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On a class of abstract fourth-order differential equations set on cusp domain

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Abstract

In this work, we concentrate on a boundary value problem set on a singular domain containing a cuspidial point. In our study, we obtain some existence and maximal regularity results. Our strategy is based on the study of a boundary value problems for a class of the complete abstract fourth-order differential equations involving fractional powers of unbounded linear operators.

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1. Origin of the Problem and Motivation

Let $\mathbb{R}^+ = [0, +\infty[$. In this work, we assume that $x = (x_1, x_2, x_3)$ is a generic point of $\mathbb{R}^3$. Let $\Pi \subseteq \mathbb{R}^3$ be a cusp domain defined by

$$\Pi := \left\{ x \in \mathbb{R}^3 : 0 < x_3 < 1, \ \left(\frac{x_1}{(x_3)^\alpha}, \frac{x_2}{(x_3)^\alpha} \right) \in \Omega \right\}, \quad \alpha > 1,$$

where $\Omega \subseteq \mathbb{R}^2$ is a bounded smooth domain.
In the cusp domain $\mathbb{R}^+ \times \Pi$, we consider the following problem

$$\frac{d^4}{dt^4} u(t,x) + (1 + \rho_4(x))(-\Delta)^{4v} u(t,x) + \sum_{j=1}^{3} \left(\rho_j(x) (-\Delta)^{jv} \right) \frac{d^{4-j}}{dt^{4-j}} u(t,x) = f(t,x), \tag{1.1}$$

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where $\upsilon \in]0,1]$ and Δ is the classical Laplace operator on $\mathbb{R}^3$ defined by $\Delta = \sum_{i=1}^{3} \partial_{x_i}^2$. The functions $\rho_j(\cdot)$, $j = 1, 2, 3, 4$, are continuous real functions defined on Π such that

$$\lim_{\kappa \to +\infty} \kappa \rho_j (.,.,\kappa) < +\infty, \quad j = 1, 2, 3, 4.$$

The right hand side of equation (1.1) is assumed to belong to the Hilbert space $L^2(\mathbb{R}^+ \times \Pi) = L^2(\mathbb{R}^+; L^2(\Pi))$. We will also accompany to (1.1) some boundary conditions and initial conditions involving Laplace operator. More precisely, we look for a solution $u(\cdot)$ satisfying

$$u|_{\mathbb{R}^+ \times \partial\Pi} = 0, \tag{1.2}$$

$$\left.\frac{du}{dt}\right|_{\{0\}\times\Pi} = 0, \ \left.\frac{d^3u}{dt^3} + b(-\Delta)^{3\upsilon}u\right|_{\{0\}\times\Pi} = 0. \tag{1.3}$$

with $b \in \mathbb{C}$. Evolution equations involving Laplace operator have significant applications in physics and engineering, especially in fluid and solid mechanics. Furthermore, in the static case this type of equations furnish us a model to study travelling waves in suspension bridges. Many studies have been carried out about some simple situations dealing with regular domains. In our study, the choice of equation (1.1) is justified by the fact that it appears in various modeling of concrete phenomena such as circulation of fluids on lungs [17], ice formation [21], and brain warping [24].
Note that problems involving higher-order PDEs and complex geometries are more difficult to solve than those with second-order PDEs and regular geometries. The approach adopted in this work is mainly based on the use of the theory of the abstract differential equations. Recall that this technique was used in many works; see, e.g., [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16] and [22]. The first step is to transform the cusp domain $\mathbb{R}^+ \times \Pi$ into a cylindrical one. To do this, we consider the following change of variables

$$\begin{aligned} \Psi : \ & \mathbb{R}^+ \times \Pi \to \mathbb{R}^+ \times Q, \\ & (t, x) \mapsto (t, \xi), \end{aligned}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ is also a new generic point of $\mathbb{R}^3$ such that

$$\xi_1 = \frac{x_1}{(x_3)^\alpha}, \qquad \xi_2 = \frac{x_2}{(x_3)^\alpha} \quad \text{and} \quad \xi_3 = \frac{(x_3)^{1-\alpha}}{\alpha - 1}. \tag{1.4}$$

Here,

$$Q = \Omega \times]\xi_{3,0}, +\infty[,$$

with

$$\xi_{3,0} = \frac{1}{\alpha - 1} > 0;$$

this means that, for every $x_3 \in]0,1[$ and $(x_1, x_2) \in \Omega$, we have

$$\begin{cases} \lim\limits_{x_3 \to 0^+} \Psi(x_1, x_2, x_3) = \Omega \times \{+\infty\}, \\ \qquad\qquad \text{and} \\ \lim\limits_{x_3 \to 1} \Psi(x_1, x_2, x_3) = \Omega \times \{\xi_{3,0}\} \end{cases}$$

In this study, we confine ourselves to the neighborhood of the origin $0_{\mathbb{R}^3}$; this means that we consider the case in which $\xi_3 > \xi_{3,0}$ is large enough. At this level, let us introduce the following change of functions

$$v(t, \xi) = u(t, x), \qquad g(t, \xi) = f(t, x).$$

According to (1.4), it is easy to check that

$$f \in L^2(\mathbb{R}^+ \times \Pi) \quad \text{if and only if} \quad \left(\frac{\gamma}{\xi_3}\right)^{\frac{-3\alpha}{2\beta}} g \in L^2(\mathbb{R}^+ \times Q)), \tag{1.5}$$

where

$$\beta = \frac{1}{\gamma} = \alpha - 1.$$

To avoid the use of weighted L^2-spaces, we opt for the use of a new change of functions given by

$$v = \left(\frac{\gamma}{\xi_3}\right)^s w, \quad h = \left(\frac{\gamma}{\xi_3}\right)^{\frac{-3\alpha}{2\beta}} g$$

with

$$s = \frac{\alpha}{\beta}\left(\frac{3}{8\upsilon} + 2\right).$$

As a direct consequence, the problem (1.1)-(1.2)-(1.3) is written as follows

$$\mathcal{P}_1(\xi_3)\frac{d^4}{dt^4}w(t,\xi) + (1+\sigma_4(\xi))(\mathcal{L})^{4\upsilon}w(t,\xi) + \sum_{j=1}^{3}\left(\sigma_j(\xi)(\mathcal{L})^{j\upsilon}\right)\frac{d^{4-j}}{dt^{4-j}}w(t,\xi) = h(t,\xi), \tag{1.6}$$

$$w|_{\mathbb{R}^+\times\partial Q} = 0, \tag{1.7}$$

and

$$\left.\frac{dw}{dt}\right|_{\{0\}\times Q} = 0, \qquad \left.\mathcal{P}_2(\xi_3)\frac{d^3w}{dt^3} + b(\mathcal{L})^{3\upsilon}w\right|_{\{0\}\times Q} = 0. \tag{1.8}$$

Here

$$\mathcal{P}_1(\xi_3) = \left(\frac{\gamma}{\xi_3}\right)^{\frac{\alpha}{\beta}\left(\frac{3}{8\upsilon}+\frac{1}{2}\right)}, \quad \mathcal{P}_2(\xi_3) = \left(\frac{\gamma}{\xi_3}\right)^{\frac{\alpha}{\beta}\left(\frac{3}{8\upsilon}+\frac{7}{8}\right)},$$

and

$$\mathcal{L} = -\Delta + \frac{1}{\xi_3}\mathcal{M},\ \xi_3 > \xi_{3,0} > 0,$$

where $\mathcal{M}$ is the second-order differential operator with smooth coefficients given by

$$\begin{aligned}
&[\mathcal{M}w](\xi)\\
&:= \frac{(\alpha\gamma)^2}{\xi_3}\left\{\xi_1^2\partial_{\xi_1}^2 w + \xi_2^2\partial_{\xi_2}^2 w + 2\xi_1\xi_2\partial_{\xi_1\xi_2}^2 w\right\}\\
&+ 2\alpha\gamma\left\{\xi_1\partial_{\xi_1\xi_3}^2 w + \xi_2\partial_{\xi_2\xi_3}^2 w\right\}\\
&+ (\alpha\gamma - 2s)\,\partial_{\xi_3} w\\
&+ \frac{\alpha\gamma}{\xi_3}\left((\alpha+1)\gamma - 2s\right)\left\{\xi_1\partial_{\xi_1} w + \xi_2\partial_{\xi_2} w\right\}\\
&- \frac{s}{\xi_3}\left\{s + 1 + \alpha\gamma\right\} w.
\end{aligned}$$

Note also that the family of functions $\sigma_j(\xi)$, $j \in \{1,2,3,4\}$ are defined as follows

$$\sigma_j(\xi) := \left(\frac{\gamma}{\xi_3}\right)^{\frac{3\alpha}{8\beta}(j-4)} \rho_j(\xi),\ j \in \{1,2,3,4\}.$$

Due to the change of variables Ψ defined by (1.4), these functions are bounded on $\mathbb{R}^+ \times Q$.
Observe that the study of (1.6)-(1.7)-(1.8) needs the investigation of the following abstract problem:

$$\frac{d^4 w(t)}{dt^4} + A^{4\upsilon}w(t) + \sum_{j=1}^{4} A_j \frac{d^{4-j}w(t)}{dt^{4-j}} = h(t),\ t \in \mathbb{R}^+, \tag{1.9}$$

endowed with the initial conditions

$$\frac{dw(0)}{dt} = 0, \ \frac{d^3w(0)}{dt^3} + Kw(0) = 0. \tag{1.10}$$

where the vector-valued functions w and h are defined by

$$\begin{aligned} w &: \mathbb{R}^+ \to H \ ; \ t \to w(t) \ ; \quad w(t)(\xi) = w(t,\xi), \\ h &: \mathbb{R}^+ \to H \ ; \ t \to h(t) \ ; \quad h(t)(\xi) = h(t,\xi), \end{aligned}$$

with $H = L^2(Q)$. Here,

$$\begin{cases} D(A_j) & := \left\{\phi \in L^2(Q) : A_j\phi \in L^2(Q), \phi|_{\partial Q} = 0\right\}, \\ (A_j\psi)(\xi) & := \left[\sigma_j(\xi)(-\Delta)^{j\upsilon}\right]\phi(\xi), \quad j = 1,2,3,4, \end{cases} \tag{1.11}$$

and

$$\begin{cases} D(A) & := \left\{\phi \in L^2(Q) : \Delta\phi \in L^2(Q), \phi|_{\partial Q} = 0\right\}, \\ (A\phi)(\xi) & := -\Delta\phi(\xi). \end{cases} \tag{1.12}$$

We define the operator K by

$$\begin{cases} D(K) & := \left\{\phi \in L^2(Q) : K\phi \in L^2(Q)\right\}, \\ (K\phi)(\xi) & := b\left[(-\Delta)^{3\upsilon}\right]\phi(\xi). \end{cases} \tag{1.13}$$

Following [19] and [5], the fractional power of the operator (1.12) is well defined. Furthermore, we have the following practical characterization of $D(A^\upsilon)$ through the classical Sobolev spaces. For the reader convenience, we recall that

$$D(A^\upsilon) = \begin{cases} H^{2\upsilon}(Q), & 0 < \upsilon < 1/4, \\ H^{1/2}_{00}(Q), & \upsilon = 1/4, \\ H^{2\upsilon}_0(Q), & 1/4 < \upsilon \le 1/2, \\ H^{2\upsilon}(Q) \cap H^1_0(Q), & 1/2 < \upsilon \le 1; \end{cases} \tag{1.14}$$

here, $H^{1/2}_{00}(Q)$ is the interpolation space defined in [20, Chapter, p. 66].

2. Statement of the Abstract Problem

In this section, a particular attention is given to the study of a general class of the abstract fourth-order differential equations with operator coefficients posed in Hilbert spaces.

2.1. Preliminaries

We consider a complex separable Hilbert space H and a self-adjoint positive-definite operator A in H. By H_υ, $\upsilon \ge 0$ we denote the scale of Hilbert spaces generated by the operator A^υ, i.e.,

$$H_\upsilon := D(A^\upsilon); \ (x,y)_\upsilon := (A^\upsilon x, A^\upsilon y), \ x, \ y \in D(A^\upsilon).$$

Consider the Hilbert space $L^2(\mathbb{R}^+; H)$ consisting of all H-valued functions defined on $\mathbb{R}^+$, endowed with its natural norm

$$\|h\|_{L^2(\mathbb{R}^+;H)} := \left(\int_0^{+\infty} \|h(t)\|_H^2 \, dt\right)^{1/2}.$$

According to Theorem 1.5.15 in [23], it is well established that A^υ is a self-adjoint positive definite operator for $\upsilon > 0$. This allows us to define the Sobolev space $W^{4,\upsilon}(\mathbb{R}^+; H)$ as follows:

$$W^{4,\upsilon}(\mathbb{R}^+; H) := \left\{w : w^{(4)} \in L^2(\mathbb{R}^+; H), \ A^{4\upsilon}w \in L^2(\mathbb{R}^+; H)\right\}. \tag{2.1}$$

endowed with the norm

$$\|w\|_{W^{4,\upsilon}(\mathbb{R}^+;H)} := \left(\left\| w^{(4)} \right\|^2_{L^2(\mathbb{R}^+;H)} + \left\| A^{4\upsilon} w \right\|^2_{L^2(\mathbb{R}^+;H)} \right)^{1/2},$$

For more details about these spaces, see [20, Chapter 1].

Let us consider the following abstract differential equation

$$\frac{d^4 w(t)}{dt^4} + A^{4\upsilon} w(t) + \sum_{j=1}^{4} A_j \frac{d^{4-j} w(t)}{dt^{4-j}} = h(t),\ t \in \mathbb{R}^+, \tag{2.2}$$

where $\upsilon \in]0,1]$, $h \in L^2(\mathbb{R}^+, H)$ and A_j, $j = 1,2,3,4$, are linear operators acting on H. We also assume that Eq. (2.2) is accompanied with the following nonhomogeneous abstract boundary conditions:

$$\frac{dw(0)}{dt} = \varphi_1,\ \frac{d^3 w(0)}{dt^3} + Kw(0) = \varphi_2, \tag{2.3}$$

with K being an element of $\mathcal{L}(H_{7\upsilon/2}, H_{\upsilon/2})$, $\varphi_1 \in H_{5\upsilon/2}$ and $\varphi_2 \in H_{\upsilon/2}$; here, $\mathcal{L}(X,Y)$ denotes the space of linear bounded operators acting from the space X to the space Y. Note here that the closed operator A^υ, $\upsilon \in]0,1[$ is boundedly invertible with inverse $A^{-\upsilon}$. For further information, we refer the reader to [3] and [22].

First of all, we seek for a regular solution for (2.2), i.e., a vectorial function $w \in W^{4,\upsilon}(\mathbb{R}^+;H)$ satisfying (2.2)-(2.3) a.e. in $\mathbb{R}^+$. Next, we provide some necessary conditions ensuring the regular solvability of our problem (2.2)-(2.3). For the reader convenience, we recall also that the problem (2.2)-(2.3) is said to be regularly solvable if and only if it admits a regular solution $w(\cdot)$ which satisfies the following conditions

$$\begin{cases} \lim\limits_{t\to 0} \left\| \dfrac{dw(t)}{dt} - \varphi_1 \right\|_{H_{5\upsilon/2}} = 0, \\ \text{and} \\ \lim\limits_{t\to 0} \left\| \dfrac{d^2 w(t)}{dt^2} w + Kw(t) - \varphi_2 \right\|_{H_{\upsilon/2}} = 0 \end{cases}$$

and for any $h \in L^2(\mathbb{R}^+;H)$, there exists $C > 0$ such that

$$\|w\|_{W^{4,\upsilon}(\mathbb{R}^+;H)} \leq C \|h\|_{L^2(\mathbb{R}^+;H)}.$$

In the current literature, we find many works considering various classes of the fourth-order operator-differential equations. For example, in [2], some optimal results about the existence and uniqueness of regular solutions have been established for the problem

$$\begin{aligned} &\frac{d^4 w(t)}{dt^4} + A^4 w(t) + \sum_{j=1}^{4} A_j \frac{d^{4-j} w(t)}{dt^{4-j}} = h(t),\ t \in \mathbb{R}, \\ &\frac{d^3 w(t)}{dt^3}(0) = 0,\ \frac{d^2 w(0)}{dt^2} - K\frac{dw(0)}{dt} = 0, \end{aligned} \tag{2.4}$$

where

- $h \in L^2(\mathbb{R}^+;H)$,
- $(A, D(A))$ is a self-adjoint positive definite operator in a Hilbert space H,
- A_j, $j \in \{1,2,3,4\}$ are, in general, linear unbounded operators,
- $K \in \mathcal{L}(H_{5/2}, H_{3/2})$.

In [1], many interesting regularity results are established for the problem

$$\begin{aligned} &\frac{d^4w(t)}{dt^4} + A^4w(t) + \sum_{j=1}^{4} A_j \frac{d^{4-j}w(t)}{dt^{4-j}} = h(t),\ t \in \mathbb{R}^+, \\ &w(0) = \varphi \in H_{7/2},\ \frac{d^2w(0)}{dt^2} - K\frac{dw(0)}{dt} = \psi \in H_{3/2}, \end{aligned} \tag{2.5}$$

with the same assumptions as above. In the same direction, in [18] we find a complete study concerning the problem

$$\begin{aligned} &\frac{d^4w(t)}{dt^4} + \rho\,(t)\,A^4w(t) + \sum_{j=1}^{4} A_j \frac{d^{4-j}w(t)}{dt^{4-j}} = h(t),\ t \in \mathbb{R}^+, \\ &w(0) = \varphi,\ \frac{dw(0)}{dt} = \psi, \end{aligned}$$

with ρ being a scalar measurable function in $\mathbb{R}^+$. In this paper, it is clear that, due to the presence of operator $A^{4\upsilon}$, $\upsilon \in \left]0, 1\right]$, the problem (2.2)-(2.3) can be viewed, in some sense, as a generalization of (2.5).

2.2. Existence of regular solution

In the sequel, the abbreviation $W_K^{4,\upsilon}(\mathbb{R}^+; H)$ stands for the space defined by

$$W_K^{4,\upsilon}(\mathbb{R}^+; H) := \left\{ w : w \in W^{4,\upsilon}(\mathbb{R}^+; H), w'(0) = 0, w'''(0) = -Kw(0) \right\},$$

where $K \in \mathcal{L}(H_{7\upsilon/2}, H_{\upsilon/2})$.

Remark 2.1. As a direct consequence of the well known Lions-Peetre interpolation, the traces

$$w'''(0) \text{ and } Kw(0)$$

are well defined, see [20, Chapter 1]. Furthermore, for $w \in W^{4,\upsilon}(\mathbb{R}^+; H)$, one has

$$w^j(0) \in D\big(A^{\upsilon(7/2-j)}\big), \quad j = 0, 1, 2, 3,$$

and the mapping

$$\begin{aligned} &W^{4,\upsilon}(\mathbb{R}^+; H) \to \textstyle\prod_{j=0}^{3} D\big(A^{\upsilon(7/2-j)}\big), \\ &w \mapsto \left\{ w^{(j)}(0) \right\},\ 0 \le j \le 3, \end{aligned}$$

is surjective; see also Theorem 3.1 in *[20, Chapter 1].*

The fist step of our strategy is based on the study of the principal part of Eq. (2.2); that is

$$\frac{d^4w(t)}{dt^4} + A^{4\upsilon}w(t) = h(t),\ t \in \mathbb{R}^+, \tag{2.6}$$

equipped with the homogeneous initial conditions

$$\frac{dw(0)}{dt} = 0,\ \frac{d^3w(0)}{dt^3} + Kw(0) = 0. \tag{2.7}$$

Towards this end, let us denote by P_0 the operator defined as follows

$$\begin{aligned} P_0 :\ & W_K^{4,\upsilon}(\mathbb{R}^+; H) \to L^2(\mathbb{R}^+; H), \\ & w \mapsto P_0w(t) = \frac{d^4w(t)}{dt^4} + A^{4\upsilon}w(t). \end{aligned} \tag{2.8}$$

We have the following:

Lemma 2.2. *Let B be the operator defined by*

$$B := A^{\upsilon/2} K A^{-7\upsilon/2}.$$

Assume that $-\sqrt{2} \notin \sigma(B)$. Then the equation

$$P_0 w(t) = 0$$

has only a zero solution.

Proof. As in [2], we look for a solution of equation

$$P_0 w(t) = 0,$$

set on the space $W^{4,\upsilon}(\mathbb{R}^+; H)$. This solution has the next standard form

$$w_0(t) = e^{\eta_1 t A^{\upsilon}} \phi_1 + e^{\eta_2 t A^{\upsilon}} \phi_2, \quad t \in \mathbb{R}^+,$$

where $(e^{\eta_1 t A^{\upsilon}})_{t \geq 0}$, $(e^{\eta_2 t A^{\upsilon}})_{t \geq 0}$ are the C_0-semigroups generated by $\eta_1 A^{\upsilon}$ and $\eta_2 A^{\upsilon}$, respectively,

$$\eta_1 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \text{ and } \eta_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i,$$

with $\phi_1,\ \phi_2 \in H_{7\upsilon/2}$. Taking into account conditions (2.7), we obtain

$$\begin{cases} \eta_1 A^{\upsilon} \phi_1 + \eta_2 A^{\upsilon} \phi_2 = 0, \\ A^{3\upsilon}(\eta_1^3 \phi_1 + \eta_2^3 \phi_2) = -K(\phi_1 + \phi_2). \end{cases} \tag{2.9}$$

A direct computation implies that

$$\phi_2 = -\frac{\eta_1}{\eta_2} \phi_1, \tag{2.10}$$

and

$$\left(\sqrt{2} I + B\right) A^{7\upsilon/2} \phi_1 = 0, \tag{2.11}$$

where I denotes the identity operator. Keeping in mind that $-\sqrt{2} \notin \sigma(B)$, this leads to $\phi_1 = 0$ and from (2.10) it results that $\phi_2 = 0$. Therefore, $w_0(t) = 0$. □

Now, we are able to state our main result concerning the solvability of problem (2.6)-(2.7):

Theorem 2.3. *Let B be the operator defined by*

$$B = A^{\upsilon/2} K A^{-7\upsilon/2},$$

and let $-\sqrt{2} \notin \sigma(B)$. Then, the problem (2.6)-(2.7) *has a unique regular solution $w \in W_K^{4,\upsilon}(\mathbb{R}^+; H)$.*

Proof. **Step 1** Thanks to Lemma 2.2, we know that the problem

$$\frac{d^4 w(t)}{dt^4} + A^{4\upsilon} w(t) = 0,\ t \in \mathbb{R}^+, \tag{2.12}$$

$$\frac{dw(0)}{dt} = 0,\ \frac{d^3 w(0)}{dt^3} = -K w(0) \tag{2.13}$$

has only zero solution in $W_K^{4,\upsilon}(\mathbb{R}^+; H)$. Let us show that the equation $P_0 w(t) = h(t)$ has a solution $w \in W_K^{4,\upsilon}(\mathbb{R}^+; H)$ for every $h \in L^2(\mathbb{R}^+; H)$. First, set

$$H(t) := \begin{cases} f(t), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Let $\hat{H}(\xi)$ be the Fourier transform of $F(t)$, i.e.,

$$\hat{H}(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} H(t) e^{-i\zeta t}\, dt,\ \zeta \in \mathbb{R}.$$

Then, performing the direct and inverse Fourier transforms, it is clear that the vector-valued function

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\zeta^4 I + A^{4\upsilon})^{-1} \left(\int_0^{+\infty} h(s) e^{-i\zeta s} ds \right) e^{i\xi t} d\zeta,\ t \in \mathbb{R}, \tag{2.14}$$

satisfies the equation

$$\frac{d^4 v(t)}{dt^4} + A^{4\upsilon} v(t) = H(t) \quad \text{a.e. in } \mathbb{R}.$$

Now, we prove that $v(\cdot)$ defined by the formula (2.14) belongs to the space $W^{4,\upsilon}(\mathbb{R}^+; H)$. By Plancherel's theorem, we have:

$$\begin{aligned} \|v\|^2_{W^{4,\upsilon}(\mathbb{R}^+;H)} &= \left\|v^{(4)}\right\|^2_{L^2(\mathbb{R}^+;H)} + \left\|A^{4\upsilon} v\right\|^2_{L^2(\mathbb{R}^+;H)} \\ &= \left\|\zeta^4 \hat{v}\right\|^2_{L^2(\mathbb{R}^+;H)} + \left\|A^{4\upsilon} \hat{v}\right\|^2_{L^2(\mathbb{R}^+;H)}; \end{aligned}$$

hence,

$$\begin{aligned} &\|v\|^2_{W^{4,\upsilon}(\mathbb{R}^+;H)} \\ &= \left\| \zeta^4 (\zeta^4 I + A^{4\upsilon})^{-1} \hat{H}(\zeta) \right\|^2_{L^2(\mathbb{R}^+;H)} + \left\| A^{4\upsilon} (\zeta^4 I + A^{4\upsilon})^{-1} \hat{H}(\zeta) \right\|^2_{L^2(\mathbb{R}^+;H)}. \end{aligned}$$

Then

$$\begin{aligned} &\|v\|^2_{W^{4,\upsilon}(\mathbb{R}^+;H)} \\ &\leq \left(\sup_{\xi \in \mathbb{R}} \left\| \zeta^4 (\xi^4 I + A^{4\upsilon})^{-1} \right\|_{\mathcal{L}(H,H)} + \sup_{\xi \in \mathbb{R}} \left\| A^{4\upsilon} (\zeta^4 I + A^{4\upsilon})^{-1} \right\|_{\mathcal{L}(H,H)} \right) \|H\|^2_{L^2(\mathbb{R}^+;H)}. \end{aligned}$$

According to the classical spectral theory of self-adjoint operators, we obtain

$$\left\| \zeta^4 (\zeta^4 I + A^{4\upsilon})^{-1} \right\|_{\mathcal{L}(H,H)} \leq \sup_{\lambda \in \sigma(A^\upsilon)} \left| \zeta^4 (\zeta^4 + \lambda^4)^{-1} \right| \leq 1,$$

and

$$\left\| A^{4\upsilon} (\zeta^4 I + A^{4\upsilon})^{-1} \right\|_{\mathcal{L}(H,H)} \leq \sup_{\lambda \in \sigma(A^\upsilon)} \left| \lambda^4 (\zeta^4 + \lambda^4)^{-1} \right| \leq 1;$$

hence $v \in W^{4,\upsilon}(\mathbb{R}; H)$.

Step 2 Put

$$w_1(t) := v(t)|_{\mathbb{R}^+}.$$

Then $w_1 \in W^{4,\upsilon}(\mathbb{R}^+; H)$ and satisfies the equation (2.6) almost everywhere in $\mathbb{R}^+$. On the other hand, the trace theorem [20, Chapter 1] yields that

$$\frac{d^j w_1(0)}{dt^4} \in H_{(7/2-j)\upsilon},\ j = 0, 1, 2, 3.$$

Similarly, as in the previous step, the solution of problem (2.6)-(2.7) can be written in the following form

$$w(t) = w_1(t) + e^{\eta_1 t A^\upsilon} \phi_1 + e^{\eta_2 t A^\upsilon} \phi_2,$$

where

$$\eta_1 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \quad \text{and} \quad \eta_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i,$$

$\phi_1,\ \phi_2 \in H_{7\upsilon/2}$; see also (2.7). Consequently, we obtain the following system

$$\begin{cases} \dfrac{dw_1(0)}{dt} + \eta_1 A^{\upsilon}\phi_1 + \eta_2 A^{\upsilon}\phi_2 = 0, \\ \dfrac{d^3 w_1(0)}{dt^3} + \eta_1^3 A^{3\upsilon}\phi_1 + \eta_2^3 A^{3\upsilon}\phi_2 = -K(w_1(0) + \phi_1 + \phi_2). \end{cases} \tag{2.15}$$

Taking into account that

$$\varphi_2 = -\frac{\eta_1}{\eta_2}\varphi_1 - \frac{1}{\eta_2} A^{-\upsilon} \frac{dw_1(0)}{dt},$$

and keeping in mind the condition $-\sqrt{2} \notin \sigma(B)$, we deduce that

$$\varphi_1 = A^{-7\upsilon/2}(\sqrt{2}I + B)^{-1} A^{7\upsilon/2}\eta \in H_{7\upsilon/2},$$

where

$$\eta = -\frac{1-i}{2} A^{-3\upsilon} \left(\frac{d^3 w_1(0)}{dt^2} + iA^{2\upsilon}\frac{dw_1(0)}{dt} + Kw_1(0) - w_1 K A^{-\upsilon}\frac{dw_1(0)}{dt} \right) \in H_{7\upsilon/2}.$$

Thus, w belongs to the space $W^{4,\upsilon}(\mathbb{R}^+; H)$ and it is a solution to the problem (2.6)-(2.7). Moreover, the operator

$$P_0 : W_K^{4,\upsilon}(\mathbb{R}^+; H) \to L^2(\mathbb{R}^+; H),$$

is bounded. In fact, we have

$$\|P_0 w\|^2_{L^2(\mathbb{R}^+;H)} = \left\| \frac{d^4 w(t)}{dt^4} + A^{4\upsilon} w(t) \right\|^2_{L^2(\mathbb{R}^+;H)} \le 2\,\|w\|^2_{W^{4,\upsilon}(\mathbb{R}^+;H)}\,.$$

Therefore, by the Banach inverse operator theorem, we deduce that operator P_0 is invertible and

$$P_0^{-1} : L^2(\mathbb{R}^+; H) \to W_K^{4,\upsilon}(\mathbb{R}^+; H).$$

Furthermore, this operator is bounded and we obtain

$$\|w\|_{W^{4,\upsilon}(\mathbb{R}^+;H)} \le C\,\|h\|_{L^2(\mathbb{R}^+;H)}\,.$$

□

As a direct consequence of Lemma 2.2 and Theorem 2.3, we have:

Corollary 2.4. *Let B be the linear operator defined by*

$$B = A^{\upsilon/2} K A^{-7\upsilon/2},$$

and let $-\sqrt{2} \notin \sigma(B)$. Then the operator P_0 defined by (2.8) *is an isomorphism.*

Let us prove now the following coercive inequality, which will be used later on:

Lemma 2.5. *Let B the linear operator defined by $B = A^{\upsilon/2} K A^{-7\upsilon/2}$, with $\operatorname{Re}(B) \ge 0$. Then, for every $w \in W_K^{4,\upsilon}(\mathbb{R}^+; H)$, the following inequality holds true:*

$$\|P_0 w\|^2_{L^2(\mathbb{R}^+;H)} \ge \|w\|^2_{W^{4,\upsilon}(\mathbb{R}^+;H)} + 2\left\| A^{2\upsilon}\frac{d^2 w}{dt^2} \right\|^2_{L^2(\mathbb{R}^+;H)}. \tag{2.16}$$

Proof. For $w \in W_K^{4,\upsilon}(\mathbb{R}^+; H)$, we have

$$\begin{aligned}&\|P_0 w\|^2_{L^2(\mathbb{R}^+;H)} \\ &= \left\|\frac{d^4 w}{dt^4}\right\|^2_{L^2(\mathbb{R}^+;H)} + \|A^{4\upsilon} w\|^2_{L^2(\mathbb{R}^+;H)} + 2\,\mathrm{Re}\left(< \frac{d^4 w}{dt^4}, A^{4\upsilon} w >_{L^2(\mathbb{R}^+;H)}\right).\end{aligned} \tag{2.17}$$

On the other hand, integrating by parts, we obtain

$$\begin{aligned}< \frac{d^4 w}{dt^4}, A^{4\upsilon} w >_{L^2(\mathbb{R}^+;H)} &= \left[< \frac{d^3 w}{dt^3}(t), A^{4\upsilon} w(t) >\right]_0^{+\infty} - \int_0^{+\infty} < \frac{d^3 w(t)}{dt^3}, A^{4\upsilon} \frac{dw(t)}{dt} > dt \\ &= < Kw(0), A^{4\upsilon} w(0) > + \int_0^{+\infty} < A^{2\upsilon} \frac{d^2 w(t)}{dt^2}, A^{2\upsilon} \frac{d^2 w(t)}{dt^2} > dt \\ &= < BA^{7\upsilon/2} w(0), A^{7\upsilon/2} w(0) > + \left\|A^{2\upsilon} \frac{d^2 w}{dt^2}\right\|^2_{L^2(\mathbb{R}^+;H)}.\end{aligned} \tag{2.18}$$

Taking into account the fact that $\mathrm{Re}\,(B) \geq 0$, the estimate (2.16) is easily deduced from relation (2.18). □

Observe here that Corollary 2.4 implies that the quantity $\|P_0 w\|_{L^2(\mathbb{R}^+;H)}$ is equivalent to $\|w\|_{W^{4,\upsilon}(\mathbb{R}^+;H)}$ in the space $W_K^{4,\upsilon}(\mathbb{R}^+; H)$. Moreover, the norms of the intermediate derivative operators

$$A^{j\upsilon} \frac{d^{4-j}}{dt^{4-j}} : W_K^{4,\upsilon}(\mathbb{R}^+; H) \to L^2(\mathbb{R}^+; H), \quad j = 1, 2, 3, 4,$$

can be estimated with respect to $\|P_0 w\|_{L^2(\mathbb{R}^+;H)}$ (by the same continuity argument used in [20]).

Theorem 2.6. *Under the assumptions of* Lemma 2.5*, the following estimates hold true*

$$\left\|A^{j\upsilon} \frac{d^{4-j} w}{dt^{4-j}}\right\|_{L^2(\mathbb{R}^+;H)} \leq a_j \|P_0 w\|_{L^2(\mathbb{R}^+;H)}, \quad j = 1, 2, 3, 4, \tag{2.19}$$

for any $w \in W_K^{4,\upsilon}(\mathbb{R}^+; H)$ *with*

$$a_0 = a_1 = a_4 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{\sqrt{2}}.$$

Proof. Let $w \in W_K^{4,\upsilon}(\mathbb{R}^+; H)$. From the equality (2.18), we have

$$\begin{aligned}&\mathrm{Re}\left(< P_0 w, A^{4\upsilon} w >_{L^2(\mathbb{R}^+;H)}\right) \\ &= \|A^{4\upsilon} w\|^2_{L^2(\mathbb{R}^+;H)} + \mathrm{Re}\left(< BA^{7\upsilon/2} w(0), A^{7\upsilon/2} w(0) >\right) + \left\|A \frac{d^2 w}{dt^2}\right\|^2_{L^2(\mathbb{R}^+;H)}.\end{aligned}$$

Then we can see that

$$\mathrm{Re}\left(< P_0 w, A^{4\upsilon} w >_{L^2(\mathbb{R}^+;H)}\right) \geq \|A^{4\upsilon} w\|^2_{L^2(\mathbb{R}^+;H)} + \left\|A^{2\upsilon} \frac{d^2 w}{dt^2}\right\|^2_{L^2(\mathbb{R}^+;H)}.$$

Applying the Cauchy-Schwarz inequality and the Young inequality, we conclude that

$$\|A^{4\upsilon} w\|^2_{L^2(\mathbb{R}^+;H)} + \left\|A^{2\upsilon} \frac{d^2 w}{dt^2}\right\|^2_{L^2(\mathbb{R}^+;H)} \leq \|P_0 w\|_{L^2(\mathbb{R}^+;H)} \left\|A^{4\upsilon} w\right\|_{L^2(\mathbb{R}^+;H)}, \tag{2.20}$$

from which we may deduce that

$$\|A^{4\upsilon} w\|^2_{L^2(\mathbb{R}^+;H)} + \left\|A^{2\upsilon} \frac{d^2 w}{dt^2}\right\|^2_{L^2(\mathbb{R}^+;H)} \leq \frac{\delta}{2} \|P_0 w\|^2_{L^2(\mathbb{R}^+;H)} + \frac{1}{2\delta} \left\|A^{4\upsilon} w\right\|^2_{L^2(\mathbb{R}^+;H)}, \tag{2.21}$$

with $\delta > 0$.

Choosing $\delta = \frac{1}{2}$ in (2.21), we get

$$\left\| A^{2\upsilon} \frac{d^2 w}{dt^2} \right\|_{L^2(\mathbb{R}^+;H)} \leq \frac{1}{2} \left\| P_0 w \right\|_{L^2(\mathbb{R}^+;H)} . \tag{2.22}$$

On the other hand, (2.20) yields

$$\left\| A^{4\upsilon} w \right\|^2_{L^2(\mathbb{R}^+;H)} \leq \left\| P_0 w \right\|_{L^2(\mathbb{R}^+;H)} \left\| A^{4\upsilon} w \right\|_{L^2(\mathbb{R}^+;H)} ,$$

which implies that

$$\left\| A^{4\upsilon} w \right\|_{L^2(\mathbb{R}^+;H)} \leq \left\| P_0 w \right\|_{L^2(\mathbb{R}^+;H)} . \tag{2.23}$$

Now the inequality (2.16) implies

$$\left\| \frac{d^4 w}{dt^4} \right\|_{L^2(\mathbb{R}^+;H)} \leq \left\| P_0 w \right\|_{L^2(\mathbb{R}^+;H)} . \tag{2.24}$$

Let us estimate now the norm $\left\| A^{3\upsilon} \frac{dw}{dt} \right\|_{L^2(\mathbb{R}^+;H)}$. Taking into account that $w \in W^{4,\upsilon}_K(\mathbb{R}^+; H)$, the use of the Cauchy-Schwarz inequality combined with inequalities (2.22)-(2.23) allows us to conclude that

$$\begin{aligned} &\left\| A^{3\upsilon} \frac{dw}{dt} \right\|^2_{L^2(\mathbb{R}^+;H)} \\ &= \left[< A^{3\upsilon} w(t), A^{3\upsilon} \frac{dw(t)}{dt} > \right]_0^{+\infty} - \int_0^{+\infty} < A^{4\upsilon} w(t), A^{2\upsilon} \frac{d^2 w(t)}{dt^2} > dt, \end{aligned}$$

so

$$\left\| A^{3\upsilon} \frac{dw}{dt} \right\|^2_{L^2(\mathbb{R}^+;H)} \leq \left\| A^{2\upsilon} \frac{d^2 w}{dt^2} \right\|_{L^2(\mathbb{R}^+;H)} \left\| A^{4\upsilon} w \right\|_{L^2(\mathbb{R}^+;H)} \leq \frac{1}{2} \left\| P_0 w \right\|^2_{L^2(\mathbb{R}^+;H)} .$$

Consequently,

$$\left\| A^{3\upsilon} \frac{dw}{dt} \right\|_{L^2(\mathbb{R}^+;H)} \leq \frac{1}{\sqrt{2}} \left\| P_0 w \right\|_{L^2(\mathbb{R}^+;H)} . \tag{2.25}$$

Finally, let us estimate the quantity $\left\| A^{\upsilon} \frac{d^2 w}{dt^2} \right\|_{L^2(\mathbb{R}^+;H)}$. We know that, for $w \in W^{4,\upsilon}(\mathbb{R}^+; H)$, we have:

$$\left\| A^{\upsilon} \frac{d^3 w}{dt^3} \right\|^2_{L^2(\mathbb{R}^+;H)} \leq 2 \left\| A^{2\upsilon} \frac{d^2 w}{dt^2} \right\|_{L^2(\mathbb{R}^+;H)} \left\| \frac{d^4 w}{dt^4} \right\|_{L^2(\mathbb{R}^+;H)} . \tag{2.26}$$

Inserting the inequalities (2.22) and (2.24) in (2.26), we get

$$\left\| A^{\upsilon} w''' \right\|_{L^2(\mathbb{R}^+;H)} \leq \left\| P_0 w \right\|_{L^2(\mathbb{R}^+;H)} , \tag{2.27}$$

which ends the proof of this theorem. □

It is worth noting that the coefficient operator A, in our boundary value problem, was considered with a positive natural power so far. From now on, we will treat our problem in general case, where the considered coefficient operators will be of the form $A^{\upsilon}, \upsilon \in (0,1)$. To this end, let us consider the following abstract Cauchy problem:

$$\frac{d^4 w(t)}{dt^4} + A^{4\upsilon} w(t) + \sum_{j=1}^{4} A_j \frac{d^{4-j} w(t)}{dt^{4-j}} = h(t), \ t \in \mathbb{R}^+, \tag{2.28}$$

$$\frac{dw(0)}{dt}=0,\ \frac{d^3w(0)}{dt^3}=-Kw(0). \tag{2.29}$$

Put

$$\begin{array}{lll} P: & W_K^{4,\upsilon}(\mathbb{R}^+;H) & \to L^2(\mathbb{R}^+;H), \\ & w(t) & \mapsto Pw(t):=\dfrac{d^4w(t)}{dt^4}+A^{4\upsilon}w(t)+\sum\limits_{j=1}^{4}A_j\dfrac{d^{4-j}w(t)}{dt^{4-j}}. \end{array} \tag{2.30}$$

The first auxiliary result concerning this operator is formulated as follows:

Lemma 2.7. *Assume that $A_jA^{-j\upsilon}\in\mathcal{L}(H,H),\ j=1,2,3,4$. Then the operator P, defined by (2.30), is bounded.*

Proof. Let $w\in W_K^{4,\upsilon}(\mathbb{R}^+;H)$. Then we have

$$\begin{aligned} \|Pw\|_{L^2(\mathbb{R}^+;H)} &\leq \|P_0w\|_{L^2(\mathbb{R}^+;H)}+\left\|\sum_{j=1}^{4}A_j\frac{d^{4-j}w(t)}{dt^{4-j}}\right\|_{L^2(\mathbb{R}^+;H)} \\ &\leq \sqrt{2}\,\|w\|_{W^{4,\upsilon}(\mathbb{R}^+;H)}+\left\|\sum_{j=1}^{4}A_j\frac{d^{4-j}w(t)}{dt^{4-j}}\right\|_{L^2(\mathbb{R}^+;H)} \\ &\leq \sqrt{2}\,\|w\|_{W^{4,\upsilon}(\mathbb{R}^+;H)}+\sum_{j=1}^{4}\left\|A_jA^{-j\upsilon}\right\|_{\mathcal{L}(H,H)}\left\|A^{j\upsilon}\frac{d^{4-j}w(t)}{dt^{4-j}}\right\|_{L^2(\mathbb{R}^+;H)}. \end{aligned}$$

Using the theorem for intermediate derivatives in [20], we deduce that

$$\|Pw\|_{L^2(\mathbb{R}^+;H)}\leq C\,\|w\|_{W^{4,\upsilon}(\mathbb{R}^+;H)}.$$

□

Let us state our essential results concerning the problem (2.28)-(2.29) performed in the space $L^2(\mathbb{R}^+;H)$.

Theorem 2.8. *Let $B=A^{\upsilon/2}KA^{-7\upsilon/2}$, let*

$$\begin{cases} \operatorname{Re}(B)\geq 0, \\ \textit{and} \\ A_jA^{-j\upsilon}\in\mathcal{L}(H,H),\ j=1,2,3,4. \end{cases}$$

Assume also that

$$a=\sum_{j=1}^{4}a_i\left\|A_jA^{-j\upsilon}\right\|_{\mathcal{L}(H,H)}<1,$$

with

$$a_1=1,\ a_2=\frac{1}{2},\ a_3=\frac{1}{\sqrt{2}},\ a_4=1.$$

Then, for every $h\in L^2(\mathbb{R}^+;H)$, the boundary value problem (2.28)-(2.29) has a unique regular solution.

Proof. First, we rewrite the boundary value problem (2.28)-(2.29) in the form of operator equation

$$P_0w(t)+(P-P_0)w(t)=h(t),\ t\in\mathbb{R}^+, \tag{2.31}$$

where $h\in L^2(\mathbb{R}^+;H)$ and $w\in W_K^{4,\upsilon}(\mathbb{R}^+;H)$.

The technical conditions

$$B=A^{\upsilon/2}KA^{-7\upsilon/2},\ \operatorname{Re}(B)\geq 0$$

ensure that the operator

$$P_0^{-1}:L^2(\mathbb{R}^+;H)\to W_K^{4,\upsilon}(\mathbb{R}^+;H)$$

is well defined. Set $w(t) := P_0^{-1}v(t)$, with $v \in L^2(\mathbb{R}^+; H)$. Then a direct computation shows that $v(t)$ satisfies the following equation

$$v(t) + (P - P_0)P_0^{-1}v(t) = h(t),\ t \in \mathbb{R}^+.$$

Keeping in mind that $v \in L^2(\mathbb{R}^+; H)$ and taking into account the estimates (2.19), one has

$$\left\|(P - P_0)P_0^{-1}v\right\|_{L^2(\mathbb{R}^+;H)} = \|(P - P_0)w\|_{L^2(\mathbb{R}^+;H)},$$

so

$$\begin{aligned}
&\left\|(P - P_0)P_0^{-1}v\right\|_{L^2(\mathbb{R}^+;H)} \\
&\le \sum_{j=1}^{4} \left\|A_j A^{-j\upsilon}\right\|_{\mathcal{L}(H,H)} \left\|A^{j\upsilon}\frac{d^{4-j}w(t)}{dt^{4-j}}\right\|_{L^2(\mathbb{R}^+;H)} \\
&\le \sum_{j=1}^{4} a_j \left\|A_j A^{-j\upsilon}\right\|_{\mathcal{L}(H,H)} \|P_0 w\|_{L^2(\mathbb{R}^+;H)}.
\end{aligned}$$

Therefore,

$$\left\|(P - P_0)P_0^{-1}v\right\|_{L^2(\mathbb{R}^+;H)} = a\,\|v\|_{L^2(\mathbb{R}^+;H)}.$$

Since $a < 1$, the operator

$$\left(I + (P - P_0)P_0^{-1}\right)^{-1}$$

is well defined in the space $L^2(\mathbb{R}^+; H)$. Consequently, the equation (2.31) is uniquely solvable in the space $W_K^{4,\upsilon}(\mathbb{R}^+; H)$, and

$$w(t) = P_0^{-1}\left(I + (P - P_0)P_0^{-1}\right)^{-1}h(t).$$

Moreover,

$$\begin{aligned}
&\|w\|_{W^{4,\upsilon}(\mathbb{R}^+;H)} \\
&\le \left\|P_0^{-1}\right\|_{\mathcal{L}(L^2(\mathbb{R}^+;H),W^{4,\upsilon}(\mathbb{R}^+;H))} \left\|(I + (P - P_0)P_0^{-1})^{-1}\right\|_{\mathcal{L}(L^2(\mathbb{R}^+;H),L^2(\mathbb{R}^+;H))} \|h\|_{L^2(\mathbb{R}^+;H)} \\
&\le C\,\|h\|_{L^2(\mathbb{R}^+;H)}.
\end{aligned}$$

□

Remark 2.9. In Theorem 2.8, the condition $\operatorname{Re}(B) \ge 0$ with $B = A^{\upsilon/2}KA^{-7\upsilon/2}$, allows us to omit the condition $-\sqrt{2} \notin \sigma(B)$.

Finally, we may get the conditions for the regular solvability of the boundary value problem (2.2)-(2.3) from Theorem 2.8:

Theorem 2.10. *Assume that all conditions of Theorem 2.8 are fulfilled. Then the boundary value problem* (2.2)-(2.3) *is regularly solvable.*

Proof. In the case $\varphi_1 = \varphi_2 = 0$, the regular solvability of the boundary value problem (2.2)-(2.3) was established.

In the case $A_j = 0$, $j = 1, 2, 3, 4$, and $h = 0$, the problem (2.2)-(2.3) is reduced to the new one given by

$$\frac{d^4w(t)}{dt^4} + A^{4\upsilon}w(t) = 0,\ t \in \mathbb{R}^+, \tag{2.32}$$

$$\frac{dw(0)}{dt} = \varphi_1,\ \frac{d^3w(0)}{dt^3} + Kw(0) = \varphi_2, \tag{2.33}$$

with $\varphi_1 \in H_{5v/2},\ \varphi_2 \in H_{v/2}$. The solution of problem (2.32)-(2.33) can be written as follows

$$w_0(t) = e^{\eta_1 t A^v}\phi_1 + e^{\eta_2 t A^v}\phi_2, \tag{2.34}$$

where

$$\eta_1 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \ \text{ and } \ \eta_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,$$

and $\phi_1,\ \phi_2$ are the unknown vectors to be determined from the conditions (2.33):

$$\begin{cases} \eta_1 A^v\phi_1 + \eta_2 A^v\phi_2 = \varphi_1, \\ A^{3v}(\eta_1^3\phi_1 + \eta_2^3\phi_2) + K(\phi_1+\phi_2) = \varphi_2. \end{cases} \tag{2.35}$$

System (2.35) yields

$$\phi_2 = \frac{1}{\eta_2}\left(A^{-v}\varphi_1 - \eta_1\phi_1\right),$$

$$(\sqrt{2}I + B)A^{-7v/2}\phi_1 = (i-1)A^{v/2}(\varphi_2 - \eta_2^2 A^{2v}\varphi_1 - \eta_1 A^{-v}\varphi_1);$$

since $-\sqrt{2} \notin \sigma(B)$, we have

$$\phi_1 = -\frac{(1-i)}{2}A^{7v/2}(\sqrt{2}I + B)A^{v/2}(\varphi_2 - \eta_2^2 A^{2v}\varphi_1 - \eta_1 A^{-v}\varphi_1),$$

thus

$$\phi_2 = \eta_1 A^{-v}\varphi_1 + (\frac{1-i}{2})A^{7v/2}(\sqrt{2}I + B)A^{v/2}(\varphi_2 - \eta_2^2 A^{2v}\varphi_1 - \eta_1 A^{-v}\varphi_1).$$

It is not difficult to show that $\phi_1,\ \phi_2 \in H_{7v/2}$. From (2.34), we obtain

$$\begin{aligned} \|w\|_{W^{4,v}(\mathbb{R}^+;H)} &\leq C\left(\|\phi_1\|_{H_{7v/2}} + \|\phi_2\|_{H_{7v/2}}\right) \\ &\leq C\left(\|\varphi_1\|_{H_{5v/2}} + \|\varphi_2\|_{H_{v/2}}\right). \end{aligned} \tag{2.36}$$

Now, we are able to study the boundary value problem (2.2)-(2.3). We will seek its solutions in the form $w(t) = v(t) + w_0(t)$, where $w_0(t)$ is a regular solution of the problem (2.32)-(2.33). Then the function $v(t)$ is the boundary value problem solution to

$$\frac{d^4v(t)}{dt^4} + A^{4v}v(t) + \sum_{j=1}^{4} A_j\frac{d^{4-j}v(t)}{dt^{4-j}} = g(t),\ t \in \mathbb{R}^+, \tag{2.37}$$

$$\frac{dv(0)}{dt} = 0,\ \frac{d^3v(0)}{dt^3} + Kv(0) = 0, \tag{2.38}$$

where

$$g(t) = -\sum_{j=1}^{4} A_j\frac{d^{4-j}w_0(t)}{dt^{4-j}} + h(t).$$

Let us estimate the quantity $\|g\|_{L^2(\mathbb{R}^+;H)}$. We have

$$\begin{aligned} \|g\|_{L^2(\mathbb{R}^+;H)} &\leq \left\|\sum_{j=1}^{4} A_j\frac{d^{4-j}w_0(t)}{dt^{4-j}}\right\|_{L^2(\mathbb{R}^+;H)} + \|h\|_{L^2(\mathbb{R}^+;H)} \\ &\leq \sum_{j=1}^{4}\|A_jA^{-jv}\|_{\mathcal{L}(H)}\left\|A^{jv}\frac{d^{4-j}w_0(t)}{dt^{4-j}}\right\|_{L^2(\mathbb{R}^+;H)} + \|h\|_{L^2(\mathbb{R}^+;H)} \\ &\leq C\left(\|\varphi_1\|_{H_{5v/2}} + \|\varphi_2\|_{H_{v/2}} + \|h\|_{L^2(\mathbb{R}^+;H)}\right). \end{aligned}$$

Thanks to Theorem 2.8 and the estimate (2.36), we have

$$\begin{aligned}\|w\|_{W^{4,\upsilon}(\mathbb{R}^+;H)} &\leq \|v\|_{W^{4,\upsilon}(\mathbb{R}^+;H)} + \|w_0\|_{W^{4,\upsilon}(\mathbb{R}^+;H)}\\ &\leq \|g\|_{L^2(\mathbb{R}^+;H)} + \|w_0\|_{W^{4,\upsilon}(\mathbb{R}^+;H)}\\ &\leq C\left(\|\varphi_1\|_{H_{5\upsilon/2}} + \|\varphi_2\|_{H_{\upsilon/2}} + \|g\|_{L^2(\mathbb{R}^+;H)}\right).\end{aligned}$$

□

3. Existence of the Solution to the Main Problem

In this section, we return to the original problem. In order to provide a comprehensive study of the problem (1.1)-(1.2)-(1.3), we need some intermediate results which can be viewed as a direct consequence of the results obtained in the previous section.

Remark 3.1. To simplify the computations involving functional spaces and make the study more comprehensible, we consider the case when $\upsilon = 1/8$. Thus, from (1.14) and (2.1), the space $W^{4,\upsilon}(\mathbb{R}^+, L^2(Q))$ is defined as follows:

$$W^{4,\upsilon}(\mathbb{R}^+, L^2(Q)) = \left\{w : \ w^{(4)} \in L^2\left(\mathbb{R}^+, H\right), \ w \in L^2\left(\mathbb{R}^+, H_0^1\left(Q\right)\right)\right\}.$$

Keeping in mind the definition of the operators $(A, D(A))$, $(A_j, D(A_j))$ and $(K, D(K))$, defined respectively by (1.12)-(1.11) and (1.13), our main result for the transformed problem (1.9)-(1.10) is formulated as follows:

Theorem 3.2. *Let $h \in L^2(\mathbb{R}^+ \times Q)$. Assume that*

$$\operatorname{Re}(b) \geq 0, \quad \sum_{j=1}^{4} \sup_{\xi\in Q} |\sigma_j(\xi)| < 1.$$

Then, the problem

$$\frac{d^4 w(t)}{dt^4} + A^{4\upsilon} w(t) + \sum_{j=1}^{4} A_j \frac{d^{4-j} w(t)}{dt^{4-j}} = h(t), \ t \in \mathbb{R}^+,$$

with

$$\frac{dw(0)}{dt} = 0, \ \frac{d^3 w(0)}{dt^3} + Kw(0) = 0,$$

has a unique regular solution $w \in W^{4,\upsilon}(\mathbb{R}^+, L^2(Q))$.

By applying the classical perturbation argument as described in [4, Section 3, p. 49], we conclude that

Theorem 3.3. *Let $h \in L^2(\mathbb{R}^+ \times Q)$. Assume that*

$$\operatorname{Re}(b) \geq 0, \qquad \sum_{j=1}^{4} \sup_{x\in\Omega} |\sigma_j(\xi)| < 1.$$

Then, the problem (1.6)-(1.7)-(1.8) *has a unique regular solution $w \in W^{4,\upsilon}(\mathbb{R}^+, L^2(Q))$.*

Consider now the inverse change of variables

$$\begin{aligned}\Psi^{-1} : \ &\mathbb{R}^+ \times Q \to \mathbb{R}^+ \times \Pi\\ &(t, \xi) \mapsto (t, x),\end{aligned}$$

with

$$x_1 = \left(\frac{\gamma}{\xi_3}\right)^{\frac{\alpha}{\beta}} \xi_1, \qquad x_2 = \left(\frac{\gamma}{\xi_3}\right)^{\frac{\alpha}{\beta}} \xi_2, \qquad x_3 = \left(\frac{\gamma}{\xi_3}\right)^{\frac{1}{\beta}}.$$

We have

$$w=\left(\frac{\gamma}{\xi_3}\right)^{-\frac{\alpha}{\beta}\left(\frac{3\alpha}{8v}+2\right)}u\left(\left(\frac{\gamma}{\xi_3}\right)^{\frac{\alpha}{\beta}}\xi_1,\left(\frac{\gamma}{\xi_3}\right)^{\frac{\alpha}{\beta}}\xi_2,\left(\frac{\gamma}{\xi_3}\right)^{\frac{1}{\beta}}\right);$$

this gives

$$w=\left(\frac{\gamma}{\xi_3}\right)^{-\frac{3\alpha}{2\beta}}(x_3)^{-\alpha\left(\frac{3}{2v}+\frac{1}{2}\right)}u. \tag{3.1}$$

In an equivalent manner,

$$\partial_{\xi_1}w=\left(\frac{\gamma}{\xi_3}\right)^{-\frac{3\alpha}{2\beta}}(x_3)^{-\alpha\left(\frac{3}{2v}-\frac{1}{2}\right)}\partial_{x_1}u, \tag{3.2}$$

and

$$\partial_{\xi_2}w=\left(\frac{\gamma}{\xi_3}\right)^{-\frac{3\alpha}{2\beta}}(x_3)^{-\alpha\left(\frac{3}{2v}-\frac{1}{2}\right)}\partial_{x_2}u. \tag{3.3}$$

Due to the fact that w, $\partial_{\xi_1}w$, and $\partial_{\xi_2}w$ are L^2-integrable in Q, (1.5) with (3.1)-(3.2)-(3.3) implies that

$$(x_3)^{-\alpha\left(\frac{3}{2v}+\frac{1}{2}\right)}u,\ (x_3)^{-\alpha\left(\frac{3}{2v}-\frac{1}{2}\right)}\partial_{x_1}u,(x_3)^{-\alpha\left(\frac{3}{2v}-\frac{1}{2}\right)}\partial_{x_2}u\in L^2(\Pi). \tag{3.4}$$

Furthemore, a direct computation shows that

$$\begin{aligned}
&\partial_{x_2}w\\
&=\left(\frac{\gamma}{\xi_3}\right)^{-s}\left[s\xi_3^{-1}u-\frac{\alpha}{\beta}\xi_1\xi_3^{-1}\left(\frac{\gamma}{\xi_3}\right)^{\frac{\alpha}{\beta}}\partial_{x_1}u-\frac{\alpha}{\beta}\xi_2\xi_3^{-1}\left(\frac{\gamma}{\xi_3}\right)^{\frac{\alpha}{\beta}}\partial_{x_2}u\right.\\
&\left.-\frac{1}{\beta}\xi_3^{-1}\left(\frac{\gamma}{\xi_3}\right)^{\frac{1}{\beta}}\partial_{x_3}u\right]\\
&=\left(\frac{\gamma}{\xi_3}\right)^{-\frac{3\alpha}{2\beta}}\left[s\xi_3^{-1}x_3^{-\alpha\left(\frac{3}{8v}+\frac{1}{2}\right)}u-\frac{\alpha}{\beta}\xi_1\xi_3^{-1}x_3^{-\alpha\left(\frac{3}{8v}-\frac{1}{2}\right)}\partial_{x_1}u-\frac{\alpha}{\beta}\xi_2\xi_3^{-1}x_3^{-\alpha\left(\frac{3}{8v}-\frac{1}{2}\right)}\partial_{x_2}u\right.\\
&\left.-\frac{1}{\gamma\beta}x_3^{-\alpha\left(\frac{3}{8v}-\frac{1}{2}\right)}\partial_{x_3}u\right].
\end{aligned}$$

Since $\partial_{\xi_3}w$ is L^2-integrable in Q, according to the previous calculations and (3.4), we obtain

$$x_3^{-\alpha\left(\frac{3}{8v}-\frac{1}{2}\right)}\partial_{x_3}u\in L^2(\Pi).$$

In summary, the following proposition has been established.

Proposition 3.4. *The fact that $w\in W^{4,v}(\mathbb{R}^+,L^2(Q))$ implies that*

$$u\in W^{4,v}(\mathbb{R}^+,L^2(\Pi)).$$

This help us to justify our main result set in the cusp domain $\mathbb{R}^+\times\Pi$:

Theorem 3.5. *Let $f\in L^2(\mathbb{R}^+\times\Pi)$. Assume that*

$$\operatorname{Re}(b)\geq 0 \ \text{and}\ \sum_{j=1}^{4}\sup_{x\in\Pi}|\rho_j(x)|<1,$$

Then, the problem

$$\frac{d^4}{dt^4}u(t,x)+(1+\rho_4(x))(-\Delta)^{4v}u(t,x)+\sum_{j=1}^{3}\left(\rho_j(x)(-\Delta)^{jv}\right)\frac{d^{4-j}}{dt^{4-j}}u(t,x)=f(t,x),$$

$$u|_{\mathbb{R}^+\times\partial\Pi} = 0,$$

$$\left.\frac{du}{dt}\right|_{\{0\}\times\Pi} = 0 \ and \quad \left.\frac{d^3u}{dt^3} + b(-\Delta)^{3\upsilon}u\right|_{\{0\}\times\Pi} = 0,$$

has a unique regular solution $u \in W^{4,\upsilon}(\mathbb{R}^+, L^2(\Pi))$.

4. Conclusion

This work provides significant insights into the boundary value problem on a singular domain involving a cuspidial point by using semigroup theory and fractional powers of linear operators on Hilbert spaces. Specifically, we have established the existence and uniqueness of solutions for the problem given by

$$\frac{d^4}{dt^4}u\,(t,x) + (1+\rho_4\,(x))(-\Delta)^{1/2}u\,(t,x) + \sum_{j=1}^{3}\left(\rho_j\,(x)\,(-\Delta)^{j/8}\right)\frac{d^{4-j}}{dt^{4-j}}u\,(t,x) = f\,(t,x)\,,$$

$$u|_{\mathbb{R}^+\times\partial\Pi} = 0,$$

and

$$\left.\frac{du}{dt}\right|_{\{0\}\times\Pi} = 0, \quad \left.\frac{d^3u}{dt^3} + b(-\Delta)^{3/8}u\right|_{\{0\}\times\Pi} = 0.$$

on the cusp domain $\mathbb{R}^+\times\Pi$, where

$$\Pi := \left\{x\in\mathbb{R}^3 : 0 < x_3 < 1, \ \left(\frac{x_1}{(x_3)^\alpha}, \frac{x_2}{(x_3)^\alpha}\right)\in\Omega\right\}, \quad \alpha > 1,$$

and

$$f \in L^2(\mathbb{R}^+\times\Pi).$$

Furthermore, we have furnished some sufficient conditions for the wellposedness and regular solvability of a class of complete abstract fourth-order differential equations

$$\frac{d^4w(t)}{dt^4} + A^{4\upsilon}w(t) + \sum_{j=1}^{4}A_j\frac{d^{4-j}w(t)}{dt^{4-j}} = h(t), \ t\in\mathbb{R}^+,$$

endowed with nonhomogeneous abstract boundary conditions

$$\frac{dw(0)}{dt} = \varphi_1 \in H_{5\upsilon/2}, \quad \frac{d^3w(0)}{dt^3} + Kw(0) = \varphi_2 \in H_{\upsilon/2},$$

where, $\upsilon \in\ ]0,1]$, A is a self-adjoint positive definite operator in a separable Hilbert space H, A_j, $j \in \{1,2,3,4\}$ are linear operators acting on H, $K \in \mathcal{L}(H_{7\upsilon/2}, H_{\upsilon/2})$ and $h \in L^2\,(\mathbb{R}^+; H)$.

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