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Modified-Laplace Transform Based Variational Iteration Method for Solving Nonlinear Fractional Order Differential Equations of Caputo-Fabrizio Type

R. A. Oderinu[a], W. A. Tijani[a], J. O. Akanni[b,c,*], K. A. Salaudeen[d], M. Taiwo[e], F. Fatmawati[f]

[a]*Department of Pure and Applied Mathematics, Ladoke Akintola University of Technology, Ogbomoso, Nigeria.*

[b]*Department of Mathematics, Saveethe School of Engineering, Seveethe Institute of Medical and Technical Sciences, Seveethe University, Chennal 602105, Tamil Nadu, India.*

[c]*Department of Mathematical and Computing sciences, Koladaisi University Ibadan, Oyo State, Nigeria.*

[d]*Department of Mathematics, Emmanuel Alayande University of Education, Oyo, Nigeria.*

[e]*Department of Physical and Chemical Sciences, Federal University of Health Sciences,Ila-oragun, Osun State Nigeria.*

[f]*Department of Mathematics, Universitas Airlangga, Kampus C Mulyorejo Surabaya 60115, Indonesia.*

Abstract

A novel hybrid computational technique, referred to as the Modified-Laplace transform based Variational Iteration Method (MLVIM), is presented in this study to efficiently address the associated challenges. This method is developed through the integration of the modified Laplace transform used to convert fractional order equations into an algebraic form and the variational iteration method, which is employed to manage nonlinear components effectively. The formulation of the method is supported by a robust theoretical framework, including convergence analysis and relevant theorems that establish its mathematical validity. To demonstrate the practical effectiveness of MLVIM, it has been applied to a series of benchmark fractional differential equations. The results of numerical experiments demonstrate that the proposed method outperforms traditional techniques in terms of accuracy, convergence, and computational efficiency. The error analysis confirms that MLVIM achieves lower approximation errors, making it a robust and precise tool for modelling complex dynamic systems. This research contributes a reliable and powerful approach to solve complex fractional models, offering significant potential for applications in science and engineering where memory-dependent behaviour is prevalent.

Keywords: Fractional Modified Laplace Transform Scheme, Variational Iteration Method, Fractional order differential equations, Caputo-Fabrizio type.
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*Corresponding author
Email addresses: raoderinu@lautech.edu.ng (R. A. Oderinu), adiguntijani111@gmail.com (W. A. Tijani), jide28@gmail.com (J. O. Akanni[b],), salaudeenka@gmail.com (K. A. Salaudeen), musilimu.taiwo@fuhsi.edu.ng (M. Taiwo), fatmawati@fst.unair.ac.id (F. Fatmawati)

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1. Introduction

Fractional order differential equations (FDE) are phenomena that occur in the fields of applied mathematics, physics, and engineering. These phenomena are commonly explained using nonlinear differential equations, which can have many forms based on the specific fractional order imposed on the FDE[1, 2, 3]. FDEs were used in the work of [4, 5] to represent both electrical and non-electrical systems with dispersed characteristics, demonstrating one of the many uses of FDE. The modelling of super capacitors, batteries, and a series of cars operating in adaptive cruise control mode provided specific instances. Caputo Fabrizio derivatives were used for different types of fractional order derivatives such as singular kernel (Caputo) derivatives, which have been used by numerous researchers to elucidate the hereditary characteristics of various systems[6, 7, 8, 9, 10]. Regarding epidemiology, [11, 12] explain that fractional order models are useful for explaining specific trends that cannot be fully explained by integer-order models. For example, it was implemented in the modelling of HIV and tuberculosis co-infection in the presence of Multi-Drug Resistant Tuberculosis (MDR-TB). This was done to differentiate between the immune systems of individuals, age, compliance with treatment, and other co-morbidities. The purpose was to incorporate actual data from specific patients [13, 14]. Nevertheless, the presence of singularities in some points in the kernel of Caputo derivatives led to the development of Caputo Fabrizio differential equations, which exhibit non-singularities at any point.

Several researchers have made efforts to employ various techniques in order to solve this particular problem, but obtaining an exact or straightforward solution is challenging due to nonlinearity. Therefore, many numerical methods, including the Adomian Decomposition Method (ADM), the Homotopy Perturbation Method (HPM), and the Variational Iteration Method (VIM), have been adopted to solve equations of this nature[15, 16] . These methods yield solutions in the form of infinite series, which converge efficiently to the exact solution. Although the aforementioned methods have shown good performance, they have a few limitations. These include the complex calculations required for the application of the Adomian polynomial, the selection and determination of small artificial parameters in HPM, and the identification of the Lagrange multiplier in VIM [17, 18, 19, 20].

In pursuit of enhanced and more effective numerical techniques, integral transforms like the Laplace transform, Kamal transform, and Aboodh have been combined with other numerical methods capable of dealing with nonlinearity[21, 22, 23, 24, 25]. Two hybridisation methods that involve the Laplace transform are its combination with ADM [26, 27, 28] and HPM . The pursuit of approximate solutions for nonlinear equation has led to modifications of integral transformations . On the one hand, integral transforms create effective operational approaches for solving initial- and boundary-value problems by transforming them into algebraic equations. However, these integral transforms can only handle linear integer-order science and engineering problems independently [29].Just recently, the work of authors in [30] updated the Laplace transform kernel to provide another integral transform that is free of singularity. The new transform, however, still has the same problems as other integral transforms, so this letter aims to improve the modified integral transform scheme to handle fractional order derivatives and nonlinearity, This study aims to further strengthen the modified Laplace transform and to provide an alternative method for solving the Caputo-Fabrizio differential equations.

2. Preliminaries / Notation

2.0 Basic of the Variational Iteration Method (VIM)

The operator representation of the general differential equation is as follows:

$$Ry(t) + Uy(t) + Ny(t) = g(t). \tag{2.1}$$

R is a linear operator with the highest derivative, U is the remaining linear operator with a derivative less than R, N is the non-linear operator and g (t) is the non-homogeneous term.

[20], VIM is built as follows:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(k)[Ry_n(k) + U\bar{y}_n(k) + N\bar{y}_n(k) - f(k)]dk. \tag{2.2}$$

Where y_0 is an initial approximation, $\bar{y}_n$ is a restricted variation, the integral part in equation (2) is the correction functional, n is the nth approximation and [14] defines the Lagrange multiplierλ as:

$$\lambda = \frac{(-1)^m (s-t)^{m-1}}{(m-1)!}. \tag{2.3}$$

where m is the highest-order derivative.

2.4.1 Definitions.
To proceed with development of the new scheme, it is important to recall certain basic definitions and notations from fractional calculus.

2.4.2 The Laplace transform of f (t)) for a piecewise continuous exponential order function is defined as:

$$L_a f(t) = \int_0^\infty e^{-st} f(t)dt \quad R(s) > 0, [10]. \tag{2.4}$$

2.4.3 The Modified-Laplace transform of f (t)) for an exponentially ordered piecewise continuous function is defined as:

$$L_a f(t) = \int_0^\infty a^{-st} f(t)dt \quad R(s) > 0, a \in (0,\infty)/\{1\}, \tag{2.5}$$

Where Eqn. (5) reduces to a simple Laplace transform for $a = e$, [30].
2.4.5 The Caputo-Fabrizio fractional derivative of a function f(t) is defined as:

$${}^{CF}D^k f(t) = \frac{M(k)}{(1-k)} \int_0^t e^{\frac{-k(t-x)}{1-k}} f'(x)dx, M(0) = M(1) = 1. \tag{2.6}$$

Where k is the fractional order of the equation and $M(k)$ is the normalization constant.

Some fundamental properties of the modified- Laplace transform:

(1) if $f(t)$=1, then $L_a(1)=\frac{1}{slog_e a}$, $(s > 0)$.

(2) if $f(t) = t$, then $L_a(t)=\frac{1}{s^2(log_e a)^2}$, $(s > 0)$.

(3) if $f(t)=t^n$, then $L_a(t^n)=\frac{n!}{s^{n+1}(log_e a)^{n+1}}$, $(s > 0, \quad n = 0, 1, 2, ...)$.

(4) if $f(t)=e^{bt}$, then $L_a(e^{bt})=\frac{1}{slog_e a - b}$, $slog_e a > |b|$.

(5) if $f(t)=sinbt$, then $L_a(sinbt)=\frac{b}{s^2(log_e a)^2 + b^2}$, $(slog_e a > 0)$.

(6) if $f(t)=cosbt$, then $L_a(cosbt)=\frac{s}{s^2(log_e a)^2 + b^2}$, $(slog_e a > 0)$

A Lemma will be established to demonstrate a crucial property of the modified Laplace transform of the Caputo-Fabrizio derivative, which will aid in the development of the new scheme.

3. Main Results

3.0 Lemma.

If f(t) is a piecewise continuous function of exponential order and the modified Laplace transform derivative of a function f(t) is given, then the modified Laplace transform of the Caputo-Fabrizio derivative is

$$L_a({}^{CF}D_t^k f(t)) = \frac{(log_e a)f(slog_e a)}{slog_e a + k(1 - slog_e a)} - (slog_e a + k(1 - slog_e a))^{-1} f(0),$$

$$z - 1 < k < z, z \in N. \tag{3.1}$$

4. Proofs

Proof

if $M(k) = 1$ in Eqn (6), then

$$ {}^{CF}D_t^k f(t) = \frac{1}{1-k}\int_0^t e^{\frac{-k(t-x)}{1-k}} f^{'}(x)dx. \tag{4.1}$$

Applying the convolution property of two functions to Eqn. (8), then:

$$ {}^{CF}D_t^k f(t) = \frac{1}{1-k}(e^{(\frac{-k}{1-k})t}) * f^{'}(x)dx. \tag{4.2}$$

Applying the modified-Laplace transform of Eqn. (9), then:

$$L_a({}^{CF}D_t^k f(t)) = \frac{1}{1-k} L_a((e^{(\frac{-k}{1-k})t}))\cdot L_a f^{'}(t), \tag{4.3}$$

Simplifying Eqn. (10) by using the properties of the modified Laplace transform,

$$L_a({}^{CF}D_t^k f(t)) = \frac{1}{1-k}(\frac{1}{slog_e a + \frac{k}{1-k}})\cdot L_a f^{'}(t), \tag{4.4}$$

further simplification of Eqn. (11), gives the following.

$$L_a({}^{CF}D_t^k f(t)) = (\frac{1}{slog_e a + k(1 - slog_e a)})\cdot L_a f^{'}(t). \tag{4.5}$$

Substituting modified-Laplace transform of the first-order derivative into Eqn. (12), that is,

$$L_a(f(t)) = (slog_e a)f(s,a) - f(0), \tag{4.6}$$

then

$$L_a({}^{CF}D_t^k f(t)) = (\frac{1}{slog_e a + k(1 - slog_e a)})((slog_e a)f(s,a) - f(0)), \tag{4.7}$$

simplifying Eqn. (14), gives

$$L_a({}^{CF}D_t^k f(t)) = \frac{(slog_e a)f(s,a)}{slog_e a + k(1 - slog_e a)} - (slog_e a + k(1 - slog_e a))^{-1} f(0). \tag{4.8}$$

In general,

$$L_a({}^{CF}D_t^k f(t)) = \frac{(slog_e a)f(s,a) - \sum_{k=1}^{z} s^{z-k}(log_e a)^{z-k} f^{(k-1)}(0)}{slog_e a + k(1 - slog_e a)}. \tag{4.9}$$

The purpose of this lemma is to allow determination of the modified Laplace transform of the fractional derivative of $f(t)$ in the Caputo-Fabrizio sense. The obtained result in equation (15) will serve as the basis for formulating the proposed approach in this study.
3.1 The derivation of the proposed scheme (MLVIM).
The proposed approach will utilise the modified Laplace transform in conjunction with the correction functional in VIM (MLVIM). Now considering the Caputo-Fabrizio form of equation (1) as

$${}^{CF}D_t^k y(t) + Uy(t) + Ny(t) = f(t), m-1 < k < m \tag{4.10}$$

Taking the modified-Laplace transform of Eqn. (17), gives the following.

$$L_a({}^{CF}D_t^k y(t) + Uy(t) + Ny(t) - f(t)) = 0 \tag{4.11}$$

Applying the modified-Laplace derivative property of the Caputo-Fabrizio type to Eqn. (18), gives

$$\frac{(slog_e a)y(slog_e a)}{slog_e a + k(1 - slog_e a)} - (slog_e a + k(1 - slog_e a))^{-1} y(0) =$$

$$-L_a(Uy(t) + Ny(t) - f(t)) \tag{4.12}$$

isolating $y(log_e a)$ in Eqn. (19), gives

$$y(slog_e a) = \frac{y(0)}{slog_e a}$$

$$+(\frac{(slog_e a + k(1 - slog_e a))}{slog_e a})L_a(-Uy(t) - Ny(t) + f(t)). \tag{4.13}$$

Taking inverse modified-Laplace transform of Eqn.(20), gives

$$y(t) = L_a^{-1}(\frac{y(0)}{slog_e a}) + (\frac{(slog_e a + k(1 - slog_e a))}{slog_e a} L_a(Uy(t) + Ny(t) - f(t))), \tag{4.14}$$

with initial approximation as

$$y_0(t) = \frac{y(0)}{slog_e a} \tag{4.15}$$

Using variational iteration formula to Eqn. (17), gives

$$y_{n+1} = y_n + \int_0^t \lambda(r)({}^{CF}D_r^k y_n(r) + Uy_n(r) + Ny_n(r) - f(r))dr \tag{4.16}$$

Applying modified-Laplace transform to Eqn. (23), gives

$$y_{n+1}(slog_e a) = y_n(slog_e a) + L_a \int_0^t \lambda(r)({}^{CF}D_r^k y_n(r) + U\bar{y}_n(r) + N\bar{y}_n(r) - f(r))dr \tag{4.17}$$

Regarding the items $L_a(U\bar{y}_n(r) + N\bar{y}_n(r)$ as a restricted variation in Eqn. (24), and differentiating Eqn.(24) with respect to $y_n(log_e a)$, gives

$$\frac{dy_{n+1}(slog_e a)}{dy_n(slog_e a)} = 1 + \lambda(\frac{slog_e a}{slog_e a + k(1 - slog_e a)}) \tag{4.18}$$

Isolating $\lambda(t)$ by setting $\frac{dy_{n+1}(slog_e a)}{dy_n(slog_e a)} = 0$ in Eqn. (25), gives

$$\lambda(t) = \frac{-1(slog_e a + k(1 - slog_e a))}{(slog_e a)} \tag{4.19}$$

substituting Eqn.(26) into Eqn. (24), gives
$y_{n+1}(slog_e a) = y_n(slog_e a) - \frac{(slog_e a + k(1 - slog_e a))}{(slog_e a)}$

$$L_a({}^{CF}D_t^k y_n(t)) + Uy_n(t) + Ny_n(t) - f(t)) \tag{4.20}$$

The consecutive approximations are derived by applying the inverse modified-Laplace transform to equation (27), resulting in the correction functional of MLVIM as
$y_{n+1}(t) = y_n(t) + L_a^{-1} \frac{-(slog_e a + k(1 - slog_e a))}{(slog_e a)}$

$$L_a({}^{CF}D_t^k y_n(t)) + Uy_n(t) + Ny_n(t) - f(t) \tag{4.21}$$

5. Applications

The above scheme is applied to obtain solutions of certain fractional order differential equations.
Illustration 4.1: Consider the nonlinear fractional order differential equation of Caputo-Fabrizio type

$${}^{CF}D_t^k y(t) = -y(t) + y^2(t), z - 1 < k < z, y(0) = \frac{1}{2}, \tag{5.1}$$

has an exact solution when k=1 as $\frac{e^{-t}}{e^{-t}+1}$ (Muhammad [31]
Taking modified-Laplace transform (L_a) of Eqn. (29) and using initial conditions gives the following.

$$y(slog_e a) = \frac{1}{(2slog_e a)} - \frac{(slog_e a + k(1 - slog_e a))}{slog_e a} L_a(-y + y^2), \tag{5.2}$$

taking inverse modified-Laplace transform (L_a^{-1}) of Eqn. (30), gives

$$y(t) = \frac{1}{2} + L_a^{-1}(\frac{(slog_e a + k(1 - slog_e a))}{slog_e a} L_a((-y + y^2))), \tag{5.3}$$

applying modified-Laplace variational iteration scheme, which is Eqn. (28) to Eqn. (31), gives correction functional as

$$y_{n+1}(t) = y_n(t) + L_a^{-1}(\frac{(slog_e a + k(1 - slog_e a))}{slog_e a} L_a({}^{CF}D_t^k y_n + y_n - y_n^2) \tag{5.4}$$

Starting with initial approximation $y_0(t) = \frac{1}{2}$ and using iteration formula for Eqn. (32), gives

$$y_1(t) = \frac{1}{2} - \frac{1}{4}kt$$

$$y_2(t) = \frac{1}{2} - \frac{1}{4}kt + \frac{1}{2}k$$

$$-\frac{1}{4}k^2 + \frac{1}{4}k^2t + \frac{1}{16}k^2t^2 + \frac{1}{48}k^3t^3 - \frac{1}{16}k^3t^2 + ... \tag{5.5}$$

Table 1: Approximate solution of MLVIM, Exact solution and Absolute errors when k = 1 of Eqn. (29)

t	MLVIM solution	Exact solution	AEMLVIM	AEKTADM [31]
0.1	0.4750002083	0.4750208126	0.0000206043	0.0000208126
0.2	0.4500066667	0.4501660027	0.0001593360	0.0001660027
0.3	0.4250506250	0.4255574831	0.0005068581	0.0005574831
0.4	0.4002133333	0.4013123399	0.0010990066	0.0013123399
0.5	0.3756510417	0.3775406688	0.0018896271	0.0025406688
0.6	0.3516200000	0.3543436938	0.0027236938	0.0043436938
0.7	0.3285104583	0.3318122279	0.0033107696	0.0068122279
0.8	0.3068266667	0.3100255190	0.0031988523	0.0100255190
0.9	0.2873018750	0.2890504974	0.0017486224	0.0140504974
1.0	0.2708333333	0.2689414214	0.0016549282	0.0189414214
MAPE:			0.0449248436	0.1595394283

AEMLVIM: Absolute error of the modified-Laplace variational iteration method.
AEKTADM: Absolute Error of Kamal Transform Adomian Decomposition Method.
MAPE: Mean Absolute Percentage Error.

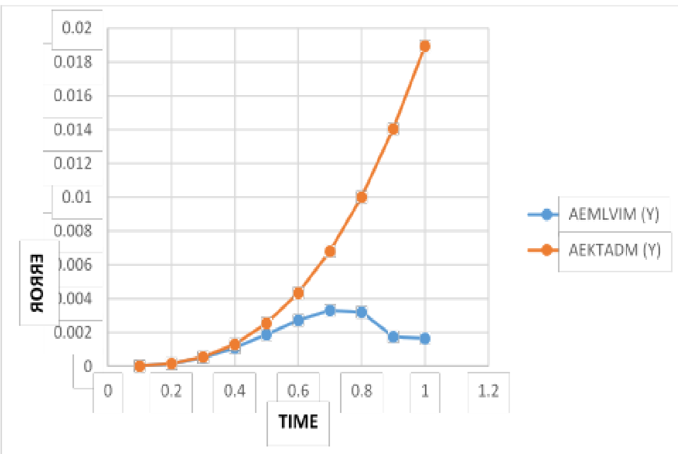

Figure 1: Absolute Error of MLVIM and KTADM for Eqn.(29)

Table 2: Solutions of MLVIM for Eqn. (29) at different values of fractional order (k)

t	k=0.6	k=0.8	k=1
0.2	0.4180360000	0.4420853333	0.4501666667
0.4	0.3762880000	0.3946826667	0.4013333333
0.6	0.3349720000	0.3483040000	0.3545000000
0.8	0.2943040000	0.3034613333	0.3106666667
1.0	0.2545000000	0.2606666667	0.2708333333

Illustration 4.2: Consider the system of the epidemic model, which is a nonlinear fractional order differential equation of the Caputo-Fabrizio type [32].

$$^{CF}D_t^k Y(t) = -\beta Y(t)I(t) \tag{5.6}$$

$$^{CF}D_t^k I(t) = \beta Y(t)I(t) - \alpha I(t) \tag{5.7}$$

$$^{CF}D_t^k R(t) = I(t), 0 < k < 1, \tag{5.8}$$

$Y(0) = 20, I(0) = 15, R(0) = 10, \beta = 0.01, \alpha = 0.002$
Taking modified-Laplace transform (L_a) of Eqns. (34),(35), (36) and, using initial conditions, gives

$$Y(slog_e a) = \frac{20}{(slog_e a)} - \frac{(slog_e a + k(1 - slog_e a))}{slog_e a} L_a(-\beta YI), \tag{5.9}$$

$$I(slog_e a) = \frac{15}{(slog_e a)} - \frac{(slog_e a + k(1 - slog_e a))}{slog_e a} L_a(-\beta YI - \alpha I), \tag{5.10}$$

$$R(slog_e a) = \frac{10}{(slog_e a)} - \frac{(slog_e a + k(1 - slog_e a))}{slog_e a} L_a(\alpha I), \tag{5.11}$$

taking inverse modified-Laplace transform (L_a^{-1}) of Eqns. (37), (38) (39), gives

$$Y(t) = 20 + L_a^{-1}(\frac{-(slog_e a + k(1 - slog_e a))}{slog_e a} L_a((-\beta YI))) \tag{5.12}$$

$$I(t) = 15 + L_a^{-1}(\frac{-(slog_e a + k(1 - slog_e a))}{slog_e a} L_a((-\beta YI - \alpha I))) \tag{5.13}$$

$$R(t) = 10 + L_a^{-1}(\frac{-(slog_e a + k(1 - slog_e a))}{slog_e a} L_a((\alpha I))) \tag{5.14}$$

Using the coupled fractional modified-Laplace variational scheme of Eqn. (28) to Eqn. (40), (41) (42) gives the correction functional as

$$Y_{n+1}(t) = Y_n(t)$$

$$+L_a^{-1}(-(slog_e a + k(1 - slog_e a))slog_e a L_a((^{CF}D_t^k Y_n + \beta Y_n I_n))) \tag{5.15}$$

$$I_{n+1}(t) = I_n(t)$$

$$+L_a^{-1}(-(slog_e a + k(1 - slog_e a))slog_e a L_a(^{CF}D_t^k I_n - \beta Y_n I_n + \alpha I_n)) \tag{5.16}$$

$$R_{n+1}(t) = R_n(t)$$

$$+L_a^{-1}(-(slog_e a + k(1 - slog_e a))slog_e a L_a{}^{CF}D_t^k R_n - \alpha I_n) \quad (5.17)$$

Starting with initial approximation $Y_0(t) = 20$, $I_0(t) = 15$, $R_{(0)} = 10$ and using iteration formula for Eqn. (43), (44) (45) gives the following.

$$Y_1(t) = 17 - 3kt + 3k \quad (5.18)$$
$$I_1(t) = 18 - 3k + 2.7kt \quad (5.19)$$
$$R_1(t) = 10.3 - 0.3k + 0.3kt \quad (5.20)$$

$$Y_2(t) = 13.94 + 9.03k - 5.979kt + 2.7182k^2t$$

$$-2.88k^2 + 0.0405k^2t^2 - 0.0855k^3t^2 + 0.261k^3t + 0.027k^3t^3 - 0.09k^3 + ... \quad (5.21)$$

$$I_2(t) = 20.7 - 8.31k + 5.26kt - 2.304k^2t$$

$$+2.52k^2 - 0.1485k^2t^2 + 0.1665k^3t^2 - 0.261k^3t - 0.027k^3t^3 + 0.09k^3 + ... \quad (5.22)$$

$$R_2(t) = 10.66 + 0.714kt - 1.02k - 0.414k^2t + 0.36k^3 + 0.027k^2t^2 + ... \quad (5.23)$$

Table 3: Approximate of MLVIM and RK4 solutions of Eqns. (34), (35) and (36) when k= 1

t	(Y)MLVIM	(I)MLVIM	(R)MLVIM	(Y)RK4	(I)RK4	(R)RK4
0.1	19.699577	15.270153	10.030270	19.699578	15.270152	10.030270
0.2	19.398416	15.540504	10.061080	19.398426	15.540494	10.061081
0.3	19.096679	15.810891	10.092430	19.096713	15.810855	10.092432
0.4	18.794528	16.081152	10.124320	18.794611	16.081065	10.124324
0.5	18.492125	16.351125	10.156750	18.492292	16.350951	10.156756
0.6	18.189632	16.620648	10.189720	18.189932	16.620340	10.189728
0.7	17.887211	16.889559	10.223230	17.887704	16.889059	10.223237
0.8	17.585024	17.157696	10.257280	17.585779	17.156938	10.257383
0.9	17.283233	17.424897	10.291870	17.284328	17.423808	10.291864
1.0	16.982000	17.691000	10.327000	16.983520	17.689502	10.326978

Illustration 4.3: Consider stiff system of nonlinear fractional order differential equation of Caputo-Fabrizio type [27].

$${}^{CF}D_t^k X(t) = -k_1 X(t) + k_2 Y(t) Z(t) \quad (5.24)$$
$${}^{CF}D_t^k Y(t) = k_3 X(t) + k_4 Y(t) Z(t) - k_5 Y^2(t) \quad (5.25)$$
$${}^{CF}D_t^k Z(t) = k_6 Y^2(t), 0 < k < 1, \quad (5.26)$$

$$X(0) = 1, Y(0) = 0, Z(0) = 0, k_1 = 0.04, k_2 = 0.001,$$

$$k_3 = 400, k_4 = 100, k_5 = 3000 \quad and \quad k_6 = 30.$$

Table 4: Absolute differences of MLVIM with NIM for Eqns. (34) (35) and (36)

t	(Y)MLVIM	(I)MLVIM	(R)MLVIM	(Y)NIM	(I)NIM	(R)NIM [32]
0.1	0.000001	0.000001	0.000000	0.000028	0.000028	0.000000
0.2	0.000010	0.000010	0.000001	0.000226	0.000226	0.000001
0.3	0.000034	0.000036	0.000002	0.000763	0.000765	0.000002
0.4	0.000083	0.000087	0.000004	0.001811	0.001815	0.000004
0.5	0.000167	0.000174	0.000006	0.003542	0.003549	0.000006
0.6	0.000302	0.000308	0.000008	0.006134	0.006140	0.000008
0.7	0.000493	0.000500	0.000007	0.009754	0.009761	0.000007
0.8	0.000755	0.000758	0.000003	0.014579	0.014582	0.000003
0.9	0.001095	0.001089	0.000006	0.020778	0.020077	0.000006
1.0	0.001520	0.001498	0.000022	0.028520	0.028498	0.000022
MAPE	0.000243	0.000271	0.000006	0.004694	0.008536	0.000006

Table 5: Solutions of MLVIM for Eqn. (34) at different values of fractional order (k)

t	k=0.6	k=0.8	k=1
0.2	17.79114234	18.69210675	19.69957700
0.4	17.28049357	18.10872858	18.79452800
0.6	16.77009363	17.52524902	18.18963200
0.8	16.26022246	16.94233165	17.58502400
1.0	15.75116000	16.36064000	16.98220000.

Table 6: Solution of MLVIM for Equation (35) at different values of fractional order (k)

t	k=0.6	k=0.8	k=1
0.2	17.09453030	16.23113933	15.54050400
0.4	17.54674099	16.74995174	16.08115200
0.6	17.99699213	17.26665370	16.62064800
0.8	18.44500378	16.78058163	17.15769600
1.0	18.89049600	18.29107200	17.69100000.

Table 7: Solutions of MLVIM for Equation (36) at different values of fractional order (k)

t	k=0.6	k=0.8	k=1
0.2	10.2338608	10.1363392	10.0610800
0.4	10.2908992	10.1996608	10.1243200
0.6	10.3487152	10.2643648	10.1897200
0.8	10.4073088	10.3304512	10.2572800
1.0	10.4666800	10.3979200	10.3270000.

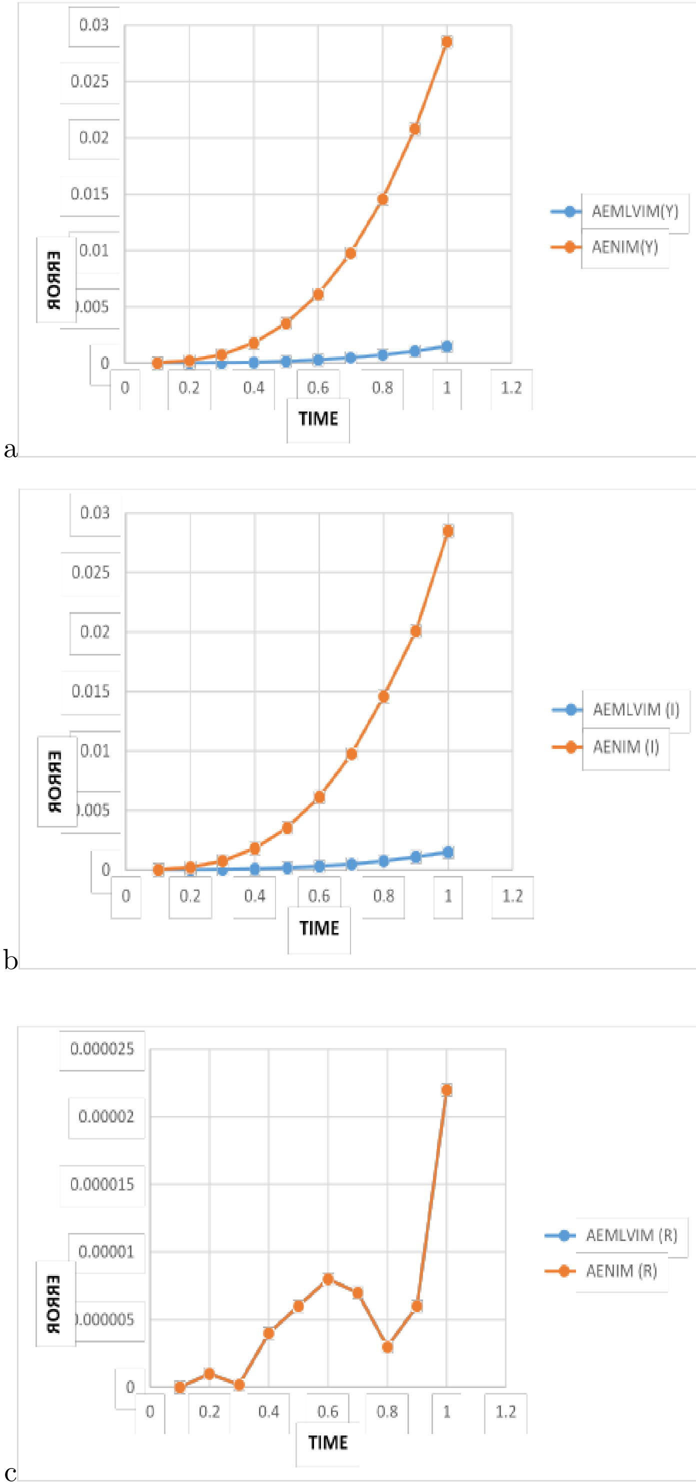

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Figure 2: Absolute Error of MLVIM and NIM for Eqns.(34, 35, 36)

Taking modified-Laplace transform (L_a) of Eqns. (52) (53) (54) and using initial conditions, gives

$$X(slog_e a) = \frac{1}{(slog_e a)} - \frac{(slog_e a + k(1 - slog_e a))}{slog_e a} L_a(-k_1 X + k_2 Y Z), \quad (5.27)$$

$$Y(slog_e a) = \frac{0}{(slog_e a)} - \frac{(slog_e a + k(1 - slog_e a))}{slog_e a} L_a(k_3 X + k_4 Y Z - k_5 Y^2), \quad (5.28)$$

$$Z(slog_e a) = \frac{0}{(slog_e a)} - \frac{(slog_e a + k(1 - slog_e a))}{slog_e a} L_a(k_6 Y^2), \quad (5.29)$$

taking inverse modified-Laplace transform (L_a^{-1}) of Eqns. (55) (56) (57), gives

$$X(t) = 1 + L_a^{-1}(\frac{-(slog_e a + k(1 - slog_e a))}{slog_e a} L_a((-k_1 X + k_2 Y Z))) \quad (5.30)$$

$$Y(t) = 0 + L_a^{-1}(\frac{-(slog_e a + k(1 - slog_e a))}{slog_e a} L_a((k_3 X + k_4 Y Z - k_5 Y^2))) \quad (5.31)$$

$$Z(t) = 0 + L_a^{-1}(\frac{-(slog_e a + k(1 - slog_e a))}{slog_e a} L_a((k_6 Y^2))) \quad (5.32)$$

Using the coupled fractional modified-Laplace variational scheme of Eqn. (28) to eqns. (58) (59) (60), gives correction functional as

$$X_{n+1}(t) = X_n(t)$$
$$+L_a^{-1}(-(slog_e a + k(1 - slog_e a))slog_e a L_a((^{CF}D_t^k X_n + k_1 x_n - k_2 Y_n Z_n))) \quad (5.33)$$
$$Y_{n+1}(t) = Y_n(t)$$
$$+L_a^{-1}(-(slog_e a + k(1 - slog_e a))slog_e a L_a(^{CF}D_t^k Y_n - k_3 X_n - k_4 Y_n Z_n)) \quad (5.34)$$
$$Z_{n+1}(t) = Z_n(t)$$
$$+L_a^{-1}(-(slog_e a + k(1 - slog_e a))slog_e a L_a{}^{CF}D_t^k Z_n - k_6 Y_n^2) \quad (5.35)$$

Starting with initial approximation $X_0(t) = 1$, $Y_0(t) = 0$, $Z_{(0)} = 0$ and using iteration formula for Eqns. (61) (62) (63), gives

$$X_1(t) = 0.96 - 0.04kt + 0.04k \quad (5.36)$$
$$Y_1(t) = 400 - 400k + 400kt \quad (5.37)$$
$$Z_1(t) = 0 \quad (5.38)$$

$$X_2(t) = 0.9216 + 0.1168k - 0.0768kt + 0.0368k^2 t$$
$$-0.0384k^2 + 0.0008k^2 t^2 + ... \quad (5.39)$$

$$Y_2(t) = -479999216 - 1439999232kt + 1439998832k + 2879999632k^2 t -$$
$$143999961k^2 - 960000008k^2 t^2 - 1440000000k^3 t +$$
$$480000000k^3 + 960000000k^3 t^2 - 160000000k^3 t^3 + ... \quad (5.40)$$

$$Z_2(t) = 4800000 + 14400000kt - 14400000k - 28800000k^2 t + 14400000k^3$$
$$+9600000k^2 t^2 + 14400000k^3 t - 4800000k^3 - 9600000k^3 t^2 + 1600000k^3 t^3 + ... \quad (5.41)$$

Table 8: Approximate of MLVIM and RK4 solutions for Eqns. (52), (53) and (54) when k= 1

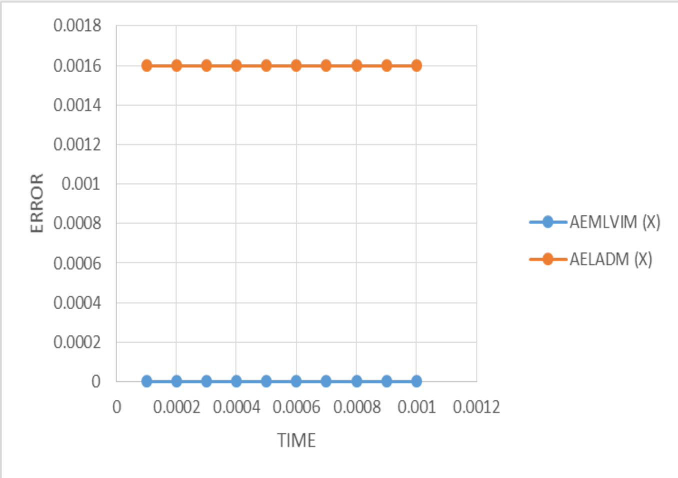
t	(X)MLVIM	(Y)MLVIM	(Z)MLVIM	(X)RK4	(Y)RK4	(Z)RK4
0.0001	0.99999600	0.03983992	0.00000160	0.99999600	0.03984068	0.00001592
0.0002	0.99999200	0.07871968	0.00001280	0.99999200	0.07874381	0.00001256
0.0003	0.99998800	0.11567928	0.00004320	0.99998800	0.11585818	0.00004141
0.0004	0.99998400	0.14975872	0.00010240	0.99998400	0.15048867	0.00009510
0.0005	0.99998000	0.17999800	0.00020000	0.99998000	0.18213874	0.00017860
0.0006	0.99997600	0.20543712	0.00034560	0.99997600	0.21052164	0.00029476
0.0007	0.99997200	0.22511608	0.00054880	0.99997200	0.23554522	0.00044425
0.0008	0.99996800	0.23807488	0.00081920	0.99996800	0.25727832	0.00062720
0.0009	0.99996400	0.24335352	0.00116640	0.99996400	0.27590886	0.00084090
0.0010	0.99996000	0.23999200	0.00160000	0.99996000	0.29170201	0.00108298

Table 9: Absolute differences of MLVIM with LADM for Eqns. (52), (53) and (54)

t	(X)MLVIM	(Y)MLVIM	(Z)MLVIM	(X)LADM	(Y)LADM	(Z)LADM [27]
0.0001	0.00000000	0.00000076	0.00001432	0.00159996	0.00015922	0.00001592
0.0002	0.00000000	0.00002413	0.00000024	0.00159992	0.00125589	0.00001256
0.0003	0.00000000	0.00017890	0.00000179	0.00159988	0.00414112	0.00004141
0.0004	0.00000000	0.00072995	0.00000730	0.00159984	0.00951003	0.00009510
0.0005	0.00000000	0.00214074	0.00002140	0.00159980	0.01785926	0.00017860
0.0006	0.00000000	0.00508452	0.00005084	0.00159976	0.02947546	0.00029476
0.0007	0.00000000	0.01042914	0.00010455	0.00159972	0.04445088	0.00044425
0.0008	0.00000000	0.01920344	0.00019200	0.00159968	0.06271658	0.00062720
0.0009	0.00000000	0.03255534	0.00032550	0.00159964	0.08408464	0.00084090
0.0010	0.00000000	0.05171001	0.00051702	0.00159960	0.10828999	0.00108298
MAPE	0.00000000	0.66406526	3.87203985	0.01599815	1.96919431	11.39288212

Table 10: Solutions of MLVIM for Equation (52) at different values of fractional order (k)

t	k=0.6	k=0.8	k=1
0.0002	0.9778494336	0.9904564220	0.9999920000
0.0004	0.9778428672	0.9904488451	0.9999840001
0.0006	0.9778363009	0.9904412672	0.9999760003
0.0008	0.9778297346	0.9904336903	0.9999680005
0.0010	0.9778231683	0.9904261125	0.9999600008.



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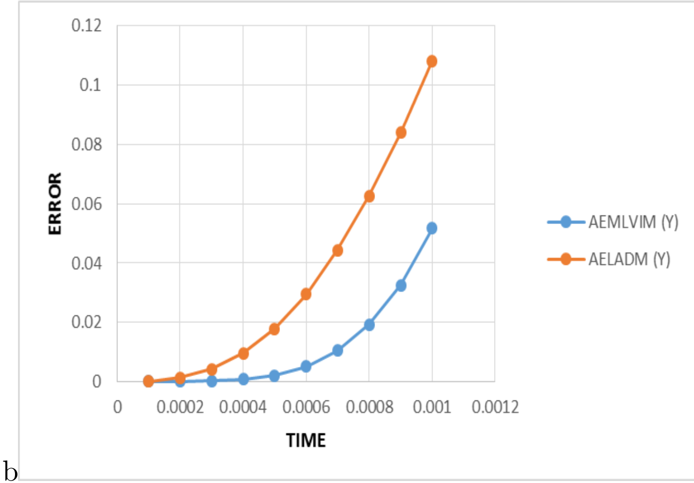

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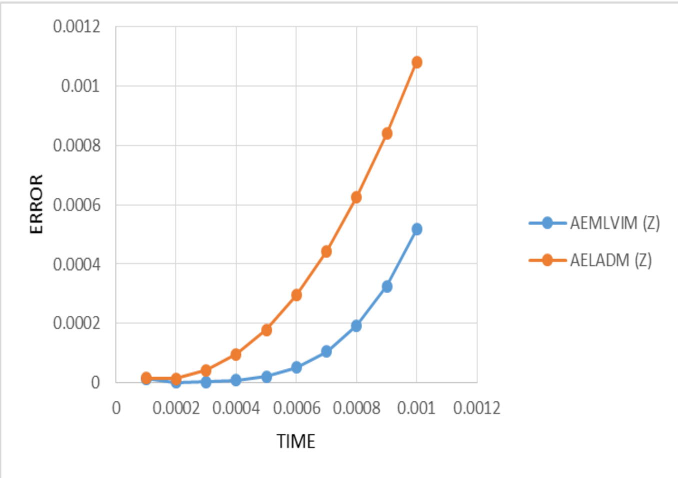

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Figure 3: Absolute Error of MLVIM and LADM for Eqn.(52, 53, 54)

Table 11: Solutions of MLVIM for Equation (53) at different values of fractional order (k)

t	k=0.6	k=0.8	k=1
0.0002	-30747432.02	-3849125.48	0.0787196800
0.0004	-30775096.90	-3858356.12	0.1497587200
0.0006	-30802772.61	-3867596.65	0.2054371200
0.0008	-30830458.51	-3876846.98	0.2380748800
0.0010	-30858156.21	-3886107.14	0.2399920000

Table 12: Solutions of MLVIM for Equation (54) at different values of fractional order (k)

t	k=0.6	k=0.8	k=1
0.0002	307476.54	38492.21	0.00001280
0.0004	307753.18	38584.52	0.00010240
0.0006	308029.94	38676.93	0.00034560
0.0008	308306.80	38769.43	0.00081920
0.0010	308583.78	38862.03	0.00160000

Discussion

A novel approach, namely MLVIM, has been devised to address nonlinear differential equations of the Caputo-Fabrizio type. This method utilises the integral transform as its foundation. To assess the effectiveness of the scheme, three different examples were examined, and the results are shown in Tables 1 to 12 with Figure 1, Figure 2 and Figure 3. Table 1 shows the results of Example 1 compared to the exact solution at k = 1, where the exact solution was attainable. The MLVIM exhibited a reduced absolute error compared to the referred solution in [31], when the Kamal transform was combined with ADM and Figure 1 visualized the absolute error for MLVIM and KTADM. This performance of this is attributed to the correction functionals derived from VIM, which have been shown to outperform the decomposition of polynomials in ADM. Furthermore, Table 2 presents the solution of Example 1 in various fractional orders. This solution illustrates the concealed impacts of fractional order in achieving solutions of classical order.
Tables 3 and 4 with Figure 2 display the outcomes of Example 2, which is a system of equations that describe the fundamental epidemic model. Table 3 presents the solution achieved using MLVIM, in comparison to the Runge-Kutta technique of order 4. On the other hand, Table 4 shows the absolute difference between MLVIM and the Runge-Kutta method, compared to the difference obtained in the reference [32]. The Runge-Kutta method was utilised as a control technique due to the lack of an exact solution of the classical order of k = 1. In this example, it was observed that MLVIM exhibited a reduced disparity compared to NIM [32]. The solution of Example 2 in fractional orders for each compartment is also presented in Tables 5, 6, and 7, respectively, with an explanation similar to that of Example 1 in terms of hereditary and nonlocal features of fractional order calculus.
The solutions for Example 3 can be found in Tables 8 and 9 together with Figure 3. Table 8 illustrates the agreement in the results achieved by MLVIM and the Runge-Kutta method. On the other hand, Table 9 and Figure 3 show the superior performance demonstrated by MLVIM compared to [27], which used the Laplace Adomian approach. The Runge-Kutta method was used as a control to measure this comparison. Tables 10, 11, and 12 demonstrate the system's behaviour at fractional orders prior to attaining a solution at the classical level. This phenomenon maintains nonlocal features and is closely aligned with the overall attributes of fractional calculus.

Conclusion

A novel method, named modified Laplace transform VIM (MLVIM), has incorporated the correction functional of VIM into the existing scheme of the modified Laplace transform. Subsequently, this approach was utilised to address three distinct problems to validate the efficacy of the technique. The results were then juxtaposed with solutions found in the existing literature, thereby confirming the superior performance of MLVIM compared to other methods.
The results for fractional orders also reveal hidden solution behaviors compared to traditional integer order calculus. This method was employed without employing linearization, discretization, or unreasonable assumptions, making it highly recommended for resolving problems that arise in the fields of biology, engineering, and the sciences.

Declaration

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