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Nonconvex Fractional Partial Hyperbolic Differential Inclusions with State-Dependent Delay

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Abstract

In this paper, we study the existence of solutions for a class of fractional-order partial hyperbolic differential inclusions with state-dependent delay, where the right-hand side is a nonconvex-valued multivalued map. The analysis is developed within an appropriate phase space and based on the Caputo fractional derivative. By employing the Covitz-Nadler fixed point theorem for multivalued contractions, we establish sufficient conditions ensuring the existence of mild solutions to the considered problem. The proposed framework extends and generalizes several known results in the literature on fractional differential inclusions with infinite delay. The obtained conditions are expressed in optimal intervals and can be applied to a wide range of models arising in physics and engineering that exhibit memory and hereditary properties.

Keywords: Partial functional differential inclusion, fractional order, solution, left-sided mixed Riemann-Liouville integral, Caputo fractional derivative, state-dependent delay, fixed point.

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1. Introduction

The first result of this paper concerns the existence of solutions to fractional-order initial value problems (IVPs) for the system

$$({}^cD_0^r u)(t,x) \in F(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}), \quad (t,x)\in J, \tag{1.1}$$

$$u(t,x)=\phi(t,x), \quad (t,x)\in \tilde{J}, \tag{1.2}$$

$$\begin{cases} u(t,0)=\varphi(t), \\ u(0,x)=\psi(x), \end{cases} \qquad (t,x)\in J, \tag{1.3}$$

where $\varphi(0)=\psi(0)$, $J:=[0,a]\times[0,b]$, $a,b,\alpha,\beta>0$, $\tilde{J}:=[-\alpha,a]\times[-\beta,b]\setminus[0,a]\times[0,b]$, and ${}^cD_0^r$ denotes the standard Caputo fractional derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$. The function $F\times C(\tilde{J},\mathbb{R}^n)\to$

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$\mathcal{P}(\mathbb{R}^n)$ is a multivalued map with compact and nonconvex values, where $\mathcal{P}$ denotes the family of all subsets of $\mathbb{R}^n$. The functions $\rho_1 \times C(\tilde{J}, \mathbb{R}^n) \to [-\alpha, a]$ and $\rho_2 \times C(\tilde{J}, \mathbb{R}^n) \to [-\beta, b]$ are given, and $\phi \in C(\tilde{J}, \mathbb{R}^n)$ is a prescribed continuous function satisfying $\phi(t,0) = \varphi(t)$ and $\phi(0,x) = \psi(x)$ for each $(t,x) \in J$. The functions $\varphi : [0,a] \to \mathbb{R}^n$ and $\psi : [0,b] \to \mathbb{R}^n$ are assumed to be absolutely continuous.

We denote by $u_{(t,x)}$ the element of $C(\tilde{J}, \mathbb{R}^n)$ defined by

$$u_{(t,x)}(s,\tau) = u(t+s, x+\tau), \quad (s,\tau) \in \tilde{J},$$

where $u_{(t,x)}(\cdot,\cdot)$ represents the history of the state u.

The second result concerns the existence of solutions to fractional-order partial differential inclusions of the form

$$({}^cD_0^r u)(t,x) \in F(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}), \quad (t,x) \in J, \tag{1.4}$$

$$u(t,x) = \phi(t,x), \quad (t,x) \in \tilde{J}', \tag{1.5}$$

$$\begin{cases} u(t,0) = \varphi(t), \\ u(0,x) = \psi(x), \end{cases} \quad (t,x) \in J, \tag{1.6}$$

where φ and ψ are as in problem (1.1)-(1.3), $\tilde{J}' := (-\infty, a] \times (-\infty, b] \setminus [0,a] \times [0,b]$, $F \times \mathcal{B} \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map with compact and nonconvex values, $\rho_1 \times \mathcal{B} \to (-\infty, a]$ and $\rho_2 \times \mathcal{B} \to (-\infty, b]$ are given functions, $\phi : \tilde{J}' \to \mathbb{R}^n$ is a given continuous function satisfying $\phi(t,0) = \varphi(t)$ and $\phi(0,x) = \psi(x)$ for each $(t,x) \in J$, and $\mathcal{B}$ denotes a phase space to be specified in Section 5.

Fractional differential equations and inclusions are powerful mathematical tools for describing memory-dependent and hereditary phenomena that appear in physics, mechanics, viscoelasticity, control theory, porous media, and electromagnetism (see [5, 8, 21, 26, 27, 28, 29]). These models provide more accurate representations of complex dynamical processes than their integer-order counterparts. In particular, fractional partial differential inclusions allow for the modeling of systems subject to uncertainties or multivalued dynamics.

Differential delay equations and inclusions, also known as functional differential equations and inclusions, have long been used to model scientific phenomena. In many cases, the delay is either a fixed constant or given as an integral, in which case it is called a distributed delay; see, for instance, the book by Hale and Verduyn Lunel [14] and the paper [13]. Many research papers and monographs have been devoted to this subject, including those by Kilbas *et al.* [23], Lakshmikantham *et al.* [25], Agarwal *et al.* [1, 2], Benchohra *et al.* [6], Helal [15, 17, 16, 18, 19, 20], and Vityuk [30], among others.

The motivation of this work arises from the need to extend existing results to problems involving *state-dependent and infinite delay* on unbounded domains, which remain less explored in the literature.

The present paper provides new existence results for fractional-order systems of the type (1.1)-(1.3) and (1.4)-(1.6). Our contributions can be summarized as follows:

- We establish *optimal existence results* for the considered systems of fractional partial differential inclusions with state-dependent delay.
- We apply *fixed point theorems for multivalued operators*, particularly the Covitz-Nadler fixed point theorem, to derive our main results.
- We extend the analysis to fractional systems defined on *unbounded domains* and involving *infinite delay*, by constructing a suitable phase space $\mathcal{B}$.
- The obtained existence conditions are expressed in *optimal intervals*, providing sharper and more general results than those found in previous related works.

This paper is organized as follows. In Section 2, we introduce notations, definitions, and preliminary results used throughout the paper. In Section 3, we discuss some properties of set-valued maps. In Section 4, we present an existence result for problem (1.1)-(1.3), based on the fixed point theorem for contraction multivalued maps due to Covitz and Nadler [9]. Finally, in Section 5, we describe the axioms of the phase space $\mathcal{B}$ and present the main results concerning the existence of solutions to problems (1.4)-(1.6).

2. Basic Ingredients

In this section, we present several definitions and auxiliary results concerning the main concepts used in this paper.

Let $C(J, \mathbb{R}^n)$ denote the Banach space of all continuous functions from J into $\mathbb{R}^n$ endowed with the norm

$$\|u\|_\infty = \sup_{(t,x)\in J} \|u(t,x)\|,$$

where $\|\cdot\|$ denotes a suitable complete norm on $\mathbb{R}^n$.

As usual, let $AC(J, \mathbb{R}^n)$ denote the space of absolutely continuous functions from J into $\mathbb{R}^n$, and let $L^1(J, \mathbb{R}^n)$ denote the space of Lebesgue-integrable functions $u : J \to \mathbb{R}^n$ endowed with the norm

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(t,x)\| dx dt,$$

where $\|\cdot\|$ again denotes a suitable complete norm on $\mathbb{R}^n$.

Definition 2.1. [31] Let $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, $\theta = (0,0)$, and $u \in L^1(J, \mathbb{R}^n)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by

$$(I^r_\theta u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} u(s,\tau) d\tau ds.$$

In particular,

$$(I^\theta_\theta u)(t,x) = u(t,x), \ (I^\sigma_\theta u)(t,x) = \int_0^t \int_0^x u(s,\tau) d\tau ds; \text{ for almost all } (t,x) \in J,$$

for almost all $(t,x) \in J$, where $\sigma = (1,1)$.

For instance, $I^r_\theta u$ exists for all $(r_1, r_2) \in (0,\infty) \times (0,\infty)$ whenever $u \in L^1(J, \mathbb{R}^n)$. Moreover, if $u \in C(J, \mathbb{R}^n)$, then $(I^r_\theta u) \in C(J, \mathbb{R}^n)$, and

$$(I^r_\theta u)(t,0) = (I^r_\theta u)(0,x) = 0, \quad (t,x) \in J.$$

Example 2.2. Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$. Then

$$I^r_\theta (t^\lambda x^\omega) = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2}, \quad \text{for almost all } (t,x) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1) \times [0,1)$. Denote by $D^2_{tx} := \frac{\partial^2}{\partial t \partial x}$ the mixed second-order partial derivative.

Definition 2.3. [31] Let $r \in (0,1] \times (0,1]$ and $u \in L^1(J, \mathbb{R}^n)$. The mixed fractional Riemann-Liouville derivative of order r of u is defined by

$$D^r_\theta u(t,x) = (D^2_{tx} I^{1-r}_\theta u)(t,x),$$

and the Caputo fractional derivative of order r of u is defined by

$$({}^c D^r_0 u)(t,x) = \left(I^{1-r}_\theta \frac{\partial^2}{\partial t \partial x} u \right)(t,x).$$

In the particular case $\sigma = (1,1)$, we have

$$(D_\theta^\sigma u)(t,x) = ({}^c D_\theta^\sigma u)(t,x) = (D_{tx}^2 u)(t,x), \text{ for almost all } (t,x) \in J.$$

Example 2.4. Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$. Then

$$D_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} t^{\lambda-r_1} x^{\omega-r_2}, \text{ for almost all } (t,x) \in J.$$

3. Some Properties of Set-Valued Maps

Let $(X, |\cdot|)$ be a Banach space. Denote:

- $\mathcal{P}(X) = \{Y \in X : Y \neq \emptyset\}$,
- $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$,
- $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$,
- $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$,
- $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$.

Definition 3.1. A multivalued map $T : X \to \mathcal{P}(X)$ is said to be convex- (resp. closed-) valued if $T(x)$ is convex (resp. closed) for all $x \in X$. It is said to be bounded on bounded sets if $T(B) = \bigcup_{x \in B} T(x)$ is bounded in X for every $B \in \mathcal{P}_b(X)$, i.e.,

$$\sup_{x \in B} \sup_{y \in T(x)} \|y\| < \infty.$$

A multivalued map $G : J \to \mathcal{P}_{cl}(\mathbb{R}^n)$ is said to be measurable if, for every $v \in \mathbb{R}^n$, the function

$$(x) \longmapsto d(v, G(x)) = \inf\{\|v - z\| : z \in G(x)\}$$

is measurable.

For more details on multivalued maps, see the books of Aubin and Cellina [3], Aubin and Frankowska [4], Deimling [11], Gorniewicz [12], Hu and Papageorgiou [22], and Kisielewicz [24].

For each $u \in C(J, \mathbb{R}^n)$, define the set of selections of F by

$$S_{F,u} = \{f \in L^1(J, \mathbb{R}^n) : f(t,x) \in F(t,x,u(t,x)) \; a.e. \; (t,x) \in J\}.$$

Let (X,d) be a metric space induced by the normed space $(X, \|\cdot\|)$. Define $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ by

$$H_d(A,B) = \max\left\{\sup_{a \in A} d(a,B), \sup_{b \in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a \in A} d(a,b)$ and $d(a,B) = \inf_{b \in B} d(a,b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [24]).

Definition 3.2. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called

(a) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(N(u), N(v)) \leq \gamma d(u,v) \quad \textit{for all } u, v \in X,$$

(b) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Theorem 3.3 (Covitz-Nadler [9])**.** *Let (X,d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then N has a fixed point.*

4. Existence Results for the Finite Delay Case

In this section, we present our main existence result for the problem (1.1)-(1.3).

For each $a, b > 0$, we consider the set $C_{(a,b)} := C([-\alpha, a] \times [-\beta, b], \mathbb{R}^n)$. We begin by defining what we mean by a solution of problem (1.1)-(1.3).

Definition 4.1. A function $u \in C_{(a,b)}$ is said to be a solution of (1.1)-(1.3) if there exists a function $f \in L^1(J, \mathbb{R}^n)$ such that $f(t,x) \in F(t, x, u_{(\rho_1(t,x,u_{(t,x)}), \rho_2(t,x,u_{(t,x)}))})$ and $({}^cD_0^r u)(t,x) = f(t,x)$ while u satisfies equations (1.3) on J and the condition (1.2) on $\tilde{J}$.

Lemma 4.2. *A function $u \in C_{(a,b)}$ is a solution of problem* (1.1)-(1.3) *if and only if u satisfies the equation*

$$u(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)\,d\tau ds,$$

for all $(t,x) \in J$ together with the condition (1.2) *on $\tilde{J}$, where*

$$z(t,x) = \varphi(t) + \psi(x) - \varphi(0).$$

Set

$$\begin{aligned} \mathcal{R} &:= \mathcal{R}_{(\rho_1^-, \rho_2^-)} \\ &= \{(\rho_1(s,\tau,u), \rho_2(s,\tau,u)) : (s,\tau,u) \in J \times C(\tilde{J}, \mathbb{R}^n),\ \rho_i(s,\tau,u) \le 0;\ i = 1,2\}. \end{aligned}$$

We always assume that ρ_1 and ρ_2 are continuous and that the function $(s,\tau) \longmapsto u_{(s,\tau)}$ is continuous from $\mathcal{R}$ into C.

Let us now introduce the following hypotheses:

- $(H1)$ $F : J \times \mathbb{R}^n \to \mathcal{P}_{cp}(\mathbb{R}^n)$ has the property that $F(\cdot, u) : J \to \mathcal{P}_{cp}(\mathbb{R}^n)$ is measurable for each $u \in \mathbb{R}^n$.
- $(H2)$ There exists $\ell \in C(J, \mathbb{R}^+)$ such that

$$H_d(F(t,x,u).F(t,x,v)) \le \ell(t,x)\|u - v\|, \text{ for any } u, v \in \mathbb{R}^n,$$

and

$$d(0, F(t,x,0)) \le \ell(t,x), \text{ a.e. } (t,x) \in J.$$

Theorem 4.3. *Assume that hypotheses $(H1)$-$(H2)$ hold. If*

$$\frac{\ell^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \tag{4.1}$$

where $\ell^* = \sup_{(t,x)\in J} \ell(t,x)$.

Then the IVP (1.1)-(1.3) *has at least one solution on* $[-\alpha, a] \times [-\beta, b]$.

Proof: We transform the problem (1.1)-(1.3) into a fixed-point problem. Consider the operators $N : C_{(a,b)} \to \mathcal{P}(C_{(a,b)})$ defined by

$$(Nu)(t,x) = h \in C_{(a,b)}$$

such that

$$h(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ z(t,x) & \\ +\dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)\,d\tau ds, & (t,x) \in J, \end{cases}$$

where $f \in S_{F, u_{(\rho_1(t,x,u), \rho_2(t,x,u))}}$.

Remark 4.4. For each $u \in C_{(a,b)}$, the set $S_{F,u}$ is nonempty since by $(H1)$, F has a measurable selection.

We now show that N satisfies the assumptions of Theorem 3.3. The proof proceeds in two steps.

Step 1: $N(u) \in \mathcal{P}_{cl}(C_{(a,b)})$ for each $u \in C_{(a,b)}$. Indeed, let $(u_n)_{n\geq 0} \in N(u)$ be such that $u_n \to \tilde{u}$ in $C_{(a,b)}$. Then, $\tilde{u} \in C_{(a,b)}$ and there exists $f_n(\cdot,\cdot) \in S_{F,u}$ such that, for each $(t,x) \in J$,

$$u_n(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f_n(s,\tau)d\tau ds,$$

and for each $(t,x) \in \tilde{J}$

$$u_n(t,x) = \phi(t,x).$$

Since F has compact values and by $(H2)$, is continuous in u, we may pass to a subsequence if necessary so that f_n converges weakly to f in $L^1_w(J,\mathbb{R})$ (the space endowed with the weak topology). By Mazur's theorem [**?**], f_n converges strongly to f; hence $f \in S_{F,u}$. Therefore, for each $(t,x) \in J$,

$$u_n(t,x) \to \tilde{u}(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)d\tau ds,$$

and for each $(t,x) \in \tilde{J}, u_n(t,x) \to \tilde{u}(t,x) = \phi(t,x)$. Hence $\tilde{u} \in N(u)$.

Step 2: We now prove that there exists $\gamma < 1$ such that

$$H_d(N(u) - N(\overline{u})) \leq \gamma ||u - \overline{u}||_\infty, \quad \textit{for each } u, \overline{u} \in C_{(a,b)}.$$

Let $u, \overline{u} \in C_{(a,b)}$ and $h \in N(u)$. Then there exists $f(t,x) \in F(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))})$ such that, for each $(t,x) \in J$,

$$h(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)d\tau ds,$$

and for each $(t,x) \in \tilde{J}, h(t,x) = \phi(t,x)$.
From $(H2)$ it follows that

$$H_d(F(t,x,u) - F(t,x,\overline{u})) \leq \ell(t,x)|u - \overline{u}|.$$

Hence there exists $w \in F(t,x,\overline{u})$ such that

$$|f(t,x) - w| \leq \ell(t,x)|u - \overline{u}|, \quad (t,x) \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R}^n)$ given by

$$U(t,x) = \{w \in \mathbb{R}^n : |f(t,x) - w| \leq \ell(t,x)|u - \overline{u}|\}.$$

Since the multivalued map $u(t,x) = U(t,x) \cap F(t,x,\overline{u})$ is measurable (see Proposition III.4 in [7]), there exists a measurable selection $\overline{f}(t,x)$ for u. Then, for each $(t,x) \in J$,

$$|f(t,x) - \overline{f}(t,x)|| \leq \ell(t,x)|u - \overline{u}|.$$

Define for each $(t,x) \in J$,

$$\overline{h}(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \overline{f}(s,\tau)d\tau ds,$$

and for each $(t,x) \in \tilde{J}, \overline{h}(t,x) = \phi(t,x)$.
Then, for each $(t,x) \in \tilde{J}, ||h - \overline{h}||_\infty = 0$, and for each $(t,x) \in J$,

$$\begin{aligned} |h(t,x) &- \overline{h}(t,x)| \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\times |f(s,\tau) - \overline{f}(s,\tau)| d\tau ds \\ &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \ell(s,\tau)|u_{(s,\tau)} - \overline{u}_{(s,\tau)}| d\tau ds \\ &\leq \frac{\ell^* ||u - \overline{u}||_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^a \int_0^b (t-s)^{r_1-1}(x-\tau)^{r_2-1} d\tau ds, \end{aligned}$$

where $\ell^* = \sup\limits_{(s,\tau)\in J} \ell(s,\tau)$.

Therefore

$$\|h - \overline{h}\|_\infty \leq \frac{\ell^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \|u - \overline{u}\|_\infty.$$

By a similar argument obtained by interchanging u and $\overline{u}$, we conclude

$$H_d(N(u) - N(\overline{u})) \leq \frac{\ell^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} ||u - \overline{u}||_\infty.$$

Hence, by (5.1), N is a contraction.

Thus, by Theorem 3.3, N has a fixed point u which is a solution of (1.1)-(1.3).

5. Existence results for the infinite delay case

5.1. The phase space $\mathcal{B}$

The concept of the phase space $\mathcal{B}$ plays a fundamental role in both the qualitative and quantitative study of functional differential equations. A common choice is a semi-normed space satisfying suitable axioms, first introduced by Hale and Kato (see [13]). For further applications, we refer the reader to the books [14**?**] and the references therein.

For any $(t,x) \in J$, let $E_{(t,x)} := [0,t] \times \{0\} \cup \{0\} \times [0,x]$. In the case $t = a$ and $x = b$, we write simply E. Consider $(\mathcal{B}, \|(\cdot,\cdot)\|_{\mathcal{B}})$ to be a seminormed linear space of functions mapping $(-\infty, 0] \times (-\infty, 0]$ into $\mathbb{R}^n$, and satisfying the following fundamental axioms, adapted from those of Hale and Kato for ordinary differential equations:

(A_1) Let $y : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n$ be continuous on J and suppose $y_{(t,x)} \in \mathcal{B}$, for all $(t,x) \in E$, then there exist constants $H, K, M > 0$ such that for any $(t,x) \in J$, the following conditions hold:

- (i) $y_{(t,x)}$ is in $\mathcal{B}$;
- (ii) $\|y(t,x)\| \leq H\|y_{(t,x)}\|_{\mathcal{B}}$,
- (iii) $\|y_{(t,x)}\|_{\mathcal{B}} \leq K \sup_{(s,\tau)\in[0,t]\times[0,x]} \|y(s,\tau)\| + M \sup_{(s,\tau)\in E_{(t,x)}} \|y_{(s,\tau)}\|_{\mathcal{B}}$,

(A_2) For the function $y(\cdot,\cdot)$ in (A_1), $y_{(t,x)}$ is a $\mathcal{B}$-valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

We now present several examples of phase spaces [10].

Example 5.1. Let $\mathcal{B}$ be the set of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ that are continuous on $[-\alpha, 0] \times [-\beta, 0]$, where $\alpha, \beta \geq 0$, equipped with the seminorm

$$\|\phi\|_{\mathcal{B}} = \sup_{(s,\tau)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s,\tau)\|.$$

Then we have $H = K = M = 1$. The quotient space $\widehat{\mathcal{B}} = \mathcal{B}/\|\cdot\|_{\mathcal{B}}$ is isometric to the space $C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n)$ of all continuous functions from $[-\alpha, 0] \times [-\beta, 0]$ into $\mathbb{R}^n$ with the supremum norm, This shows that partial differential functional equations with finite delay are included in our axiomatic model.

Example 5.2. Let $\gamma \in \mathbb{R}$ and let C_γ denote the set of all continuous functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ for which the limit $\lim_{\|(s,\tau)\|\to\infty} e^{\gamma(s+\tau)}\phi(s, \tau)$ exists, equipped with the norm

$$\|\phi\|_{C_\gamma} = \sup_{(s,\tau)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(s+\tau)}\|\phi(s,\tau)\|.$$

Then $H = 1$ and $K = M = \max\{e^{-\gamma(a+b)}, 1\}$.

Example 5.3. Let $\alpha, \beta, \gamma \geq 0$ and define the seminorm

$$\|\phi\|_{CL_\gamma} = \sup_{(s,\tau)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s,\tau)\| + \int_{-\infty}^{0}\int_{-\infty}^{0} e^{\gamma(s+\tau)}\|\phi(s,\tau)\|d\tau ds.$$

The space CL_γ consists of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ that are continuous on $[-\alpha, 0] \times [-\beta, 0]$ measurable on $(-\infty, -\alpha] \times (-\infty, 0] \cup (-\infty, 0] \times (-\infty, -\beta]$, and satisfy $\|\phi\|_{CL_\gamma} < \infty$. In this case,

$$H = 1,\ K = \int_{-\alpha}^{0}\int_{-\beta}^{0} e^{\gamma(s+\tau)}d\tau ds,\ M = 2.$$

5.2. Main results

In this section, we begin by defining what we mean by a solution of the problem (1.4)-(1.6). Let the space

$$\Omega := \{u : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n : u_{(t,x)} \in \mathcal{B}\ for\ (t, x) \in E\ and\ u|_J\ is\ continuous\}.$$

Definition 5.4. A function $u \in \Omega$ is said to be a solution of (1.4)-(1.6) if there exists a function $f \in L^1(J, \mathbb{R}^n)$ such that $f(t, x) \in F(t, x, u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))})$, $({}^cD_0^r u)(t, x) = f(t, x)$, and u satisfies equations (1.6) on J and condition (1.5) on $\tilde{J}$.

Set $\mathcal{R}' := \mathcal{R}'_{(\rho_1^-,\rho_2^-)}$

$$= \{(\rho_1(s, \tau, u), \rho_2(s, \tau, u)) : (s, \tau, u) \in J \times \mathcal{B},\ \rho_i(s, \tau, u) \leq 0;\ i = 1, 2\}.$$

We always assume that $\rho_1 : J \times \mathcal{B} \to (-\infty, a]$, $\rho_2 : J \times \mathcal{B} \to (-\infty, b]$ are continuous, and that the function $(s, \tau) \longmapsto u_{(s,\tau)}$ is continuous from $\mathcal{R}'$ into $\mathcal{B}$.

We now introduce the following hypothesis:

(H_ϕ) There exists a continuous bounded function $L : \mathcal{R}'_{(\rho_1^-,\rho_2^-)} \to (0, \infty)$ such that

$$\|\phi_{(s,\tau)}\|_{\mathcal{B}} \leq L(s, \tau)\|\phi\|_{\mathcal{B}},\ \text{for all } (s, \tau) \in \mathcal{R}'.$$

In the sequel, we use the following generalization of a consequence of the phase space axioms ([**?**], Lemma 2.1).

Lemma 5.5. *If $u \in \Omega$, then*

$$\|u_{(s,\tau)}\|_{\mathcal{B}} = (M + L')\|\phi\|_{\mathcal{B}} + K \sup_{(\theta,\eta)\in[0,\max\{0,s\}]\times[0,\max\{0,\tau\}]} \|u(\theta,\eta)\|,$$

where

$$L' = \sup_{(s,\tau)\in\mathcal{R}'} L(s,\tau).$$

Our main result in this section is based on the fixed point theorem due to Covitz and Nadler.

Theorem 5.6. *Assume (H_ϕ) and that the following hypotheses hold:*

$(H1)$ *$F : J \times \mathcal{B} \to \mathcal{P}_{cp}(\mathbb{R})$ satisfies that $F(\cdot, u) : J \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $u \in \mathcal{B}$.*

$(H2)$ *There exists $\ell \in C(J, \mathbb{R}^+)$ such that*

$$H_d(F(t,x,u).F(t,x,v)) \le \ell(t,x)\|u - v\|_{\mathcal{B}}, \text{ for all } u, v \in \mathcal{B},$$

and

$$d(0, (F(t,x,0)) \le \ell(t,x), \quad a.e. \ (t,x) \in J.$$

If

$$\frac{K\ell^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \tag{5.1}$$

Then the IVP (1.4)-(1.6) has at least one solution on $(-\infty, a] \times (-\infty, b]$.

Proof: We transform the problem (1.4)-(1.6) into a fixed point problem. Consider the operator $A : \Omega \to \mathcal{P}(\Omega)$ defined by

$$(Au)(t,x) = h \in \Omega$$

where

$$h(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ z(t,x) & \\ +\dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)\,d\tau ds, \ f \in S_{F,u} & (t,x) \in J. \end{cases}$$

Let $v(\cdot,\cdot) : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n$ be defined by,

$$v(t,x) = \begin{cases} z(t,x), & (t,x) \in J. \\ \phi(t,x), & (t,x) \in \tilde{J}, \end{cases}$$

Then $v_{(t,x)} = \phi$ for all $(t,x) \in E$.
For each $w \in C(J, \mathbb{R}^n)$ with $w(t,x) = 0$ for $(t,x) \in E$ we denote by $\overline{w}$ the function

$$\overline{w}(t,x) = \begin{cases} w(t,x) & (t,x) \in J. \\ 0, & (t,x) \in \tilde{J}, \end{cases}$$

If $u(\cdot,\cdot)$ satisfies the integral equation

$$u(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)\,d\tau ds,$$

then $u(t,x)=\overline{w}(t,x)+v(t,x)$ for $(t,x)\in J$, which implies $u_{(t,x)}=\overline{w}_{(t,x)}+v_{(t,x)}$, for all $(t,x)\in J$. The function $w(\cdot,\cdot)$ thus satisfies

$$w(t,x)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}f(s,\tau)d\tau ds,$$

where $f\in S_{F,\overline{w}+v}$. Set

$$C_0=\{w\in C(J,\mathbb{R}^n):\ w(t,x)=0\ \text{ for } (t,x)\in E\},$$

and let $\|\cdot\|_{(a,b)}$ denote the seminorm in C_0 defined by

$$\|w\|_{(a,b)}=\sup_{(t,x)\in E}\|w_{(t,x)}\|_{\mathcal{B}}+\sup_{(t,x)\in J}\|w(t,x)\|=\sup_{(t,x)\in J}\|w(t,x)\|,\ w\in C_0.$$

Then C_0 is a Banach space with norm $\|.\|_{(a,b)}$. Define the operator $N:\ C_0\to\mathcal{P}(C_0)$ by

$$(Nw)(t,x)=h\in C_0,$$

where

$$h(t,x)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}f(s,\tau)d\tau ds,$$

with $f\in S_{F,\overline{w}_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}+v_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}}$. Clearly, A has a fixed point if and only if N has a fixed point.

Remark 5.7. For each $w\in C_0$, the set $S_{F,\overline{w}+v}$ is nonempty since by $(H1)$, F admit a measurable selection.

We now show that N satisfies the assumptions of Theorem 3.3. The proof is given in two steps.

Step 1: $N(w)\in\mathcal{P}_{cl}(C_0)$ for each $w\in C_0$. Indeed, let $(w_n)_{n\geq 0}\in N(w)$ such that $w_n\to\tilde{w}$ in C_0. Then, $\tilde{w}\in C_0$ and there exist $f_n(\cdot,\cdot)\in S_{F,\overline{w}+v}$ such that for each $(t,x)\in J$,

$$w_n(t,x)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}f_n(s,\tau)d\tau ds,$$

and

$$w_n(t,x)=\phi(t,x),\quad \textit{for each } (t,x)\in\tilde{J}.$$

Since F has compact values and by $(H2)$, we may extract a subsequence such that $f_n(\cdot,\cdot)$ converges weakly to f in $L^1_w(J,\mathbb{R})$. By Mazur's theorem, $f_n(\cdot,\cdot)$ converges strongly to f, hence $f\in S_{F,\overline{w}+v}$. It follows that $w_n(t,x)\to\tilde{w}(t,x)$ uniformly, so $\tilde{w}\in N(w)$.

Step 2: We show that there exists $\gamma<1$ such that

$$H_d(N(w)-N(w^*))\leq\gamma||w-w^*||_{(a,b)},\quad \textit{for each } w,w^*\in C_0.$$

Steps identical, ending with

$$H_d(N(w)-N(w^*))\leq\frac{K\ell^*a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}||\overline{w}-\overline{w^*}||_{(a,b)}.$$

Thus, by condition (5.1), N is a contraction and by Theorem 3.3, N has a fixed point w which is solution to (1.1)-(1.3).

6. Conclusion

In this paper, we have investigated the existence of solutions for two classes of fractional-order systems involving state-dependent delays and multivalued right-hand sides. The first class concerns fractional initial value problems defined on a bounded domain, while the second extends the analysis to partial differential inclusions on an unbounded domain with infinite delay. The existence results were obtained using fixed point techniques for multivalued operators, in particular the Covitz-Nadler fixed point theorem. The phase space $\mathcal{B}$ introduced in this work provides a suitable framework for studying such delayed fractional systems.

The main contributions of this paper can be summarized as follows:

- Establishment of optimal existence results for fractional partial differential inclusions with state-dependent delay (finite and infinite cases).
- Application of fixed point principles to multivalued mappings with compact and nonconvex values.
- Formulation of existence conditions in optimal intervals, which generalize and improve several known results in the literature.

Benefits and limitations. The results presented here offer a general theoretical framework applicable to many models in physics, engineering, and biological sciences where fractional dynamics and memory effects play a crucial role. In particular, the inclusion of nonconvex multivalued terms allows for the modeling of uncertain or discontinuous behaviors. However, the current analysis is limited to systems satisfying compactness and continuity conditions of the multivalued maps, and no explicit numerical or stability analysis was carried out. Relaxing these assumptions or extending the results to noncompact cases remains an open problem.

Future work. In future research, it would be of interest to study fractional inclusion systems involving impulses, stochastic effects, or boundary value problems in higher-dimensional domains. Additionally, the theoretical results obtained here could be complemented by numerical simulations and applications to concrete physical models to better illustrate their significance.

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