2}{\theta_1^2 + \sup\limits_{t\in[0,t]}|T|^2} + \frac{2\mu_3^2 \sup\limits_{t\in[0,t]}|E|^2}{\theta_2^2 + \sup\limits_{t\in[0,t]}|E|^2} + 4p^2 + 4\phi_2^2 \sup_{t\in[0,t]}|T|^2\right| \left|I-\widetilde{I}\right|^2\\
&\leq \left|\frac{2q^2\|T\|^2}{\theta_1^2+\|T\|^2} + \frac{2\mu_3^2\|E\|^2}{\theta_2^2+\|E\|^2} + 4p^2 + 4\phi_2^2\|T\|^2\right| \left|I-\widetilde{I}\right|^2\\
&\leq m_3 \left|I-\widetilde{I}\right|^2.
\end{aligned}$$

$$\begin{aligned}
\left|f_4(t,E) - f_4\left(t,\widetilde{E}\right)\right|^2 &= \left|\varepsilon_2\left(E-\widetilde{E}\right)F - w_2\left(E-\widetilde{E}\right)\right|^2 \qquad (2.11)\\
&\leq 2\varepsilon_2^2 \sup_{t\in[0,t]}|F|^2 \left|E-\widetilde{E}\right|^2 + 2w_2^2\left|E-\widetilde{E}\right|^2\\
&\leq \left(2\varepsilon_2^2\|F\|_\infty^2 + 2w_2^2\right)\left|E-\widetilde{E}\right|^2\\
&\leq m_4\left|E-\widetilde{E}\right|^2.
\end{aligned}$$

$$\begin{aligned}
\left|f_5(t,F) - f_5\left(t,\widetilde{F}\right)\right|^2 &= \left|\kappa_3 F(1-\varepsilon_3 F)\right|^2 \qquad (2.12)\\
&\leq 2\kappa_3^2\left|F-\widetilde{F}\right|^2\left(1+\varepsilon_3^2 \sup_{t\in[0,t]}\left|F-\widetilde{F}\right|^2\right)\\
&\leq 2\kappa_3^2\left|F-\widetilde{F}\right|^2\left(1+\varepsilon_3^2\left\|F-\widetilde{F}\right\|_\infty^2\right)\\
&\leq m_5\left|F-\widetilde{F}\right|^2.
\end{aligned}$$

For the function $g(x,t)$, we have

$$|g_i(x,t) - g_i(\widetilde{x},t)|^2 \leq \sigma_i |x-\widetilde{x}|^2. \tag{2.13}$$

To proceed with the proof, we will demonstrate that the second condition of the theorem is satisfied[6]. To accomplish our goal, a similar procedure will be pursued as mentioned above. We will start with the function $f_1(t,H)$

$$\begin{aligned}
|f_1(t,H)|^2 &= |H(\kappa_1 - \delta_1 H - \eta T) - \mu_1 E H|^2 \qquad (2.14)\\
&\leq 2|H|^2\left(3\kappa_1^2 + 3\delta_1^2\|H\|_\infty^2 + 3\eta^2\|T\|_\infty^2\right) + 2\mu_1^2\|E\|_\infty^2|H|^2\\
&\leq 2\left(1 + \left(3\kappa_1^2 + 3\delta_1^2\|H\|_\infty^2 + 3\eta^2\|T\|_\infty^2 + \mu_1^2\|E\|_\infty^2\right)|H|^2\right)\\
&\leq \widetilde{m_1}\left(1+|H|^2\right),
\end{aligned}$$

such that $\varrho_1 = \left(3\kappa_1^2 + 3\delta_1^2 \|H\|_\infty^2 + 3\eta^2 \|T\|_\infty^2 + \mu_1^2 \|E\|_\infty^2\right) < 1.$

$$\begin{aligned} |f_2(t,T)|^2 &= |T(\kappa_2 - \delta_2 T - \phi_1 I) + \mu_2 HE + \varepsilon_1 TF|^2 \\ &\leq 3|T|^2\left(3\kappa_2^2 + 3\delta_2^2 \|T\|_\infty^2 + 3\phi_1^2 \|I\|_\infty^2\right) + 3\mu_2^2 \|H\|_\infty^2 \|E\|_\infty^2 + 3\varepsilon_1^2 \|F\|_\infty^2 |T|^2 \\ &\leq 3\mu_2^2 \|H\|_\infty^2 \|E\|_\infty^2 + \left(9\kappa_2^2 + 9\delta_2^2 \|T\|_\infty^2 + 9\phi_1^2 \|I\|_\infty^2 + 3\varepsilon_1^2 \|F\|_\infty^2\right)|T|^2 \\ &\leq 3\mu_2^2 \|H\|_\infty^2 \|E\|_\infty^2 \left(1 + \frac{\left(9\kappa_2^2 + 9\delta_2^2 \|T\|_\infty^2 + 9\phi_1^2 \|I\|_\infty^2 + 3\varepsilon_1^2 \|F\|_\infty^2\right)}{3\mu_2^2 \|H\|_\infty^2 \|E\|_\infty^2}|T|^2\right) \\ &\leq \widetilde{m}_2\left(1 + |T|^2\right), \end{aligned} \tag{2.15}$$

where $\varrho_2 = \dfrac{\left(9\kappa_2^2 + 9\delta_2^2 \|T\|_\infty^2 + 9\phi_1^2 \|I\|_\infty^2 + 3\varepsilon_1^2 \|F\|_\infty^2\right)}{3\mu_2^2 \|H\|_\infty^2 \|E\|_\infty^2} < 1$ and $\widetilde{m}_2 = 3\mu_2^2 \|H\|_\infty^2 \|E\|_\infty^2.$

$$\begin{aligned} |f_3(t,I)|^2 &= \left|\Lambda + I\left(\frac{qT}{\theta_1 + T} - \frac{\mu_3 E}{\theta_2 + E} - p - \phi_2 T\right)\right|^2 \\ &\leq 2\Lambda^2 + |I|^2\left(\frac{2q^2 \|T\|_\infty^2}{\theta_1^2 + \|T\|_\infty^2} + \frac{2\mu_3^2 \|E\|_\infty^2}{\theta_2^2 + \|E\|_\infty^2} + 4p^2 + 4\phi_2^2 \|T\|_\infty^2\right) \\ &\leq 2\Lambda^2\left(1 + \frac{\left(\dfrac{2q^2 \|T\|_\infty^2}{\theta_1^2 + \|T\|_\infty^2} + \dfrac{2\mu_3^2 \|E\|_\infty^2}{\theta_2^2 + \|E\|_\infty^2} + 4p^2 + 4\phi_2^2 \|T\|_\infty^2\right)}{2\Lambda^2}|I|^2\right) \\ &\leq \widetilde{m_3}\left(1 + |I|^2\right), \end{aligned} \tag{2.16}$$

where $\widetilde{m_3} = 2\Lambda^2$ and $\varrho_3 = \left(\dfrac{2q^2 \|T\|_\infty^2}{\theta_1^2 + \|T\|_\infty^2} + \dfrac{2\mu_3 \|E\|_\infty^2}{\theta_2^2 + \|E\|_\infty^2} + 4p^2 + 4\phi_2^2 \|T\|_\infty^2\right) < 1.$

$$\begin{aligned} |f_4(t,E)|^2 &= |w_1 + \varepsilon_2 EF - w_2 E|^2 \\ &\leq 3w_1^2 + 3\varepsilon_2^2 |E|^2 \|F\|_\infty^2 + 3w_2^2 |E|^2 \\ &\leq 3w_1^2\left(1 + \frac{3\varepsilon_2^2 \|F\|_\infty^2 + 3w_2^2}{3w_1^2}|E|^2\right) \\ &\leq \widetilde{m_4}\left(1 + |E|^2\right), \end{aligned} \tag{2.17}$$

noting that $\widetilde{m_4} = 3w_1^2$ and $\varrho_4 = \dfrac{\varepsilon_2 \|F\|_\infty^2 + w_2^2}{w_1^2} < 1,$

$$\begin{aligned} |f_5(t,F)|^2 &= |\kappa_3 F(1 - \varepsilon_3 F)|^2 \\ &\leq 2\kappa_3^2 |F|^2\left(1 + \varepsilon_3^2 \|F\|_\infty^2\right) \\ &\leq 2\left(1 + \varepsilon_3^2 \|F\|_\infty^2\right)\left(1 + \kappa_3^2 |F|^2\right) \\ &\leq \widetilde{m_5}\left(1 + |F|^2\right), \end{aligned} \tag{2.18}$$

where $\widetilde{m_5} = \left(1 + \varepsilon_3^2 \|F\|_\infty^2\right)$ and $\varrho_5 = \kappa_3^2 < 1$. For the function $g(x,t)$, we obtain

$$\begin{aligned} |g_i(x,t)|^2 &= \sigma_i^2 |x|^2 \\ &\leq \sigma_i^2 \left(1 + |x|^2\right) \\ &\leq \overline{m}_i \left(1 + |x|^2\right), \end{aligned} \tag{2.19}$$

where $\overline{m}_i = \sigma_i^2$. The system has a unique solution under the condition

$$\max\{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5\} < 1, \tag{2.20}$$

It is worth noting that the Picard iteration can be employed to demonstrate the existence of the solution model. We will not address this here in order to proceed with further analysis. However, more information about the proof can be found in [6]. □

2.1. Global positive solution of stochastic model

In this subsection, we analyze the global positivity of the stochastic model that takes into account the interactions between tumors, estrogen, fat, and immune cells[21]. To achieve this aim, we will make use of the theorem presented below [1-6].

Theorem 2.2. *Let us consider a stochastic tumor growth model with initial data*

$$\widetilde{T}(0) = (H(0),\ T(0),\ I(0),\ E(0),\ F(0)) \in \mathbb{R}_+^5.$$

Then, there exists a non-negative solution $\widetilde{T}(t) = (H(t),\ T(t),\ I(t),\ E(t),\ F(t))$ *for* $\forall\ t \geq 0$ *such that the solution of the system will remain within* $\mathbb{R}_+^5$ *almost surely.*

Proof. Since the coefficients of the system satisfy the local Lipschitz condition, then the stochastic system has a unique solution $\widetilde{T}(t) \in \mathbb{R}_+^5$ for $t \in [0, \tau_e]$, where τ_e is known as the explosion time[1-6]. Our aim is to prove that this solution is global. In other words, we need to demonstrate that $\tau_e = \infty$ it probability one for each $\gamma \geq \gamma_0$, $\gamma \in \mathbb{N}$, We define the stopping

$$\tau_\gamma = \left\{ t \in (0, \tau_e) : \min\left\{\widetilde{T}(t)\right\} \leq \frac{1}{\gamma} \text{ or } \max\left\{\widetilde{T}(t)\right\} \geq \gamma \right\}, \tag{2.21}$$

assuming that γ is sufficiently large such that $\widetilde{T}(0) \in \left\{\frac{1}{\tau_\gamma}, \tau_\gamma\right\}$.By the definition of the stopping time, τ_γ is increasing, where $\gamma \to \infty$. Let $\tau_\infty = \lim\limits_{\gamma \to \infty} \tau_\gamma$ then $\tau_\infty \leq \tau_e$. Then we must show that $\tau_\infty = \infty$ with probability one, then $\tau_e = \infty$ almost surely and $\widetilde{T}(t) \in \mathbb{R}_+^5$ for each $t \geq 0$. Where it is incorrect, there exists a pair $\psi(t) \geq 0$ and $\varepsilon \in (0,1)$ such that

$$P\{r_\infty \leq \psi\} > \varepsilon. \tag{2.22}$$

By the Lyapunov functional methodology, one can conclude that the stochastic system admits the existence of a global positive solution. Thus, we define a C^2-function $\Phi : \mathbb{R}_+^5 \to \mathbb{R}$ given by

$$\Phi(H,\ T,\ I,\ E,\ F) = H + T + I + E + F - 5 - (\ln H + \ln T + \ln I + \ln E + \ln F). \tag{2.23}$$

Obviously, $\Phi \geq 0$ since $c-1-\ln c \geq 0$ for all $c>0$. Applying the Ito formula, we can obtain the following result

$$\begin{aligned} d\Phi\left(H,\ T,\ I,\ E,\ F\right) &= \left(1-\frac{1}{H}\right)dt+\sigma_1\left(H-1\right)dB_1\left(t\right) \\ &\quad +\left(1-\frac{1}{T}\right)dt+\sigma_2\left(T-1\right)dB_2\left(t\right) \\ &\quad +\left(1-\frac{1}{I}\right)dt+\sigma_3\left(I-1\right)dB_3\left(t\right) \\ &\quad +\left(1-\frac{1}{E}\right)dt+\sigma_4\left(E-1\right)dB_4\left(t\right) \\ &\quad +\left(1-\frac{1}{F}\right)dt+\sigma_5\left(F-1\right)dB_5\left(t\right) \\ &= \alpha\Phi\left(H,\ T,\ I,\ E,\ F\right)dt+\sigma_1\left(H-1\right)dB_1\left(t\right)+\sigma_2\left(T-1\right)dB_2\left(t\right) \\ &\quad +\sigma_3\left(I-1\right)dB_3\left(t\right)+\sigma_4\left(E-1\right)dB_4\left(t\right)+\sigma_5\left(F-1\right)dB_5\left(t\right), \end{aligned} \tag{2.24}$$

noting that $d\Phi:\mathbb{R}_+^5\to\mathbb{R}$.Replacing the right-side function of the system (1.14) into the above equation leads to

$$\begin{aligned} d\Phi &= \left(1-\frac{1}{H}\right)\left\{H\left(\kappa_1-\delta_1 H-\eta T\right)-\mu_1 HE\right\}+\frac{\sigma_1^2}{2} \\ &\quad +\left(1-\frac{1}{T}\right)\left\{T\left(\kappa_2-\delta_2 T-\phi_1 I\right)+\mu_2 HE+\varepsilon_1 TF\right\}+\frac{\sigma_2^2}{2} \\ &\quad +\left(1-\frac{1}{I}\right)\left\{\Lambda+I\left(\frac{qT}{\theta_1+T}-\frac{\mu_3 E}{\theta_2+E}-p-\phi_2 T\right)\right\}+\frac{\sigma_3^2}{2} \\ &\quad +\left(1-\frac{1}{E}\right)\left\{w_1+\varepsilon_2 EF-w_2 E\right\}+\frac{\sigma_4^2}{2} \\ &\quad +\left(1-\frac{1}{F}\right)\left\{\kappa_3 F\left(1-\varepsilon_3 F\right)\right\}+\frac{\sigma_5^2}{2} \\ &\leq \kappa_1 H+\delta_1+\eta T+\mu_1 E+\kappa_2 T+\mu_2 HE+\varepsilon_1 TF+\delta_2\phi_1\frac{I}{T} \\ &\quad \Lambda+I\frac{qT}{\theta_1+T}+I\frac{\mu_3 E}{\theta_2+E}+\frac{p}{I}+\frac{\phi_2 T}{I}+w_1+\varepsilon_2 EF \\ &\quad +w_2+\kappa_3 F+\kappa_3\varepsilon_3 F+\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2}{2} \\ &= K, \end{aligned} \tag{2.25}$$

where K is positive and not dependent on time and $\widetilde{T}\left(t\right)$. Thus we have the following

$$\begin{aligned} d\Phi\left(H,\ T,\ I,\ E,\ F\right) &\leq Kdt+\sigma_1\left(H-1\right)dB_1\left(t\right)+\sigma_2\left(T-1\right)dB_2\left(t\right) \\ &\quad +\sigma_3\left(I-1\right)dB_3\left(t\right)+\sigma_4\left(E-1\right)dB_4\left(t\right) \\ &\quad +\sigma_5\left(F-1\right)dB_5\left(t\right). \end{aligned} \tag{2.26}$$

Applying the integral to the equation above and then taking the expected value E on both sides, we obtain

$$E\left[\Phi\left(\widetilde{T}\left(\tau_\gamma\wedge\psi\right)\right)\right]\leq E\left[\Phi\left(\widetilde{T}\left(0\right)\right)\right]+K\psi. \tag{2.27}$$

Setting $F_\gamma=\tau_\gamma\leq\psi$ for $\gamma\geq\gamma_0$ and then $P\left(F_\gamma\right)\geq\varepsilon$.

For each $w \in F_{\gamma}$ there exist at least one of $H(\tau_{\gamma} \wedge \psi)$, $T(\tau_{\gamma} \wedge \psi)$, $I(\tau_{\gamma} \wedge \psi)$, $E(\tau_{\gamma} \wedge \psi)$ *or* $F(\tau_{\gamma} \wedge \psi)$ equaling γ or $\frac{1}{\gamma}$. Then, we obtain

$$\Phi\left(\widetilde{T}(\tau_{\gamma} \wedge \psi)\right) \geq (\gamma - 1 - \ln\gamma) \wedge \left(\frac{1}{\gamma} - 1 - \ln\gamma\right). \tag{2.28}$$

In view of (2.27), one obtains

$$\begin{aligned} E\Phi\left(\widetilde{T}(0)\right) + K\psi &\geq E\left(1_{F_w}\Phi\left(\widetilde{T}(\tau_{\gamma} \wedge \psi)\right)\right) \\ &\geq \varepsilon\left\{(\gamma - 1 - \ln\gamma) \wedge \left(\frac{1}{\gamma} - 1 - \ln\gamma\right)\right\}, \end{aligned} \tag{2.29}$$

where 1_{F_w} stands for the indicator function of F_{γ}. As $\gamma \to \infty$, we obtain

$$\infty > E\Phi\left(\widetilde{T}(0)\right) + K\psi = \infty, \tag{2.30}$$

which is a contradiction. Thus, we conclude that $\tau_{\infty} = \infty$ with probability one, which completes the proof. □

3. Numerical simulation

3.1. Numerical solution of tumor growth model with deterministic and stochastic approaches:

In this subsection, we present the numerical solution of the tumor growth model with deterministic and stochastic approaches. Using the same notations as in equation (2.4), the deterministic model is represented by:

$$\begin{aligned} x'(t) &= f(x,t), \\ x(0) &= x_0, \end{aligned} \tag{3.1}$$

and stochastic model is as follows:

$$dx = f(x,t)\,dt + \sigma_i x dB_i(t). \tag{3.2}$$

Integration above, we obtain

$$\begin{aligned} x(t) &= x_0 + (1-\alpha) f(x,t) + \alpha \int_0^t f(s,x)\,ds \\ &\quad + (1-\alpha)\sigma_i x dB_i(t) + \alpha \int_0^t \sigma_i x(s)\,dB_i(t). \end{aligned} \tag{3.3}$$

Considering at $t = t_n$ and $t = t_{n+1}$ and replacing the function $f(x,t)$ by its Newton polynomial [22] we obtain the following numerical scheme

$$\begin{aligned} x^{n+1} &= x^n + (1-\alpha)\left(f\left(t_{n+1}, \widetilde{x}^{n+1}\right) - f\left(t_n, \widetilde{x}^n\right)\right) + \frac{(1-\alpha)}{h}\sigma_i\left(\widetilde{x}^{n+1} - \widetilde{x}^n\right) \\ &\quad (B_i(t_{n+1}) - B(t_n)) + \alpha h\left(\frac{5h}{12} f\left(t_{n-2}, x^{n-2}\right) - \frac{4}{3} f\left(t_{n-1}, x^{n-1}\right) + \frac{23}{12} f(t_n, x_n)\right) \\ &\quad + \sigma_i x(c_n)\left(B_i(t_{n+1}) - B_i(t_n)\right), \end{aligned} \tag{3.4}$$

where $c_n \in [t_n, t_{n+1}]$. We need to calculate the predictor term due to the definition of Caputo-Fabrizio fractional derivative[18]. Using Euler approximation, the predictor term is obtained as

$$\widetilde{x}^{n+1} = x_0 + (1-\alpha) f(t_n, x^n) + \alpha h \sum_{k=0}^{n} f\left(t_k, x^k\right). \tag{3.5}$$

Remark 3.1. We recover stochastic for classical when $\alpha = 1$. When the stochastic constant $\sigma_i = 0$, we retrieve the deterministic case for Caputo-Fabrizio, and also the classical case when $\alpha = 1$.

For the Atangana Baleanu case[19], the model is modified by:

$$ {}_0^{AB}D_t^{\alpha} x(t) = f(t,x) + \sigma_i x dB_i(t). \tag{3.6}$$

Applying the integral, we have

$$\begin{aligned} x(t) &= x_0 + (1-\alpha) f(x,t) + \frac{\alpha}{\Gamma(\alpha)} \int_0^t f(s,x)(t-s)^{\alpha-1} ds \\ &\quad + (1-\alpha)\sigma_i x dB_i(t) + \frac{\alpha}{\Gamma(\alpha)} \int_0^t \sigma_i x(s) dB_i(s). \end{aligned} \tag{3.7}$$

Then, the numerical scheme is obtained as follows:

$$x^{n+1} = \begin{cases} x(0) + (1-\alpha) f\left(t_{n+1}, \widetilde{x}^{n+1}\right) + \frac{\alpha h^{\alpha}}{\Gamma(\alpha+1)} \sum_{k=2}^{n} f\left(t_{k-2}, x^{k-2}\right) \Pi_{n,k}^{1} \\ + \frac{\alpha h^{\alpha}}{\Gamma(\alpha+2)} \sum_{k=2}^{n} \left[f\left(t_{k-1}, x^{k-1}\right) - f\left(t_{k-2}, x^{k-2}\right) \right] \Pi_{n,k}^{2} \\ + \frac{\alpha h^{\alpha}}{\Gamma(\alpha+3)} \sum_{k=2}^{n} \left[f\left(t_k, x^k\right) - 2f\left(t_{k-1}, x^{k-1}\right) + f\left(t_{k-2}, x^{k-2}\right) \right] \Pi_{n,k}^{3}, \ t_0 \le t \le T \\ + \sum_{k=0}^{n} \sigma_i x(c_k) \left(B_i(t_{k+1}) - B_i(t_k) \right), \end{cases} \tag{3.8}$$

where

$$\begin{aligned} \Pi_{n,k}^{1} &= \left[(n-k+1)^{\alpha} - (n-k)^{\alpha} \right], \\ \Pi_{n,k}^{2} &= \begin{bmatrix} (n-k+1)^{\alpha}(n-k+3+2\alpha) \\ -(n-k)^{\alpha}(n-k+3+3\alpha) \end{bmatrix}, \\ \Pi_{n,k}^{3} &= \begin{bmatrix} (n-k+1)^{\alpha} \begin{pmatrix} 2(n-k)^2 + (3\alpha+10)(n-k) \\ +2\alpha^2 + 9\alpha + 12 \end{pmatrix} \\ -(n-k)^{\alpha} \begin{pmatrix} 2(n-k)^2 + (5\alpha+10)(n-k) \\ +6\alpha^2 + 18\alpha + 12 \end{pmatrix} \end{bmatrix}. \end{aligned} \tag{3.9}$$

For the second scenario, we consider:

$$ {}_0^{C}D_t^{\alpha} x(t) = f(x,t) + \sigma_i x dB_i(t), \tag{3.10}$$

where the derivative is with Caputo fractional derivative[17]. In this case, the numerical scheme is as follows:

$$x^{n+1} = \begin{cases} x(0) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{k=2}^{n} f\left(t_{k-2}, x^{k-2}\right) \Pi_{n,k}^{1} \\ + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{k=2}^{n} \left[f\left(t_{k-1}, x^{k-1}\right) - f\left(t_{k-2}, x^{k-2}\right) \right] \Pi_{n,k}^{2} \\ + \frac{h^{\alpha}}{\Gamma(\alpha+3)} \sum_{k=2}^{n} \left[f\left(t_k, x^k\right) - 2f\left(t_{k-1}, x^{k-1}\right) + f\left(t_{k-2}, x^{k-2}\right) \right] \Pi_{n,k}^{3}, \ t_0 \le t \le T \\ + \sum_{k=0}^{n} \sigma_i x(c_k) \left(B_i(t_{k+1}) - B_i(t_k) \right) \end{cases} \tag{3.11}$$

Figures 1-3 illustrate numerical simulations of both deterministic and stochastic approaches for each class in the model.

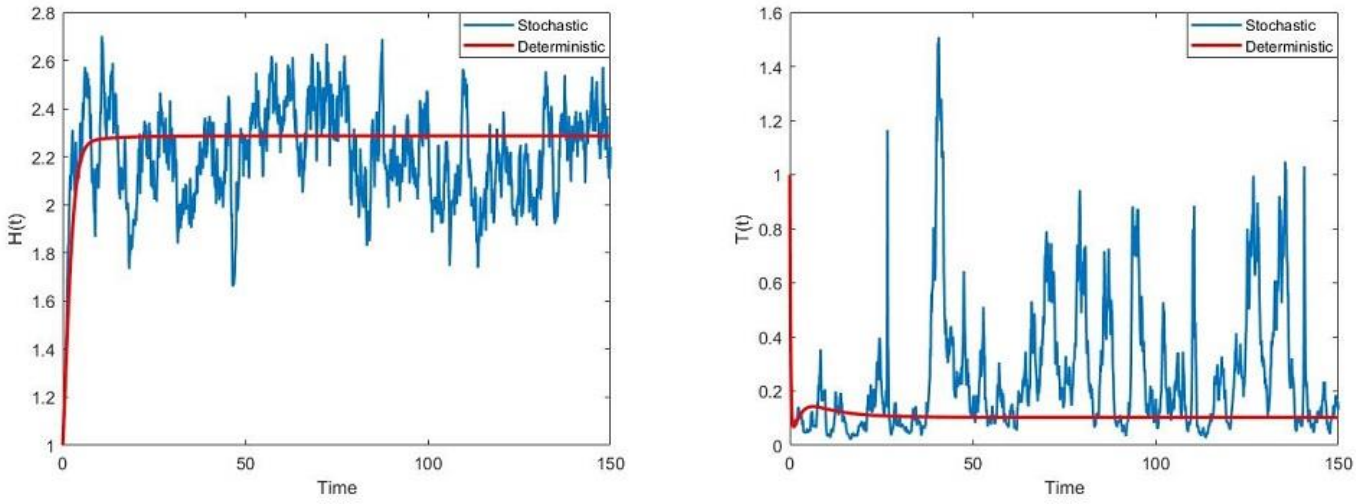

Figure 1: The graphical representation for the classes $H(t)$ and $T(t)$

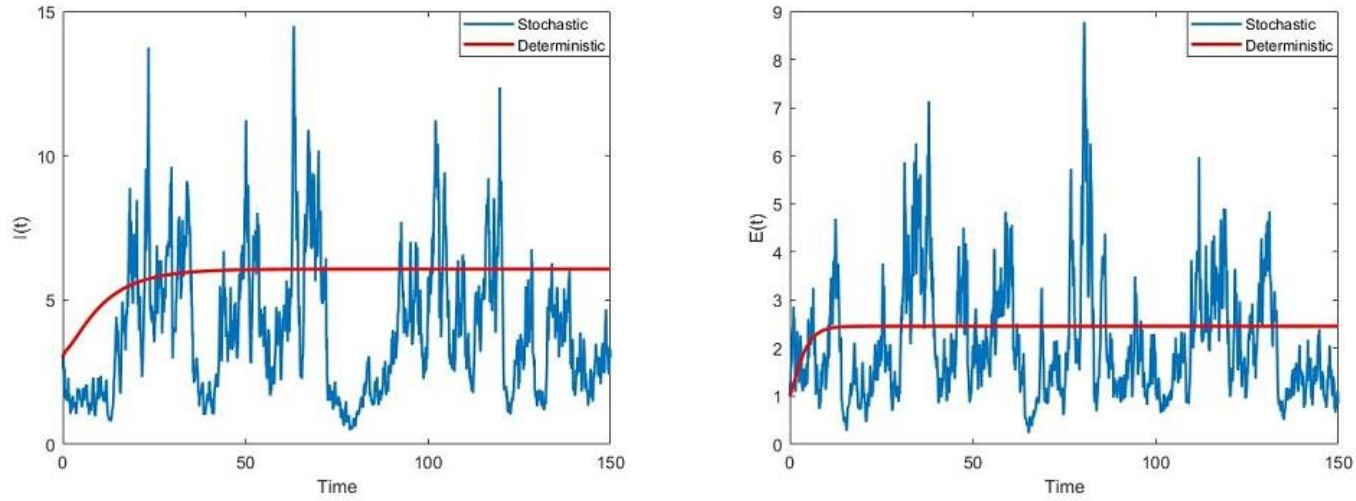

Figure 2: The graphical representation for the classes $I(t)$ and $E(t)$

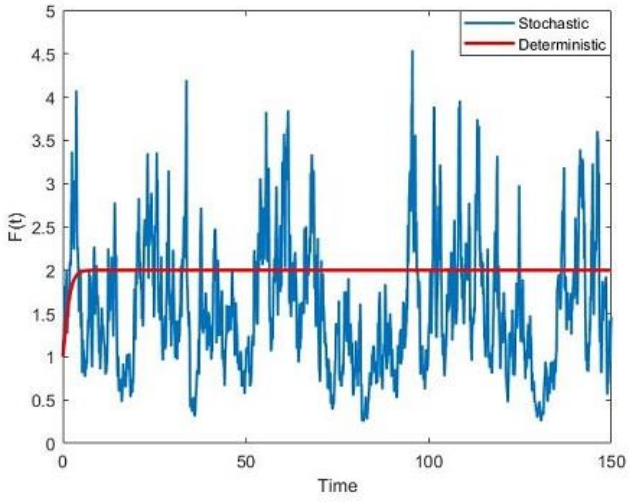

Figure 3: The graphical representation for the class $F(t)$.

4. Numerical solution of the deterministic-stochastic tumor growth model

Natural processes can exist that can facilitate the evolution from memory-less behaviors to behaviors that include memory or nonlocal behaviors. Moreover, because nature is highly intricate, numerous real-world problems involve overlapping processes that can be addressed through stochastic-fractional or fractional-stochastic methodologies. The piecewise model discussed here seeks to model real-world scenarios where initially there are memoryless processes followed by processes with memory, specifically focusing on power law memory or Mittag-Leffler process. Therefore, in this section, we aim to construct the numerical solution for the deterministic-stochastic tumor growth model, estimated by employing the piecewise derivative [20]. This derivative is useful in modeling real-world problems with crossover processes, as it suggests the use of differential operators that can handle different processes at different time intervals. Moreover, the fact that many differential operators have been defined in the literature and the possibility of using them together with the help of this derivative [20]. The use of the piecewise derivative can be chosen according to the behavior of the process, and different scenarios can be taken into account in order to observe different outcomes of the system.

Here, we will explore two scenarios for the tumor growth model. First scenario will be to consider model with stochastic and Atangana-Baleanu cases due to its nonlocal property effect, second scenario is to consider model with Caputo case and stochastic components.

We start with first scenario which is represented by

$$\begin{aligned} dx &= f(x,t)\,dt + \sigma_i x dB_i(t),\ 0 \le t \le t_0. \\ {}_{t_0}^{AB}D_t^{\alpha} x(t) &= f(x,t), t_0 \le t \le T. \end{aligned} \tag{4.1}$$

Applying the associated integral, we obtain

$$x(t) = \begin{cases} \left\{ x(0) + \int_0^t f(x,s)\,ds + \int_0^t \sigma_i x(s)\,dB_i(s),\ 0 \le t \le t_0, \right. \\ \left\{ x(t_0) + (1-\alpha) f(x,t) + \dfrac{\alpha}{\Gamma(\alpha)} \int_0^t f(x,s)(t-s)^{\alpha-1}\,ds, t_0 \le t \le T \right. \end{cases}. \tag{4.2}$$

We consider above at $t = t_{n+1}$, and replace the function $f(x,t)$ by its Newton polynomial [22]. Thus, we

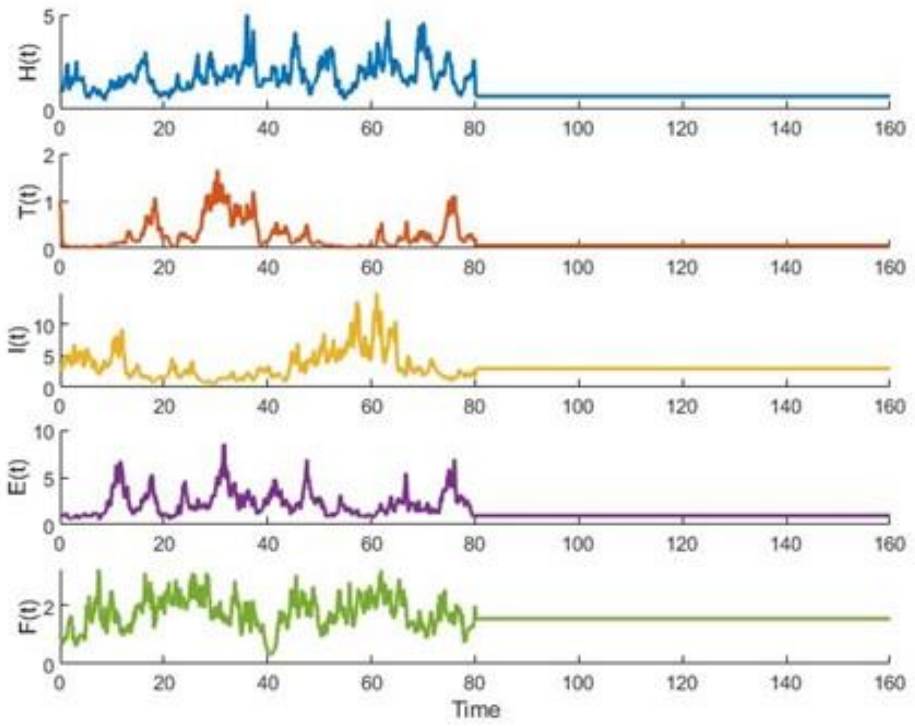

Figure 4: The graphical representation for each class with stochastic-deterministic approaches. For $\alpha = 0.99$, $\sigma_1 = 0.14$, $\sigma_2 = 0.15$, $\sigma_3 = 0.16$, $\sigma_4 = 0.17$, $\sigma_5 = 0.18$.

obtain

$$x^{n+1} = \begin{cases} \left\{ x(0) + h\sum_{k=0}^{m} f\left(t_k, x^k\right) + \sum_{k=0}^{m} \sigma_i x(c_k)\left(B_i(t_{k+1}) - B_i(t_k)\right), \ 0 \le t \le t_0, \right. \\ \left\{ \begin{array}{c} x(t_0) + (1-\alpha) f\left(t_{n+1}, \widetilde{x}^{n+1}\right) \\ + \dfrac{\alpha h^\alpha}{\Gamma(\alpha+2)} \sum_{k=m+3}^{n} f\left(t_{k-2}, x^{k-2}\right) \Pi_{n,k}^1 \\ + \dfrac{\alpha h^\alpha}{\Gamma(\alpha+1)} \sum_{k=m+3}^{n} \left[f\left(t_{k-1}, x^{k-1}\right) - f\left(t_{k-2}, x^{k-2}\right) \right] \Pi_{n,k}^2 \\ + \dfrac{\alpha h^\alpha}{\Gamma(\alpha+3)} \sum_{k=m+3}^{n} \left[\begin{array}{c} f\left(t_k, x^k\right) - 2f\left(t_{k-1}, x^{k-1}\right) \\ + f\left(t_{k-2}, x^{k-2}\right) \end{array} \right] \Pi_{n,k}^3, t_0 \le t \le T. \end{array} \right. \end{cases} \tag{4.3}$$

where the predictor component is calculated as

$$\widetilde{x}_{n+1} = x_0 + (1-\alpha) f(t_n, x^n) + \frac{\alpha}{\Gamma(\alpha)} \sum_{k=0}^{n} f\left(t_k, x^k\right) \Pi_{n,k}^1. \tag{4.4}$$

Figures 4-5 display the outcomes of numerical simulations conducted on the stochastic-deterministic model, which combines stochastic processes within the interval $[0, 80]$ and deterministic processes within the interval $[80, 160]$ using piecewise derivatives.

We now consider stochastic and Caputo case

$$\begin{array}{rcl} dx & = & f(x,t)\,dt + \sigma_i x dB_i(t), \ 0 \le t \le t_0. \\ {}_{t_0}^{C}D_t^{\alpha} x(t), t_0 & \le & t \le T. \end{array} \tag{4.5}$$

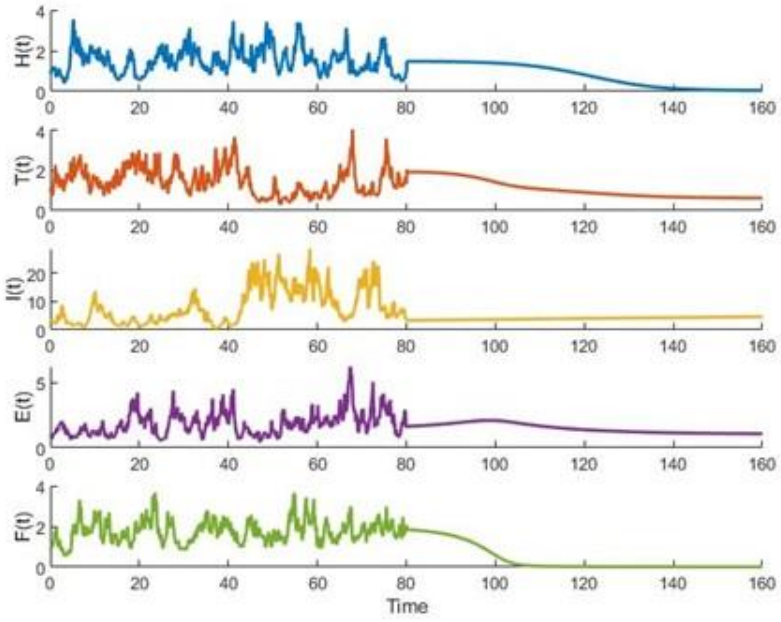

Figure 5: The graphical representation for each class with stochastic-deterministic approaches. For $\alpha = 0.95$, $\sigma_1 = 0.14$, $\sigma_2 = 0.15$, $\sigma_3 = 0.16$, $\sigma_4 = 0.17$, $\sigma_5 = 0.18$.

Applying the associated integral, we obtain

$$x(t) = \begin{cases} \left\{ x(0) + \int_0^t f(x,s)\,ds + \int_0^t \sigma_i x(s)\,dB_i(s),\ 0 \le t \le t_0, \right. \\ \left\{ x(t_0) + \dfrac{1}{\Gamma(\alpha)} \int_0^t f(x,s)\,(t-s)^{\alpha-1}\,ds, t_0 \le t \le T \right. \end{cases}. \tag{4.6}$$

We consider above at $t = t_{n+1}$, and replace the function $f(x,t)$ by its Newton polynomial. Thus, we obtain

$$x^{n+1} = \begin{cases} \left\{ x(0) + h\sum_{k=0}^{m} f\left(t_k, x^k\right) + \sum_{k=0}^{m} \sigma_i x(c_k)\left(B_i(t_{k+1}) - B_i(t_k)\right),\ 0 \le t \le t_0, \right. \\ \left\{ \begin{array}{c} x(t_0) + \dfrac{h^\alpha}{\Gamma(\alpha+1)} \sum_{k=m+3}^{n} f\left(t_{k-2}, x^{k-2}\right) \Pi_{n,k}^1 \\ + \dfrac{h^\alpha}{\Gamma(\alpha+2)} \sum_{k=m+3}^{n} \left[f\left(t_{k-1}, x^{k-1}\right) - f\left(t_{k-2}, x^{k-2}\right) \right] \Pi_{n,k}^2 \\ + \dfrac{h^\alpha}{\Gamma(\alpha+3)} \sum_{k=m+3}^{n} \left[\begin{array}{c} f\left(t_k, x^k\right) - 2f\left(t_{k-1}, x^{k-1}\right) \\ + f\left(t_{k-2}, x^{k-2}\right) \end{array} \right] \Pi_{n,k}^3, t_0 \le t \le T \end{array} \right. \end{cases}. \tag{4.7}$$

The predictor component is calculated as

$$\widetilde{x}_{n+1} = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} f\left(t_k, x^k\right) \Pi_{n,k}^1. \tag{4.8}$$

Figures 6-7 illustrate the results of numerical simulations from a stochastic-deterministic model that incorporates stochastic processes within the interval $[0, 60]$ and deterministic processes within the interval $[60, 160]$ using piecewise derivatives.

5. Conclusion

As a result of introducing the theory of fractional calculus, new doors have been opened for the development of fractional differential operators and their application to real-world problems. We obtained the

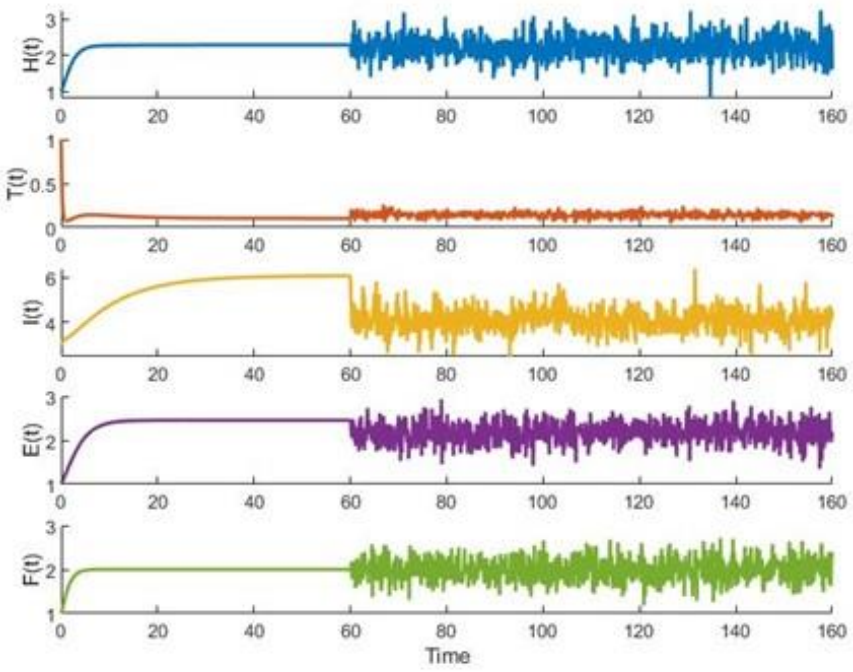

Figure 6: The graphical representation for each class with deterministic-stochastic approaches. For $\alpha = 0.99$, $\sigma_1 = 0.14$, $\sigma_2 = 0.22$, $\sigma_3 = 0.25$, $\sigma_4 = 0.11$, $\sigma_5 = 0.13$.

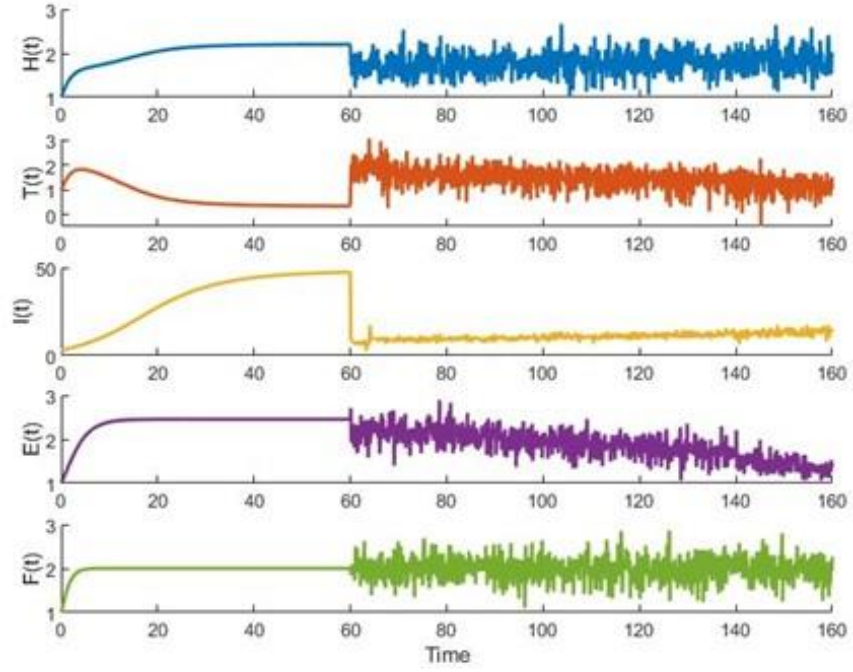

Figure 7: The graphical representation for each class with deterministic-stochastic approaches. For $\alpha = 0.95$, $\sigma_1 = 0.14$, $\sigma_2 = 0.22$, $\sigma_3 = 0.25$, $\sigma_4 = 0.11$, $\sigma_5 = 0.13$.

conditions under which the solution of the stochastic model exist and unique. Considering that real-world problems have crossover tendencies, the deterministic-stochastic model has been taken into account as a result of the need to use a piecewise differential operator. Thus, simulations that provide more realistic and different scenarios for the model considered are presented. To verify the obtained analytical results, several numerical simulations are carried out by employing a numerical method based on the Newton polynomial. When the simulations are examined, it has been concluded that more realistic and different scenarios have been obtained for the considered model. We believe that this study, which combines deterministic and stochastic approaches, will make significant contributions to the literature as it can be applied to models in different disciplines.

Author contributions

All authors contributed to the study conception and design. All authors read and approved the final manuscript.

Data availability

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Conflict of interest

The authors have no relevant financial or non-financial interests to disclose.

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