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Journal of Prime Research in Mathematics, 22(2) (2026), 105–112

Journal of Prime Research in Mathematics

Cyclic Dynamics of Operator Tuples Under Semigroup Actions on Banach Spaces

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Abstract

This article investigates the concept of $\mathcal{G}$-cyclicity for tuples of commuting bounded linear operators on separable infinite-dimensional Banach spaces. We characterize $\mathcal{G}$-cyclic tuples and introduce the related notion of $\mathcal{G}$-transitivity. Furthermore, we establish sufficient conditions—called the $\mathcal{G}$-cyclic tuple criterion—under which a tuple becomes $\mathcal{G}$-cyclic. These results extend the classical theory of hypercyclic and supercyclic operators to a broader semigroup setting. Illustrative examples, structural results, and applications to quotient and direct sum operators are also provided.

Keywords: $\mathcal{G}$-cyclcic operators, tuple of operators.
2010 Mathematics Subject Classification:. 47A16, 47B37

1. Introduction

The study of linear dynamics on Banach spaces has undergone significant development through the notions of hypercyclicity and supercyclicity, which characterize the dense orbit behavior of linear operators. A bounded linear operator T on a separable Banach space X is called hypercyclic if there exists a vector $x \in X$ such that the orbit

$$\mathrm{Orb}(T, x) = \{T^n x : n \geq 0\}$$

is dense in X. The first construction of such an operator was given by Rolewicz [9], who showed that certain scalar multiples of the backward shift on $\ell^p(\mathbb{N})$ are hypercyclic.

A natural extension of this concept, known as supercyclicity, was introduced by Hilden and Wallen [6]. In this setting, one considers scalar multiples of orbits: an operator T is supercyclic if there exists $x \in X$ such that the projective orbit $\mathbb{C}\,\mathrm{Orb}(T, x)$ is dense in X. Later, Zeana [10] introduced the notion of diskcyclicity, where the scalar multiples are restricted to the closed unit disk. For further developments in these areas, see [1, 2, 4, 5, 7].

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Received : 23 June 2025*; Accepted:* 12 December 2025*; Published Online:* 17 July 2026

Building on these ideas, the concept of $\mathcal{G}$-cyclicity replaces the scalar field with a multiplicative semigroup $\mathcal{G} \subseteq \mathbb{C}$. An operator T is said to be $\mathcal{G}$-cyclic if there exists $x \in X$ such that the $\mathcal{G}$-orbit

$$\mathcal{G}\,\mathrm{Orb}(T,x) = \{\lambda T^n x : \lambda \in \mathcal{G},\, n \geq 0\}$$

is dense in X. This framework allows one to interpolate between various notions of cyclicity depending on the choice of $\mathcal{G}$; see [3, 8] for related discussions.

Let $\mathcal{T} = (T_1, \ldots, T_n)$ be a commuting n-tuple of continuous linear operators on a Banach space X, and let

$$\mathcal{F} = \{T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} : k_i \geq 0,\, 1 \leq i \leq n\}$$

be the semigroup generated by $\mathcal{T}$. The tuple $\mathcal{T}$ is called hypercyclic if there exists $x \in X$ such that

$$\mathrm{Orb}(\mathcal{T}, x) = \{Tx : T \in \mathcal{F}\}$$

is dense in X.

Hypercyclic, supercyclic, and diskcyclic tuples have been studied analogously, each defined through its respective extended orbit. In this paper, we extend the concept of $\mathcal{G}$-cyclicity to tuples of commuting operators. Specifically, we investigate the conditions under which a tuple $\mathcal{T} = (T_1, \ldots, T_n)$ is $\mathcal{G}$-cyclic; that is, when the combined $\mathcal{G}$-orbit under all operators in the tuple is dense in X. We establish that $\mathcal{G}$-cyclicity for tuples is equivalent to a dynamical mixing property, which we refer to as $\mathcal{G}$-transitivity. Moreover, we provide a general $\mathcal{G}$-cyclic tuple criterion that offers sufficient conditions for $\mathcal{G}$-cyclicity based on the convergence behavior of certain auxiliary sequences and mappings.

Finally, through illustrative examples, we demonstrate that $\mathcal{G}$-cyclicity is a genuine extension of the classical notions of hypercyclicity and supercyclicity. We also analyze the stability of this property under standard operator constructions such as invertibility, invariant subspaces, and direct sums, showing that these structures preserve or induce $\mathcal{G}$-cyclicity under appropriate conditions.

2. Main results

Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a commuting n-tuple of operators acting on an infinite dimensional Banach space X, and let $\mathcal{G}$ be a multiplication semigroup with identity in $\mathbb{C}$. Denote by $\mathcal{F}_{\mathcal{T}} = \{T_1^{k_1}, T_2^{k_2}, ..., T_n^{k_n} : k_i \geq 0, i = 1, ..., n\}$ the semigroup generated by $\mathcal{T}$. For $x \in X$, the $\mathcal{G}$-orbit of x under $\mathcal{T}$ is defined by $\mathcal{G}\mathrm{Orb}(\mathcal{T}, x) = \{\alpha S x : \alpha \in \mathcal{G}, S \in \mathcal{F}_{\mathcal{T}}\}$. We also define $\mathcal{G}^{-1} = \{1/g : g \in \mathcal{G}; g \neq 0\}$ which is itself a multiplication semigroup with identity in $\mathbb{C}$ whenever $\mathcal{G}$ is. For simplicity, we will assume $n = 2$ from now on; the proof for the general case follow analogously.

Definition 2.1. A vector $x \in X$ is called a $\mathcal{G}$-cyclic vector for a tuple of operators $\mathcal{T}$ if the set

$$\mathcal{G}\mathrm{Orb}(\mathcal{T}, x) = \{\lambda T_1^{n_1} T_2^{n_2} \cdots T_k^{n_k} x : \lambda \in \mathcal{G}, n_i \geq 0\}$$

is dense in X. In this case, the tuple $\mathcal{T}$ is called $\mathcal{G}$-cyclic. The collection of all $\mathcal{G}$-cyclic vectors for $\mathcal{T}$ is denoted by $\mathcal{G}C(\mathcal{T})$.

Remark 2.2. Let $\mathcal{T} = (T_1, T_2)$ be a tuple of commuting operators on X. Suppose, without loss of generality, that T_1 does not have dense range. Then either both $\mathcal{T}$ and T_2 are $\mathcal{G}$-cyclic, or both are not $\mathcal{G}$-cyclic. This observation allows us to consider only tuples of operators with dense range.

Indeed, suppose x is a $\mathcal{G}$-cyclic vector for the tuple $\mathcal{T}$. Define

$$B = \{\lambda T_1^{k_1} T_2^{k_2} x : \lambda \in \mathcal{G}, k_1 > 0, k_2 \geq 0\}, \quad H = \{\lambda T_2^{k_2} x : \lambda \in \mathcal{G}, k_2 \geq 0\}.$$

Since $B \subseteq B \cup H$, we have

$$X = \overline{B} \subseteq \overline{B \cup H},$$

and hence $X = \overline{B \cup H}$. Moreover, $\mathrm{int}(\overline{B}) = \emptyset$, so it follows that $X = \overline{H}$.

To see this, suppose H were not dense. Since B is meager and H is not dense, $B \cup H$ is also meager, contradicting the Baire Category Theorem. Therefore, H is dense in X, and so x is a $\mathcal{G}$-cyclic vector for T_2.

Definition 2.3. An n-tuple $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ is called $\mathcal{G}$-transitive if for every pair of nonempty open subsets U and V of X, there exist $\lambda \in \mathcal{G}$ and nonnegative integers $m_i \in \mathbb{N}_0$ for $i = 1, 2, \ldots, n$ such that

$$T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n}(\lambda U) \cap V \neq \emptyset,$$

or equivalently, there exists $\lambda \in \mathcal{G}^{-1}$ such that

$$T_1^{-m_1} T_2^{-m_2} \cdots T_n^{-m_n}(\lambda U) \cap V \neq \emptyset.$$

Example 2.4. Let θ be an irrational multiple of 2π, and let I denote the identity operator on $\mathbb{C}$. Let

$$\mathcal{G} = \{2^k : k \in \mathbb{N}_0\} \quad \text{and} \quad \mathcal{T} = \left(\frac{1}{3} I, e^{i\theta} I\right).$$

Then $\mathcal{T}$ is $\mathcal{G}$-cyclic but not hypercyclic on $\mathbb{C}$.

Proof. For any $x \in \mathbb{C}$, the $\mathcal{G}$-orbit of x under $\mathcal{T}$ is

$$\mathcal{G}\mathrm{Orb}(\mathcal{T}, x) = \left\{ \frac{2^m}{3^n} e^{ik\theta} x : m, n, k \in \mathbb{N}_0 \right\}.$$

Consider the set

$$L = \left\{ \log \frac{2^m}{3^n} : m, n \in \mathbb{N}_0 \right\} = \{ m \log 2 - n \log 3 : m, n \in \mathbb{N}_0 \}.$$

Since $\log 2$ and $\log 3$ are logarithms of distinct primes, their ratio $\frac{\log 2}{\log 3}$ is irrational. By Dirichlet's approximation theorem, the set L is dense in $\mathbb{R}$. Consequently,

$$e^L = \left\{ \frac{2^m}{3^n} : m, n \in \mathbb{N}_0 \right\}$$

is dense in $\mathbb{R}^+$, and therefore in $\mathbb{C}$ modulo the phase factor $e^{ik\theta}$.

Since $\theta/(2\pi)$ is irrational, the set $\{e^{ik\theta} : k \in \mathbb{N}_0\}$ is dense in the unit circle $\mathbb{T} \subset \mathbb{C}$. Combining this density with the density of e^L in $\mathbb{R}^+$, we conclude that

$$\mathcal{G}\mathrm{Orb}(\mathcal{T}, x)$$

is dense in $\mathbb{C}$ for any $x \neq 0$. Hence, $\mathcal{T}$ is $\mathcal{G}$-cyclic.

However, the orbit

$$\mathrm{Orb}(\mathcal{T}, x) = \left\{ 2^m e^{ik\theta} x : m, k \in \mathbb{N}_0 \right\}$$

is not dense in $\mathbb{C}$. Indeed, for any $x \neq 0$, the modulus satisfies $|2^m e^{ik\theta} x| = 2^m |x| \geq |x| > 0$, so the origin is never approximated. Therefore, $\mathcal{T}$ is not hypercyclic. $\square$

Remark 2.5. In Example 2.4, the multiplicative semigroup $\mathcal{G} = \{2^k : k \in \mathbb{N}_0\}$ plays a crucial role in generating density. The factors $2^m/3^n$ arising from the combined action of the tuple $\left(\frac{1}{3} I, e^{i\theta} I\right)$ and $\mathcal{G}$ form a dense subset of $\mathbb{R}^+$ because $\log 2 / \log 3$ is irrational. Hence, multiplication by elements of $\mathcal{G}$ allows the orbit to approximate all magnitudes in $\mathbb{C}$, whereas the orbit $\{2^m e^{in\theta} x : m, n \geq 0\}$ remains discrete and non-dense.

Example 2.6. Let $\mathcal{T} = (aI, bI, A)$, where I is the identity operator on $\mathbb{C}$, A is a continuous linear operator on $\mathbb{C}$, and let $\mathcal{G}$ be any multiplicative semigroup with identity in $\mathbb{C}$. If the tuple $\mathcal{T}$ is $\mathcal{G}$-cyclic, then A is supercyclic.

Proof. Let $x \in \mathcal{G}C(\mathcal{T})$ be a $\mathcal{G}$-cyclic vector for $\mathcal{T}$. Then the $\mathcal{G}$-orbit of x under $\mathcal{T}$ is

$$\mathcal{G}\mathrm{Orb}(\mathcal{T},x) = \left\{\lambda\, a^k b^m A^n x : k,m,n \in \mathbb{N}_0,\ \lambda \in \mathcal{G}\right\},$$

which is dense in $\mathbb{C}$ by assumption.

Let $\alpha = \lambda a^k b^m$, then the set

$$\{\alpha A^n x : n \in \mathbb{N}_0,\ \alpha \in \mathbb{C}\}$$

is dense in $\mathbb{C}$, showing that x is a supercyclic vector for A. Hence, A is a supercyclic operator. □

Theorem 2.7. *Suppose $\mathcal{T} = (T_1, T_2)$ is a $\mathcal{G}$-cyclic tuple of continuous linear operators on a separable Banach space X, and let $\{B_j\}_{j\in\mathbb{N}}$ be a countable basis of open sets for X. Then the set of $\mathcal{G}$-cyclic vectors*

$$\mathcal{G}C(\mathcal{T}) = \bigcap_{j\in\mathbb{N}}\left(\bigcup_{\lambda\in\mathcal{G}^{-1}}\ \bigcup_{m_1,m_2\geq 0} \lambda T_1^{-m_1}T_2^{-m_2}(B_j)\right)$$

is a G_δ subset of X.

Proof. A vector $x \in X$ is in $\mathcal{G}C(\mathcal{T})$ if and only if its $\mathcal{G}$-orbit under $\mathcal{T}$,

$$\mathcal{G}\mathrm{Orb}(\mathcal{T},x) = \{\alpha T_1^{m_1}T_2^{m_2}x : m_1,m_2 \geq 0,\ \alpha \in \mathcal{G}\},$$

is dense in X.

By definition of density in terms of a countable basis $\{B_j\}$, this is equivalent to saying that for every $j \in \mathbb{N}$, there exist $\alpha \in \mathcal{G}$ and integers $m_1, m_2 \geq 0$ such that

$$\alpha T_1^{m_1}T_2^{m_2}x \in B_j.$$

Equivalently, we can rewrite this condition as

$$x \in \alpha^{-1}T_1^{-m_1}T_2^{-m_2}(B_j),$$

where $\alpha^{-1} \in \mathcal{G}^{-1}$.

Hence, $x \in \bigcup_{\lambda\in\mathcal{G}^{-1}}\bigcup_{m_1,m_2\geq 0}\lambda T_1^{-m_1}T_2^{-m_2}(B_j)$ for each j, and taking the intersection over all j gives

$$x \in \bigcap_{j\in\mathbb{N}}\left(\bigcup_{\lambda\in\mathcal{G}^{-1}}\ \bigcup_{m_1,m_2\geq 0} \lambda T_1^{-m_1}T_2^{-m_2}(B_j)\right).$$

Since each $\bigcup_{\lambda\in\mathcal{G}^{-1}}\bigcup_{m_1,m_2\geq 0}\lambda T_1^{-m_1}T_2^{-m_2}(B_j)$ is open, the set $\mathcal{G}C(\mathcal{T})$ is a countable intersection of open sets, i.e., a G_δ set. □

Lemma 2.8. *Let $\mathcal{T} = (T_1, T_2)$ be a $\mathcal{G}$-cyclic tuple of continuous linear operators acting on a separable infinite dimensional Banach space X. If $\mathcal{G}C(\mathcal{T}) \neq \emptyset$, then $\mathcal{G}C(\mathcal{T})$ is dense in X.*

Proof. Let $\{B_n : n \in \mathbb{N}\}$ be a family of open balls in X with rational radius and centers in a countable dense subset of X. By the continuity of the operators T_1 and T_2, the sets

$$A_j = \bigcup\{\lambda T_1^{-k_1}T_2^{-k_2}(B_j) : k_1,k_2 \geq 0,\ \lambda \in \mathcal{G}^{-1}\}$$

are open.

Since for any two open sets U and V, there exist $k_1, k_2 \geq 0$ and $\lambda \in \mathcal{G}$ such that

$$\lambda T_1^{k_1}T_2^{k_2}(U) \cap V \neq \emptyset,$$

it follows that

$$\mathcal{G}C(\mathcal{T}) = \bigcap_{j\in\mathbb{N}} A_j,$$

which is dense in X. This completes the proof. □

Theorem 2.9. *Let $\mathcal{T} = (T_1, T_2)$ be a $\mathcal{G}$-cyclic tuple of continuous linear operators acting on a separable infinite dimensional Banach space X. If $\mathcal{F}_{\mathcal{T}}$ is a bounded set, then*

$$T_1^n T_2^p \to 0 \quad \text{as } n, p \to \infty.$$

Proof. Assume, for contradiction, that there exists $x \in \mathcal{G}C(\mathcal{T})$ such that

$$\lim_{n,p\to\infty} T_1^n T_2^p x \neq 0.$$

Then, there exists $r > 0$ such that $\|T_1^{k_j} T_2^{r_j} x\| > r$ for all $j \geq 0$.

Since $\mathcal{F}_T$ is bounded, there exists $M > 0$ such that

$$\|T_1^n T_2^p\| \leq M \quad \text{for all } n, p \geq 0.$$

Let $y \in X$ with $\|y\| = 1$ such that $\lim_{n,p\to\infty} T_1^n T_2^p y \neq 0$. By density of $\mathcal{G}$-orbit of x, there exist scalars $c_j \in \mathcal{G}$ and integers $k_j, r_j \geq 0$ such that

$$c_j T_1^{k_j} T_2^{r_j} x \to y \quad \text{as } j \to \infty. \tag{2.1}$$

Hence, there exists $k_0 \in \mathbb{N}$ such that for all $j \geq k_0$ we have $\|c_j T_1^{k_j} T_2^{r_j} x - y\| < 1/2$, which implies

$$\|c_j T_1^{k_j} T_2^{r_j} x\| > 1/2 \quad \text{and} \quad (2M\|x\|)^{-1} \leq \frac{1}{2}\|T_1^{k_j} T_2^{r_j} x\|^{-1} < |c_j| \text{ for all } j \geq k_0.$$

Define $\alpha = r(3M^2\|x\|)^{-1}$. By Equation (2.1), there exist $j_0 > k_0$, $c_{j_0} \in \mathcal{G}$, and $k_{j_0}, r_{j_0} \in \mathbb{N}$ such that

$$\|c_{j_0} T_1^{k_{j_0}} T_2^{r_{j_0}} x - y\| < \alpha/2. \tag{2.2}$$

Since $\lim_{n,p\to\infty} T_1^n T_2^p y \neq 0$, there exist $n_0, p_0 \in \mathbb{N}$ such that

$$\|T_1^{n_0} T_2^{p_0} y\| \leq M\alpha/2.$$

Put

$$L = c_{j_0} T_1^{k_{j_0}+n_0} T_2^{r_{j_0}+p_0} x - T_1^{n_0} T_2^{p_0} y.$$

Then, by Equation (2.2),

$$\|L\| = \|T_1^{n_0} T_2^{p_0}(c_{j_0} T_1^{k_{j_0}} T_2^{r_{j_0}} x - y)\| \leq \|T_1^{n_0} T_2^{p_0}\| \, \|c_{j_0} T_1^{k_{j_0}} T_2^{r_{j_0}} x - y\| < M\alpha/2.$$

Hence,

$$\begin{aligned} r(2M\|x\|)^{-1} &\leq \|c_{j_0} T_1^{k_{j_0}+n_0} T_2^{r_{j_0}+p_0} x\| \\ &\leq \|L\| + \|T_1^{n_0} T_2^{p_0} y\| \\ &< M\alpha/2 + M\alpha/2 = M\alpha \leq r(3M\|x\|)^{-1}, \end{aligned}$$

which is a contradiction.

Therefore, our assumption is false, and we must have

$$\lim_{n,p\to\infty} T_1^n T_2^p x = 0 \quad \text{for all } x \in \mathcal{G}C(\mathcal{T}).$$

Finally, by density of $\mathcal{G}C(\mathcal{T})$ in X, it follows that

$$T_1^n T_2^p x \to 0 \quad \text{for all } x \in X.$$

□

Theorem 2.10. *Let $\mathcal{T} = (T_1, T_2)$ be a $\mathcal{G}$-transitive tuple of commuting continuous operators on X. Then $\mathcal{T}$ is $\mathcal{G}$-cyclic.*

Proof. Suppose $\mathcal{T}$ is $\mathcal{G}$-transitive and let $\{B_j\}_{j\in\mathbb{N}}$ be a countable open basis of X. Then for every open set $U \subset X$ and for each $j \in \mathbb{N}$, there exist integers $m_1, m_2 \geq 0$ and a scalar $\lambda \in \mathcal{G}^{-1}$ such that

$$T_1^{-m_1} T_2^{-m_2}(\lambda B_j) \cap U \neq \emptyset.$$

Define

$$A_j = \bigcup_{\lambda\in\mathcal{G}^{-1}} \bigcup_{m_1,m_2\geq 0} \lambda T_1^{-m_1} T_2^{-m_2}(B_j).$$

Each set A_j is dense in X since it intersects every open set U. By the Baire Category Theorem,

$$\bigcap_{j\in\mathbb{N}} A_j = \mathcal{G}C(\mathcal{T})$$

is dense in X. Therefore, $\mathcal{T}$ is $\mathcal{G}$-cyclic. □

Theorem 2.11. *Let $\mathcal{T} = (T_1, T_2)$ be an n-tuple of operators on X. The following statements are equivalent:*

1. *$\mathcal{T}$ is $\mathcal{G}$-transitive.*
2. *For each $x, y \in X$, there exist sequences $\{x_j\}_{j\in\mathbb{N}} \subseteq X$, integers $m_1, m_2 \in \mathbb{N}$, and scalars $\{\lambda_j\}_{j\in\mathbb{N}} \subseteq \mathcal{G}$, such that*
$$x_j \to x \quad \text{and} \quad T_1^{m_1} T_2^{m_2}(\lambda_j x_j) \to y.$$
3. *For each $x, y \in X$ and for each neighborhood W of zero in X, there exist $z \in X$, $m_1, m_2 \in \mathbb{N}$, and $\lambda \in \mathcal{G}$ such that*
$$x - z \in W \quad \text{and} \quad T_1^{m_1} T_2^{m_2}(\lambda z) - y \in W.$$

Proof. (1) ⇒ (2): Let $x, y \in X$. For $j \in \mathbb{N}$, define $B_j = B_{\frac{1}{j}}(x)$ and $B'_j = B_{\frac{1}{j}}(y)$. Since $\mathcal{T}$ is $\mathcal{G}$-transitive, there exist integers $m_1, m_2 \in \mathbb{N}$ and scalars $\{\lambda_j\}_{j\in\mathbb{N}} \subseteq \mathcal{G}$ such that

$$T_1^{m_1} T_2^{m_2}(\lambda_j B_j) \cap B'_j \neq \emptyset.$$

Hence, for each $j \geq 1$, there exists $x_j \in B_j$ with

$$T_1^{m_1} T_2^{m_2}(\lambda_j x_j) \in B'_j.$$

Thus $\|x_j - x\| < 1/j$ and $\|T_1^{m_1} T_2^{m_2}(\lambda_j x_j) - y\| < 1/j$, which implies $x_j \to x$ and $T_1^{m_1} T_2^{m_2}(\lambda_j x_j) \to y$ as $j \to \infty$.

(2) ⇒ (3): Let $x, y \in X$ and let W be a neighborhood of zero. By (2), there exist sequences $\{x_j\} \subseteq X$, integers m_1, m_2, and $\{\lambda_j\} \subseteq \mathcal{G}$ such that $x_j \to x$ and $T_1^{m_1} T_2^{m_2}(\lambda_j x_j) \to y$. Choosing a sufficiently large J, we can set $z = x_J$ and $\lambda = \lambda_J$ to satisfy $x - z \in W$ and $T_1^{m_1} T_2^{m_2}(\lambda z) - y \in W$.

(3) ⇒ (1): Let $U, V \subset X$ be nonempty open sets, and let W be a neighborhood of zero. Take $x \in U$ and $y \in V$. By (3), there exist $z \in X$, integers m_1, m_2, and $\lambda \in \mathcal{G}$ such that $x - z \in W$ and $T_1^{m_1} T_2^{m_2}(\lambda z) - y \in W$. Then $z \in U$ and $T_1^{m_1} T_2^{m_2}(\lambda z) \in V$, implying

$$T_1^{m_1} T_2^{m_2}(\lambda U) \cap V \neq \emptyset,$$

which shows that $\mathcal{T}$ is $\mathcal{G}$-transitive. □

Proposition 2.12. *Suppose that $\mathcal{T} = (T_1, T_2)$ is a $\mathcal{G}$-cyclic tuple on X. Then:*

1. *If x_0 is a $\mathcal{G}$-cyclic vector for $\mathcal{T}$, then so is $T_1^n T_2^m x_0$ for all $n, m \in \mathbb{N}_0$.*
2. *If T_1 and T_2 are invertible, then the tuple (T_1^{-1}, T_2^{-1}) is $\mathcal{G}^{-1}$-cyclic on X.*

3. *If $\mathcal{M}$ is a closed invariant subspace for $\mathcal{T}$, then the quotient tuple $\mathcal{T}/\mathcal{M}$ is $\mathcal{G}$-cyclic on $X/\mathcal{M}$.*

Proof. 1. Let $x_0 \in \mathcal{G}C(\mathcal{T})$. For nonnegative integers k, j, we have

$$\begin{aligned}\overline{\mathcal{G}\mathrm{Orb}(\mathcal{T}, T_1^k T_2^j x_0)} &= \overline{\{\alpha T_1^n T_2^m T_1^k T_2^j x_0 : n, m \geq 0, \alpha \in \mathcal{G}\}}\\ &= \overline{\{\alpha T_1^k T_2^j T_1^n T_2^m x_0 : n, m \geq 0, \alpha \in \mathcal{G}\}}\\ &= \overline{T_1^k T_2^j \{\alpha T_1^n T_2^m x_0 : n, m \geq 0, \alpha \in \mathcal{G}\}}\\ &\supseteq T_1^k T_2^j \left(\overline{\{\alpha T_1^n T_2^m x_0 : n, m \geq 0, \alpha \in \mathcal{G}\}}\right)\\ &= T_1^k T_2^j (X) = X.\end{aligned}$$

Hence, $T_1^k T_2^j x_0 \in \mathcal{G}C(\mathcal{T})$ and the set of $\mathcal{G}$-cyclic vectors is dense in X.

2. Let $U, V \subset X$ be nonempty open sets and $x_0 \in \mathcal{G}C(\mathcal{T})$. Then there exist $\lambda \in \mathcal{G}$ and nonnegative integers k_1, k_2 such that
$$\lambda T_1^{k_1} T_2^{k_2} (U) \cap V \neq \emptyset.$$
Equivalently,
$$\lambda^{-1} T_1^{-k_1} T_2^{-k_2} (V) \cap U \neq \emptyset.$$
Since $\lambda^{-1} \in \mathcal{G}^{-1}$, Theorem 2.7 implies that (T_1^{-1}, T_2^{-1}) is $\mathcal{G}^{-1}$-cyclic.

3. Let $x_0 \in \mathcal{G}C(\mathcal{T})$ and $x \in X$. Using the quotient norm $\|x + \mathcal{M}\|_{X/\mathcal{M}} = \inf_{y \in \mathcal{M}} \|x + y\|_X$, we have
$$\|\lambda T_1^{k_1} T_2^{k_2} x_0 - x + \mathcal{M}\|_{X/\mathcal{M}} \leq \|\lambda T_1^{k_1} T_2^{k_2} x_0 - x\| \to 0,$$
for every $\lambda \in \mathcal{G}$ and $k_1, k_2 \geq 0$. Equivalently,
$$\|(\lambda T_1^{k_1} T_2^{k_2} x_0 + \mathcal{M}) - (x + \mathcal{M})\| \to 0.$$
Since $\mathcal{T}/\mathcal{M}(x + \mathcal{M}) = \mathcal{T}x + \mathcal{M}$, it follows that $x_0 + \mathcal{M}$ is a $\mathcal{G}$-cyclic vector for $\mathcal{T}/\mathcal{M}$ on $X/\mathcal{M}$.

□

Theorem 2.13 ($\mathcal{G}$-cyclic Tuples Criterion). *Let $\mathcal{T} = (T_1, T_2)$ be a tuple of operators on X. Suppose there exist two dense subsets $Y, Z \subseteq X$, increasing sequences of positive integers $\{m_i\}, \{n_i\}$, and a sequence of functions $\{S_i : Z \to X\}$ such that:*

1. *$T_1^{m_i} T_2^{n_i} S_i z \to z$ for all $z \in Z$.*
2. *For all $z \in Z$ and $y \in Y$, there exists a sequence $\{\lambda_i\}_{i \in \mathbb{N}} \subseteq \mathcal{G}$ such that*
$$\|\lambda_i^{-1} S_i z\| \to 0 \quad \textit{and} \quad \|\lambda_i T_1^{m_i} T_2^{n_i} y\| \to 0.$$

Then $\mathcal{T}$ is $\mathcal{G}$-cyclic.

Proof. Let U and V be two nonempty open subsets of X. Choose $y \in Y \cap U$ and $z \in Z \cap V$. By hypothesis, there exist sequences $\{m_i\}, \{n_i\}$ and functions $\{S_i\}$ satisfying the stated conditions.

Define
$$w_i = y + \lambda_i^{-1} S_i z, \quad i \in \mathbb{N}.$$
Then
$$\lambda_i T_1^{m_i} T_2^{n_i} w_i = \lambda_i T_1^{m_i} T_2^{n_i} y + T_1^{m_i} T_2^{n_i} S_i z.$$
Since $\|w_i - y\| = \|\lambda_i^{-1} S_i z\| \to 0$, we have $w_i \in U$ for sufficiently large i. Moreover, by the hypotheses:
$$\begin{aligned}\|\lambda_i T_1^{m_i} T_2^{n_i} w_i - z\| &= \|\lambda_i T_1^{m_i} T_2^{n_i} y + T_1^{m_i} T_2^{n_i} S_i z - z\|\\ &\leq \|\lambda_i T_1^{m_i} T_2^{n_i} y\| + \|T_1^{m_i} T_2^{n_i} S_i z - z\| \to 0.\end{aligned}$$
Hence, for sufficiently large i, $\lambda_i T_1^{m_i} T_2^{n_i} w_i \in V$, which shows that
$$V \cap \lambda_i T_1^{m_i} T_2^{n_i} (U) \neq \emptyset.$$
Therefore, $\mathcal{T}$ is $\mathcal{G}$-transitive, and consequently $\mathcal{T}$ is $\mathcal{G}$-cyclic. □

Corollary 2.14. *If $\mathcal{T} = (T_1, T_2)$ satisfies the $\mathcal{G}$-cyclicity criterion, then so does $(T_1 \oplus T_1, T_2 \oplus T_2)$, hence it is a $\mathcal{G}$-cyclic pair on $X \oplus X$.*

Proof. Let $\mathcal{T} = (T_1, T_2)$ satisfy the $\mathcal{G}$-cyclicity criterion. Then there exist dense subsets $Y, Z \subseteq X$, increasing sequences of positive integers $\{m_i\}, \{n_i\}$, and a sequence of functions $\{S_i : Z \to X\}$ such that:

1. $T_1^{m_i} T_2^{n_i} S_i z \to z$ for all $z \in Z$.
2. For all $z \in Z$ and $y \in Y$, there exists a sequence $\{\lambda_i\}_{i \in \mathbb{N}} \subseteq \mathcal{G}$ such that $\|\lambda_i^{-1} S_i z\| \to 0$ and $\|\lambda_i T_1^{m_i} T_2^{n_i} y\| \to 0$.

Consider $Y \oplus Y$ and $Z \oplus Z$ in $X \oplus X$, which are dense. Define $A_i : Z \oplus Z \to X \oplus X$ by

$$A_i(z_1, z_2) = (S_i z_1, S_i z_2), \quad (z_1, z_2) \in Z \oplus Z.$$

Then, for each $(z_1, z_2) \in Z \oplus Z$,

$$(T_1^{m_i} T_2^{n_i}, T_1^{m_i} T_2^{n_i}) A_i(z_1, z_2) = (T_1^{m_i} T_2^{n_i} S_i z_1, T_1^{m_i} T_2^{n_i} S_i z_2) \to (z_1, z_2),$$

so that

$$(T_1 \oplus T_1)^{m_i} (T_2 \oplus T_2)^{n_i} A_i(z_1, z_2) \to (z_1, z_2).$$

Next, let $(y_1, y_2) \in Y \oplus Y$. By hypothesis, there exists $\{\lambda_i\} \subseteq \mathcal{G}$ such that

$$\|\lambda_i T_1^{m_i} T_2^{n_i} y_1\| \to 0 \quad \text{and} \quad \|\lambda_i T_1^{m_i} T_2^{n_i} y_2\| \to 0.$$

Hence,

$$\|\lambda_i (T_1 \oplus T_1)^{m_i} (T_2 \oplus T_2)^{n_i} (y_1, y_2)\| \to 0.$$

Furthermore,

$$\|\lambda_i^{-1} A_i(z_1, z_2)\| = \|(\lambda_i^{-1} S_i z_1, \lambda_i^{-1} S_i z_2)\| \to 0.$$

Therefore, $(T_1 \oplus T_1, T_2 \oplus T_2)$ satisfies the $\mathcal{G}$-cyclicity criterion on $X \oplus X$, which completes the proof. □

3 Conclusion

In this paper, we have introduced and studied the notion of $\mathcal{G}$-cyclicity for tuples of commuting bounded linear operators on Banach spaces. We established the concept of $\mathcal{G}$-transitivity and proved its equivalence to $\mathcal{G}$-cyclicity, extending classical results on hypercyclic and supercyclic tuples to a more general semigroup framework. Moreover, we derived a general $\mathcal{G}$-cyclic tuple criterion that provides sufficient conditions for $\mathcal{G}$-cyclicity in terms of convergence of auxiliary sequences and mappings. We also examined structural properties of $\mathcal{G}$-cyclic tuples under invertibility, invariant subspaces, and direct sums.

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