Available online at http://jprm.sms.edu.pk/
Journal of Prime Research in Mathematics, 21(1) (2025), 71–81

Journal of Prime Research in Mathematics

Refinements of the Fractional Hermite-Hadamard Inequality via Arbitrary Means

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Abstract

This paper aims to refine the fractional Hermite-Hadamard inequality by employing arbitrary means defined on a given interval. Using weighted integral techniques and properties of convex functions, new bounds are established which improve existing results.

Keywords: Hermite-Hadamard inequality, convex functions, Riemann-Liouville fractional integrals, means.
2010 Mathematics Subject Classification: 26D10;26D15

1. Introduction

The classical Hermite-Hadamard inequalities:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which hold for all convex functions f defined on an interval $I \subset \mathbb{R}$. If f is concave on I, the inequalities in (1.1) are reversed.

In 2013, Sarikaya et al. [2] extended the Hermite–Hadamard inequality to the framework of fractional calculus and established the following fractional version of inequality (1.1):

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] \leq \frac{f(a)+f(b)}{2}, \tag{1.2}$$

which also holds for all convex functions f on the interval $[a,b]$. The inequality (1.2) is reversed if f is concave on $[a,b]$. Here, $J_{a^+}^\alpha f(b)$ and $J_{b^-}^\alpha f(a)$ are the left- and right-sided Riemann-Liouville fractional

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Received : 25 June 2025; *Accepted:* 07 September 2025; *Published Online:* 10 September 2025

integrals, respectively, defined for $\alpha > 0$ by:

$$J_{a^+}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1} f(x)\, dx,$$
$$J_{b^-}^{\alpha} f(a) = \frac{1}{\Gamma(\alpha)} \int_a^b (x-a)^{\alpha-1} f(x)\, dx,$$

where $\Gamma(\cdot)$ is the Gamma function. For $\alpha = 0$, we have $J_{a^+}^{0} f(b) = J_{b^-}^{0} f(a) = f(x)$.

The generalization of classical integral inequalities for convex functions using the framework of fractional calculus has become an active area of research in recent years. In particular, the extension of Hermite–Hadamard type inequalities through fractional integral operators such as those of Riemann–Liouville and Hadamard has not only enriched the theoretical landscape but also broadened the applicability of such inequalities (see [2], [3]). These generalizations are especially useful in estimating errors in numerical solutions of fractional differential equations, in numerical analysis, and in control theory.

Furthermore, by extending the analysis to more generalized classes of functions such as log-convex, s-convex, and other related convexities stronger and more diverse results have been obtained (see [1]). Within this context, the tools provided by fractional calculus offer a powerful framework that surpasses classical analytical methods, allowing for the modeling of complex and anomalous phenomena encountered in various applied sciences.

For further recent developments on Hermite–Hadamard type inequalities for means, we refer the reader to [4]–[5].

The main objective of this paper is to refine and improve the fractional Hermite–Hadamard inequality given in (1.2) by utilizing arbitrary means defined on the interval I.

2. Main Theorem

Recall that $M(a,b)$ is a mean on $I \subset \mathbb{R}$ if the inequality

$$min\{a,b\} \leq M(a,b) \leq max\{a,b\}$$

holds for each $a, b \in I$.

Lemma 2.1. *Let $a < b$ and fix $\alpha > 0$. Define*

$$p(t) = (t-a)^{\alpha-1} + (b-t)^{\alpha-1}, \qquad t \in (a,b),$$

and for $M \in (a,b)$ put

$$I_a^M := \int_a^M p(t)\, dt, \qquad I_a^b := \int_a^b p(t)\, dt, \qquad \theta := \frac{M-a}{b-a} \in (0,1).$$

Then the following statements are equivalent:

1. *For every convex function $f : [a,b] \to \mathbb{R}$ the inequality*

$$\frac{f(b)-f(a)}{I_a^b} \geq \frac{f(M)-f(a)}{I_a^M} \tag{2.1}$$

holds.

2. *The integral ratio satisfies*

$$\frac{I_a^M}{I_a^b} \geq \frac{M-a}{b-a} = \theta. \tag{2.2}$$

3. *In terms of θ,*

$$\theta^{\alpha} - (1-\theta)^{\alpha} \geq 2\theta - 1. \tag{2.3}$$

Moreover, (2.3) *(hence* (2.2) *and* (2.1)*) is satisfied for all* $\theta \in (0,1)$ *when* $\alpha = 1$ *or* $\alpha = 2$*. For other* α *values the sign of the left side relative to* $2\theta - 1$ *depends on whether* $\theta \leq \frac{1}{2}$ *or* $\theta \geq \frac{1}{2}$ *(see proof).*

Proof. First compute the integrals explicitly. A direct antiderivative calculation gives

$$I_a^M = \int_a^M \left((t-a)^{\alpha-1} + (b-t)^{\alpha-1}\right) dt = \frac{(M-a)^\alpha + (b-a)^\alpha - (b-M)^\alpha}{\alpha},$$

and

$$I_a^b = \int_a^b p(t)\, dt = \frac{2(b-a)^\alpha}{\alpha}.$$

Hence

$$\frac{I_a^M}{I_a^b} = \frac{\theta^\alpha + 1 - (1-\theta)^\alpha}{2}, \qquad \theta = \frac{M-a}{b-a}.$$

The equivalence between (2) and (3) is now algebraic: (2.2) is exactly

$$\frac{\theta^\alpha + 1 - (1-\theta)^\alpha}{2} \geq \theta \iff \theta^\alpha - (1-\theta)^\alpha \geq 2\theta - 1,$$

which is (2.3).

It remains to show (1) $\iff$ (2).

(2) $\Rightarrow$ *(1).* Assume (2.2) holds. Let f be any convex function on $[a,b]$. Convexity gives the standard bound

$$f(M) \leq f(a) + \frac{M-a}{b-a}\big(f(b) - f(a)\big) = f(a) + \theta\,(f(b) - f(a)),$$

hence

$$f(M) - f(a) \leq \theta\,(f(b) - f(a)).$$

Multiply both sides by the positive number I_a^b and rearrange using (2.2):

$$(f(M) - f(a))I_a^b \leq \theta(f(b) - f(a))I_a^b \leq (f(b) - f(a))I_a^M,$$

where the last inequality follows from $I_a^M \geq \theta I_a^b$. Thus

$$(f(b) - f(a))I_a^M \geq (f(M) - f(a))I_a^b,$$

which is equivalent to (2.1). This proves (1).

(1) $\Rightarrow$ *(2).* Assume (2.1) holds for every convex f. Take the affine function $f(x) = x$. Then

$$\frac{f(b) - f(a)}{I_a^b} = \frac{b-a}{I_a^b}, \qquad \frac{f(M) - f(a)}{I_a^M} = \frac{M-a}{I_a^M},$$

so (2.1) becomes

$$\frac{b-a}{I_a^b} \geq \frac{M-a}{I_a^M},$$

equivalently $I_a^M / I_a^b \geq (M-a)/(b-a) = \theta$, which is (2.2). Thus (2) holds. This completes the proof of equivalence. □

Corollary 2.2. *Under the hypotheses of Lemma 2.1, the inequality*

$$\frac{f(b)-f(a)}{I_a^b} \ \geq \ \frac{f(M)-f(a)}{I_a^M}$$

holds for every $M \in (a,b)$ when $\alpha = 1$ or $\alpha = 2$. In these cases, the condition (2.3) reduces to an identity, and the ratio of integrals satisfies

$$\frac{I_a^M}{I_a^b} = \frac{M-a}{b-a}.$$

Consequently, for $\alpha = 1$ we obtain the classical slope inequality

$$\frac{f(b)-f(a)}{b-a} \geq \frac{f(M)-f(a)}{M-a},$$

and for $\alpha = 2$ the same inequality holds after the analogous normalization of the integrals.

Proof. For $\alpha = 1$, the weight function is constant, $p(t) = 2$ for all $t \in (a,b)$, hence

$$I_a^M = 2(M-a), \qquad I_a^b = 2(b-a),$$

so that $\dfrac{I_a^M}{I_a^b} = \dfrac{M-a}{b-a}$. Substituting into (2.1) gives

$$\frac{f(b)-f(a)}{2(b-a)} \geq \frac{f(M)-f(a)}{2(M-a)} \quad \Longleftrightarrow \quad \frac{f(b)-f(a)}{b-a} \geq \frac{f(M)-f(a)}{M-a},$$

which is the standard consequence of convexity: the slope of the secant line is non-decreasing on $[a,b]$.

For $\alpha = 2$, a similar computation shows

$$\frac{I_a^M}{I_a^b} = \frac{M-a}{b-a},$$

and the same reduction to the secant slope inequality applies. □

Remark 2.3. The condition (2.3) is sharp: for values of α other than 1 or 2, the inequality (2.1) may fail for certain choices of M. For example, taking the affine function $f(x) = x$ and a suitable $M \neq \frac{a+b}{2}$ provides a counterexample when the condition (2.3) is violated. Hence, (2.3) precisely characterizes the validity of (2.1).

Theorem 2.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be convex on an interval I with $a < b$, where $a, b \in I$. For $\alpha > 0$ define*

$$p(t) = (t-a)^{\alpha-1} + (b-t)^{\alpha-1}, \qquad t \in (a,b),$$

and, for $M \in (a,b)$, set

$$I_a^M := \int_a^M p(t)\,dt, \qquad I_a^b := \int_a^b p(t)\,dt.$$

Define the functional

$$F_f^\alpha(M;a,b) := \frac{1}{2}f(M) + \frac{1}{2\int_a^b p(t)\,dt}\Big[f(a)\int_a^M p(t)\,dt + f(b)\int_M^b p(t)\,dt\Big].$$

Then the following chain of inequalities holds for every convex f:

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\big] = \frac{\int_a^b p(t)f(t)\,dt}{\int_a^b p(t)\,dt} \leq F_f^\alpha(M;a,b).$$

Moreover, the upper bound

$$F_f^{\alpha}(M;a,b) \le \frac{f(a)+f(b)}{2}$$

holds for every convex f if and only if the equivalent conditions of Lemma 2.1 hold (i.e. (2.2) *or* (2.3)*). In particular, the upper bound is unconditional for all $M \in (a,b)$ when $\alpha = 1$ or $\alpha = 2$.*

Proof. Using the classical Riemann–Liouville definitions, we have

$$J_{a^+}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} f(t)\,dt, \qquad J_{b^-}^{\alpha} f(a) = \frac{1}{\Gamma(\alpha)} \int_a^b (t-a)^{\alpha-1} f(t)\,dt.$$

Summing and simplifying gives

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a) \right] = \frac{\alpha}{2(b-a)^{\alpha}} \int_a^b p(t) f(t)\,dt,$$

and since $\int_a^b p(t)\,dt = \frac{2(b-a)^{\alpha}}{\alpha}$, the right-hand side equals $\frac{\int_a^b p(t)f(t)\,dt}{\int_a^b p(t)\,dt}$, which proves the identity.

The function p is symmetric about the midpoint $(a+b)/2$:

$$p(a+u) = u^{\alpha-1} + (b-a-u)^{\alpha-1} = p(b-u), \quad 0 < u < b-a.$$

For a nonnegative weight symmetric about the midpoint and a convex function, the weighted Hermite–Hadamard–Fejer inequality gives

$$\frac{\int_a^b p(t)f(t)\,dt}{\int_a^b p(t)\,dt} \le \frac{f(a)+f(b)}{2}.$$

Moreover, decomposing the weighted average over $[a,b]$ as a convex combination of the weighted averages over $[a,M]$ and $[M,b]$ yields

$$\frac{\int_a^b pf}{\int_a^b p} = \frac{\int_a^M p}{\int_a^b p} \cdot \frac{\int_a^M pf}{\int_a^M p} + \frac{\int_M^b p}{\int_a^b p} \cdot \frac{\int_M^b pf}{\int_M^b p}.$$

Applying the standard (unweighted) Hermite–Hadamard inequality to each subinterval and collecting terms gives exactly the definition of $F_f^{\alpha}(M;a,b)$ as an upper bound for the weighted average:

$$\frac{\int_a^b p(t)f(t)\,dt}{\int_a^b p(t)\,dt} \le F_f^{\alpha}(M;a,b).$$

This establishes the first chain of inequalities.

For the upper bound, $F_f^{\alpha}(M;a,b)$ can be rewritten as

$$F_f^{\alpha}(M;a,b) = \frac{f(a)+f(b)}{2} - \frac{1}{2} \int_a^M p(t)\,dt \left(\frac{f(b)-f(a)}{\int_a^b p(t)\,dt} - \frac{f(M)-f(a)}{\int_a^M p(t)\,dt} \right).$$

Hence, $F_f^{\alpha}(M;a,b) \le \frac{f(a)+f(b)}{2}$ holds for every convex f if and only if

$$\frac{f(b)-f(a)}{\int_a^b p(t)\,dt} \ge \frac{f(M)-f(a)}{\int_a^M p(t)\,dt}.$$

By Lemma 2.1, this is equivalent to the integral condition $\frac{I_a^M}{I_a^b} \ge \frac{M-a}{b-a}$ or to the θ-condition $\theta^{\alpha} - (1-\theta)^{\alpha} \ge 2\theta - 1$. Necessity is immediate: if (2.2) fails, the affine function $f(x) = x$ provides a convex counterexample. This completes the equivalence and the proof of the theorem. □

Remark 2.5.
1. The left identity and the lower bound are unconditional and follow from the symmetry of p and standard weighted Hermite–Hadamard–Fejér arguments.
2. The right inequality $F_f^{\alpha}(M;a,b) \leq (f(a)+f(b))/2$ is *equivalent* (for all convex f) to the integral ratio condition of Lemma 2.1; in particular it holds for every M when $\alpha = 1$ or $\alpha = 2$.
3. If one needs the right inequality for a specific (fixed) convex function f, it may hold even if the integral condition fails; however the equivalence above describes the exact uniform (over all convex f) criterion.

Remark 2.6 (Special cases: $\alpha = 1$ and $\alpha = 2$). Consider Theorem 2.4 for the special choices $\alpha = 1$ and $\alpha = 2$:

- **Case** $\alpha = 1$**:** The weight function becomes constant,

$$p(t) = (t-a)^0 + (b-t)^0 = 2,$$

so the weighted integral reduces to the standard average:

$$\frac{\int_a^b p(t)f(t)\,dt}{\int_a^b p(t)\,dt} = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

The functional reduces to

$$F_f(M;a,b) = \frac{1}{2}f(M) + \frac{1}{2(b-a)}\Big[(M-a)f(a) + (b-M)f(b)\Big],$$

and the Hermite–Hadamard type inequality

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq F_f(M;a,b) \leq \frac{f(a)+f(b)}{2}$$

holds for every $M \in (a,b)$, as first proved by [4].

- **Case** $\alpha = 2$**:** The weight function becomes

$$p(t) = (t-a) + (b-t) = b-a,$$

which is again constant, and the functional simplifies to

$$F_f^2(M;a,b) = \frac{1}{2}f(M) + \frac{1}{2(b-a)}\Big[(M-a)f(a) + (b-M)f(b)\Big],$$

identical in form to the $\alpha = 1$ case. Hence, the inequality

$$\frac{\Gamma(3)}{2(b-a)^2}\left[J_{a+}^2 f(b) + J_{b-}^2 f(a)\right] \leq \frac{\int_a^b p(t)f(t)\,dt}{\int_a^b p(t)\,dt} \leq F_f^2(M;a,b) \leq \frac{f(a)+f(b)}{2}$$

holds for all $M \in (a,b)$.

In both cases, the upper bound is valid for every M because the condition $\theta^{\alpha} - (1-\theta)^{\alpha} \geq 2\theta - 1$ is satisfied for all $\theta \in (0,1)$.

Theorem 2.7. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be convex on an interval I, and let $a < b$ with $a, b \in I$. For $\alpha > 0$, define*

$$p(t) = (t-a)^{\alpha-1} + (b-t)^{\alpha-1}, \qquad t \in (a,b),$$

and for any $M \in (a,b)$, set

$$F_f^{\alpha}(M;a,b) := \frac{1}{2}f(M) + \frac{1}{2\int_a^b p(t)\,dt}\left[f(a)\int_a^M p(t)\,dt + f(b)\int_M^b p(t)\,dt\right].$$

Then the following chain of inequalities holds for all M satisfying Lemma 2.1 condition:

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\le\frac{\int_a^b p(t)f(t)\,dt}{\int_a^b p(t)\,dt}\le F_f^{\alpha}(M;a,b)\le\frac{f(a)+f(b)}{2}.$$

Moreover, for the special choice

$$M=f^{-1}\Big(\frac{f(a)+f(b)}{2}\Big),$$

we have $f(M)=(f(a)+f(b))/2$ and

$$F_f^{\alpha}(M;a,b)=\frac{f(a)+f(b)}{2}-\frac{f(b)-f(a)}{4(b-a)^{\alpha}}\Big[(M-a)^{\alpha}-(b-M)^{\alpha}\Big],$$

which is valid for any $\alpha>0$ without requiring Lemma 2.1 condition.

Proof. By definition of the left and right Riemann–Liouville fractional integrals,

$$J_{a^+}^{\alpha}f(b)=\frac{1}{\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}f(t)\,dt,\quad J_{b^-}^{\alpha}f(a)=\frac{1}{\Gamma(\alpha)}\int_a^b(t-a)^{\alpha-1}f(t)\,dt.$$

Hence,

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]=\frac{\int_a^b p(t)f(t)\,dt}{\int_a^b p(t)\,dt},$$

because $\int_a^b p(t)\,dt=2(b-a)^{\alpha}/\alpha$.

Decompose the weighted integral over $[a,b]$ as a convex combination of the subintervals $[a,M]$ and $[M,b]$:

$$\frac{\int_a^b p(t)f(t)\,dt}{\int_a^b p(t)\,dt}=\frac{\int_a^M p(t)\,dt}{\int_a^b p(t)\,dt}\cdot\frac{\int_a^M p(t)f(t)\,dt}{\int_a^M p(t)\,dt}+\frac{\int_M^b p(t)\,dt}{\int_a^b p(t)\,dt}\cdot\frac{\int_M^b p(t)f(t)\,dt}{\int_M^b p(t)\,dt}.$$

Applying the weighted Hermite–Hadamard–Fejér inequality on each subinterval yields

$$\frac{\int_a^M p(t)f(t)\,dt}{\int_a^M p(t)\,dt}\le\frac{f(a)+f(M)}{2},\qquad\frac{\int_M^b p(t)f(t)\,dt}{\int_M^b p(t)\,dt}\le\frac{f(M)+f(b)}{2}.$$

Consequently,

$$\frac{\int_a^b p(t)f(t)\,dt}{\int_a^b p(t)\,dt}\le F_f^{\alpha}(M;a,b).$$

Rewriting $F_f^{\alpha}(M;a,b)$ gives

$$F_f^{\alpha}(M;a,b)=\frac{f(a)+f(b)}{2}-\frac{1}{2}\int_a^M p(t)\,dt\left(\frac{f(b)-f(a)}{\int_a^b p(t)\,dt}-\frac{f(M)-f(a)}{\int_a^M p(t)\,dt}\right),$$

so by Lemma 2.1, the upper bound $F_f^{\alpha}(M;a,b)\le(f(a)+f(b))/2$ holds if and only if $\theta^{\alpha}-(1-\theta)^{\alpha}\ge2\theta-1$ with $\theta=(M-a)/(b-a)$.

For the special choice $M=f^{-1}((f(a)+f(b))/2)$, we have $f(M)=(f(a)+f(b))/2$, which implies

$$F_f^{\alpha}(M;a,b)=\frac{f(a)+f(b)}{2}-\frac{f(b)-f(a)}{4(b-a)^{\alpha}}[(M-a)^{\alpha}-(b-M)^{\alpha}],$$

independent of the condition in Lemma 2.1. This completes the proof. □

Remark 2.8 (Special case $\alpha = 1$). When $\alpha = 1$ in Theorem 2.9, the fractional integrals reduce to the classical Riemann integrals, i.e.,

$$J^1_{a^+} f(b) = \int_a^b f(t)\,dt, \quad J^1_{b^-} f(a) = \int_a^b f(t)\,dt.$$

Moreover, setting

$$M = A_f(a,b) := f^{-1}\left(\frac{f(a)+f(b)}{2}\right), \quad A(a,b) := \frac{a+b}{2},$$

the inequality in Theorem 2.9 becomes

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2} - \frac{f(b)-f(a)}{2(b-a)}\big(A_f(a,b) - A(a,b)\big),$$

which coincides with the result provided by Simic in [4].

Theorem 2.9. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex and invertible function on the interval I with endpoints $a, b \in I$, $a < b$. For any $\alpha > 0$ and $\theta \in [0,1]$, define*

$$M := f^{-1}\left(\theta \cdot \frac{f(a)+f(b)}{2} + (1-\theta) \cdot \frac{1}{b-a}\int_a^b f(t)\,dt\right).$$

Then $M \in [a,b]$, and the following inequality holds:

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)\right] \le \frac{f(a)+f(b)}{2} - \frac{(M-a)^\alpha + (b-a)^\alpha - (b-M)^\alpha}{4(b-a)^\alpha}(f(b)-f(a)) + \frac{f(M)-f(a)}{2},$$

where $J^\alpha_{a^+}$ and $J^\alpha_{b^-}$ denote the left and right Riemann-Liouville fractional integrals of order α.

Proof. Since f is convex and invertible on $[a,b]$, it is continuous and strictly monotonic, hence bijective onto its image $f([a,b])$. The values

$$\frac{f(a)+f(b)}{2} \quad \text{and} \quad \frac{1}{b-a}\int_a^b f(t)\,dt$$

both lie in $[f(a), f(b)]$ (or $[f(b), f(a)]$ depending on monotonicity). Since $\theta \in [0,1]$, the convex combination

$$\theta \cdot \frac{f(a)+f(b)}{2} + (1-\theta) \cdot \frac{1}{b-a}\int_a^b f(t)\,dt$$

also lies in $[f(a), f(b)]$. Hence, $M := f^{-1}(\cdot)$ is well-defined and satisfies $M \in [a,b]$.

Now consider the fractional-type functional

$$F^\alpha_f(M; a, b) := \frac{f(a)+f(b)}{2} - \frac{1}{2}\int_a^M p(t)\,dt \cdot \left(\frac{f(b)-f(a)}{\int_a^b p(t)\,dt} - \frac{f(M)-f(a)}{\int_a^M p(t)\,dt}\right),$$

where $p(t) := (t-a)^{\alpha-1} + (b-t)^{\alpha-1}$ for $\alpha > 0$.

The integrals of $p(t)$ can be computed as:

$$\int_a^b p(t)\,dt = \int_a^b (t-a)^{\alpha-1}dt + \int_a^b (b-t)^{\alpha-1}dt = \frac{(b-a)^\alpha}{\alpha} + \frac{(b-a)^\alpha}{\alpha} = \frac{2(b-a)^\alpha}{\alpha},$$

and

$$\int_a^M p(t)\,dt = \int_a^M (t-a)^{\alpha-1}dt + \int_a^M (b-t)^{\alpha-1}dt = \frac{(M-a)^\alpha}{\alpha} + \frac{(b-a)^\alpha - (b-M)^\alpha}{\alpha}.$$

Substituting into F_f^α, we get:

$$\begin{aligned} F_f^\alpha(M;a,b) &= \frac{f(a)+f(b)}{2} \\ &\quad - \frac{1}{2}\cdot\frac{(M-a)^\alpha+(b-a)^\alpha-(b-M)^\alpha}{\alpha}\left(\frac{\alpha(f(b)-f(a))}{2(b-a)^\alpha} - \frac{\alpha(f(M)-f(a))}{(M-a)^\alpha+(b-a)^\alpha-(b-M)^\alpha}\right) \\ &= \frac{f(a)+f(b)}{2} - \frac{(M-a)^\alpha+(b-a)^\alpha-(b-M)^\alpha}{4(b-a)^\alpha}(f(b)-f(a)) + \frac{f(M)-f(a)}{2} \end{aligned}$$

which completes the proof. □

Corollary 2.10. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex and invertible function on the interval I with endpoints $a, b \in I$, $a < b$. For any $\theta \in [0,1]$, define*

$$M := f^{-1}\left(\theta\cdot\frac{f(a)+f(b)}{2} + (1-\theta)\cdot\frac{1}{b-a}\int_a^b f(t)\,dt\right).$$

Then the following inequality holds:

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2} - \frac{M-a}{2(b-a)}(f(b)-f(a)) + \frac{f(M)-f(a)}{2}.$$

Proof. The inequality follows immediately by setting $\alpha = 1$ in Theorem 2.9 and simplifying using the properties of the classical integrals and the Gamma function, noting that

$$\Gamma(2) = 1, \quad J_{a^+}^1 f(b) = \int_a^b f(t)\,dt, \quad J_{b^-}^1 f(a) = \int_a^b f(t)\,dt.$$

The details of the simplification are as shown in the proof of Theorem 2.9, yielding the stated inequality. □

Theorem 2.11. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I with $a < b$, $a, b \in I$. If $N = N(a,b)$ is a mean point in I, then the following double inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le G_f^\alpha(N;a,b) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right],$$

where

$$\begin{aligned} G_f^\alpha(N;a,b) \;:&= \frac{1}{2(b-a)^\alpha}\left\{[(b-a)^\alpha + (N-a)^\alpha - (b-N)^\alpha]\cdot f\left(\frac{a+N}{2}\right)\right. \\ &\quad \left. + [(b-a)^\alpha + (b-N)^\alpha - (N-a)^\alpha]\cdot f\left(\frac{N+b}{2}\right)\right\}. \end{aligned}$$

Proof. Let

$$p(t) = (t-a)^{\alpha-1} + (b-t)^{\alpha-1}, \quad \text{for some } \alpha > 0.$$

By the left part of Hermite-Hadamard-Fejer inequality applied separately on $[a, N]$ and $[N, b]$, we have:

$$f\left(\frac{a+N}{2}\right) \le \frac{\int_a^N p(t)f(t)dt}{\int_a^N p(t)dt} \quad \text{and} \quad f\left(\frac{N+b}{2}\right) \le \frac{\int_N^b p(t)f(t)dt}{\int_N^b p(t)dt}.$$

Multiplying each inequality by the corresponding weight and summing yields:

$$G_f^\alpha(N;a,b) \le \frac{\int_a^b p(t)f(t)dt}{\int_a^b p(t)dt}.$$

By convexity of f, for $\lambda := \frac{\int_a^N p(t)dt}{\int_a^b p(t)dt}$, we have

$$f\left(\lambda\frac{a+N}{2}+(1-\lambda)\frac{N+b}{2}\right)\leq G_f^{\alpha}(N;a,b).$$

Since

$$\lambda\frac{a+N}{2}+(1-\lambda)\frac{N+b}{2}\leq\frac{a+b}{2},$$

and f is convex (and thus midpoint convex), it follows that

$$f\left(\frac{a+b}{2}\right)\leq f\left(\lambda\frac{a+N}{2}+(1-\lambda)\frac{N+b}{2}\right)\leq G_f^{\alpha}(N;a,b).$$

The remaining inequalities follow from the Hermite-Hadamard-Fejer inequality applied on subintervals. □

Remark 2.12. If we choose as $\alpha=1$ in Theorem 2.11, it follows that $p(t)=b-a$,

$$f\left(\frac{a+b}{2}\right)\leq G_f(N;a,b)\leq\frac{1}{b-a}\int_a^b f(t)\,dt,$$

where

$$G_f(N;a,b):=\frac{1}{b-a}\left[(N-a)\cdot f\left(\frac{a+N}{2}\right)+(b-N)\cdot f\left(\frac{N+b}{2}\right)\right],$$

which is proved by Simic in [4].

Example 2.13. Let $f:[0,1]\to\mathbb{R}$ be the convex function defined by

$$f(x)=x^p,\quad p>-\frac{1}{2}.$$

Consider the fractional order $\alpha=\frac{1}{2}$. Then, the fractional integrals appearing in Theorem 2.11 can be expressed in terms of the Beta function as follows:

$$J_{0^+}^{1/2}f(1)=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^1(1-t)^{-1/2}t^p\,dt=\frac{1}{\sqrt{\pi}}B\left(p+1,\frac{1}{2}\right),$$

and

$$J_{1^-}^{1/2}f(0)=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^1 t^{-1/2}t^p\,dt=\frac{1}{\sqrt{\pi}}\int_0^1 t^{p-\frac{1}{2}}\,dt=\frac{1}{\sqrt{\pi}}\cdot\frac{1}{p+\frac{1}{2}},$$

where $B(\cdot,\cdot)$ denotes the Beta function defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Substituting these into the upper bound of Theorem 2.11 with $a=0$ and $b=1$, we obtain

$$G_f^{1/2}(N;0,1)\leq\frac{\Gamma\left(\frac{3}{2}\right)}{2}\left[J_{0^+}^{1/2}f(1)+J_{1^-}^{1/2}f(0)\right]=\frac{\frac{\sqrt{\pi}}{2}}{2}\left[\frac{1}{\sqrt{\pi}}B\left(p+1,\frac{1}{2}\right)+\frac{1}{\sqrt{\pi}}\cdot\frac{1}{p+\frac{1}{2}}\right],$$

which simplifies to

$$G_f^{1/2}(N;0,1)\leq\frac{1}{4}\left[B\left(p+1,\frac{1}{2}\right)+\frac{1}{p+\frac{1}{2}}\right].$$

Moreover, the lower bound in the inequality reads

$$f\left(\frac{0+1}{2}\right)=\left(\frac{1}{2}\right)^{p}\leq G_{f}^{1/2}(N;0,1).$$

Hence, for the power function $f(x)=x^{p}$, the inequality in Theorem 2.11 with fractional order $\alpha=\frac{1}{2}$ becomes

$$\left(\frac{1}{2}\right)^{p}\leq G_{f}^{1/2}(N;0,1)\leq\frac{1}{4}\left[B\left(p+1,\frac{1}{2}\right)+\frac{1}{p+\frac{1}{2}}\right]$$

where

$$G_{f}^{\frac{1}{2}}(N;0,1)=\frac{1}{2}\left\{\left(1+\sqrt{N}-\sqrt{1-N}\right)\left(\frac{N}{2}\right)^{p}+\left(1+\sqrt{1-N}-\sqrt{N}\right)\left(\frac{1+N}{2}\right)^{p}\right\}.$$

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