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New Results Concerned with Fixed Point Theory in Controlled Metric Spaces

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Abstract

This paper introduces a new sufficient condition for Cauchyness on sequences in controlled metric spaces. This sufficient condition can improve many pre-existing fixed point results under weaker hypotheses. Applying this sufficient condition, we establish some fixed point theorems involving nonlinear Ćirić type contraction via control functions in controlled metric spaces and present numerical examples to show the validity.

Keywords: Fixed point; Cauchy sequence; Controlled metric space; Ćirić type contraction.
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1. Introduction

The fixed point theory is very important and effective tool in mathematics and other science. Particularly, it is widely applied in nonlinear analysis. Banach's contraction principle [1] has become a fundamental tool which provides existence of solutions for different types of differential equations and integral equations.

Many authors have generalized the Banach's result under various contraction conditions [2, 3, 4, 5, 6]. Moreover, metric space has been extended into many abstract spaces. The notion of b-metric space defined by Bakhtin [7] is a significant generalization of metric space. In 2017, by using the idea of b-metric space, Kamran et al. [8] initiated the notion of extended b-metric space. In 2018, Mlaiki et al. [9] proposed another generalization of b-metric space and named it controlled metric space.

Cauchyness on sequences plays a key role in proving the existence of fixed points in complete spaces. In 2017, Suzuki [10] and Miculescu and Mihail [11] proposed a useful sufficient condition for the Cauchyness on sequences in b-metric spaces. In 2018, Alqahtani et al. [12] discussed Cauchyness on sequences in extended b-metric spaces. After that in 2020, Mitrovic et al. [13] improved the result for Cauchyness on sequences in [12], and their result is also generalization of the results of Suzuki [10] and Miculescu and Mihail [11]. Their results, which provide sufficient conditions for Cauchyness on sequences, give existence of fixed points under weaker assumptions in b-metric spaces and extended b-metric spaces, and improve many of the fixed point results presented previously.

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Given the strong motivation in previous studies on the sufficient condition for Cauchyness on sequences in fixed point theory, this paper aims to extend the results of Suzuki [10], Miculescu and Mihail [11] and Mitrovic et al. [13] to the controlled metric space. We establish a valid sufficient condition for a sequence in controlled metric space to be a Cauchy sequence and give some examples to show the validity of the obtained results. While the extended b-metric was defined by multiplying the whole right-hand side of the triangular inequality by a function, the controlled metric was defined by multiplying each term on the right-hand side of the triangular inequality by a function. Therefore, it is more difficult and complicated to prove a sufficient condition for Cauchyness on sequences in controlled metric spaces. And our result can be used to improve fixed point theorems in controlled metric spaces. Indeed, many previous fixed point results in controlled metric spaces have been established under certain assumptions. For example, Banach type result (Theorem 1 in [9]), Kannan type result ([14]), Reich type result (Theorem 8 in [15]), Fisher type result ([16]), Geraghty type result (Theorem 2.1 in [17]), rational type fixed point results (Theorem 4 and Theorem 5 in [18], Theorem 2.1 in [19]), nonlinear Banach type result (Theorem 2.1 in [20]), nonlinear Kannan type result (Theorem 2.3 in [20]), nonlinear Chatterjea type result (Theorem 2.5 in [20]). Using our result concerned with the sufficient condition for Cauchyness on sequences, we can show that these fixed point results hold even under weaker assumptions. Applying the obtained results, we also prove some fixed point theorems involving nonlinear Ćirić type contractions via some control functions, which give a partial answer to the open questions presented in Mlaiki et al. [20].

2. Preliminaries

Definition 2.1. [7]. Let $\Xi \neq \emptyset$ and $s \geq 1$. A mapping $\varpi_b : \Xi \times \Xi \to [0, \infty)$ is called a b-metric if $\forall \varsigma, \rho, \sigma \in \Xi$, it satisfies:

$(b1)$ $\varpi_b(\varsigma, \rho) = 0 \Leftrightarrow \varsigma = \rho$;

$(b2)$ $\varpi_b(\varsigma, \rho) = \varpi_b(\rho, \varsigma)$;

$(b3)$ $\varpi_b(\varsigma, \rho) \leq s[\varpi_b(\varsigma, \sigma) + \varpi_b(\sigma, \rho)]$.

The pair (Ξ, ϖ_b) is called a b-metric space (b-*MS* for short).

Definition 2.2. [8]. Let $\Xi \neq \emptyset$ and $e : \Xi \times \Xi \to [1, \infty)$. A mapping $\varpi_{Eb} : \Xi \times \Xi \to [0, \infty)$ is called an extended b-metric if $\forall \varsigma, \rho, \sigma \in \Xi$, it satisfies:

$(Eb1)$ $\varpi_{Eb}(\varsigma, \rho) = 0 \Leftrightarrow \varsigma = \rho$;

$(Eb2)$ $\varpi_{Eb}(\varsigma, \rho) = \varpi_{Eb}(\rho, \varsigma)$;

$(Eb3)$ $\varpi_{Eb}(\varsigma, \rho) \leq e(\varsigma, \rho)[\varpi_{Eb}(\varsigma, \sigma) + \varpi_{Eb}(\sigma, \rho)]$.

The pair (Ξ, ϖ_{Eb}) is called an extended b-metric space (Eb-*MS* for short).

Definition 2.3. [9]. Let $\Xi \neq \emptyset$ and $\vartheta : \Xi \times \Xi \to [1, \infty)$. A mapping $\varpi_c : \Xi \times \Xi \to [0, \infty)$ is called a controlled metric if $\forall \varsigma, \rho, \sigma \in \Xi$, it satisfies:

(C1) $\varpi_c(\varsigma, \rho) = 0 \Leftrightarrow \varsigma = \rho$;

(C2) $\varpi_c(\varsigma, \rho) = \varpi_c(\rho, \varsigma)$;

(C3) $\varpi_c(\varsigma, \rho) \leq \vartheta(\varsigma, \sigma)\varpi_c(\varsigma, \sigma) + \vartheta(\sigma, \rho)\varpi_c(\sigma, \rho)$.

The pair $(\Xi, \varpi_c)$is called a controlled metric space (*CMTS* for short).

Definition 2.4. [9]. Let (Ξ, ϖ_c) be a *CMTS* and $\{\varsigma_t\}$ be a sequence in Ξ. Then

1. $\{\varsigma_t\}$ is called convergent to $\varsigma \in \Xi$ if $\forall \varepsilon > 0$, there is $N = N(\varepsilon) \in \mathbb{N}$ such that $\varpi_c(\varsigma_t, \varsigma) < \varepsilon$, $\forall t \geq N$. This is written as $\lim_{t\to\infty} \varsigma_t = \varsigma$;
2. $\{\varsigma_t\}$ is called Cauchy if $\forall \varepsilon > 0$, there is some $N = N(\varepsilon) \in \mathbb{N}$ such that $\varpi_c(\varsigma_t, \varsigma_s) < \varepsilon$,
$$\forall t, s \geq N;$$
3. (Ξ, ϖ_c) is called complete if every Cauchy sequence in Ξ converges in Ξ.

Miculescu and Mihail [11] and Suzuki [10] obtained the following result in b-*MS*.

Lemma 2.5. *[10, 11]. Let $\{\varsigma_t\}$ be a sequence in a b-MS (Ξ, ϖ_b). Assume that there exists $\lambda \in [0, 1)$ satisfying*

$$\varpi_b(\varsigma_t, \varsigma_{t+1}) \leq \lambda \cdot \varpi_b(\varsigma_{t-1}, \varsigma_t),$$

for every $t \in \mathbb{N}$, then $\{\varsigma_t\}$ is Cauchy.

Alqahtani et al. [12] provided the following result in *Eb-MS*.

Lemma 2.6. *[12]. Let $\{\varsigma_t\}$ be a sequence in an Eb-MS (Ξ, ϖ_{Eb}). Assume that there exists $\lambda \in [0, 1)$ satisfying*

$$\varpi_{Eb}(\varsigma_t, \varsigma_{t+1}) \leq \lambda \cdot \varpi_{Eb}(\varsigma_{t-1}, \varsigma_t),$$

for every $t \in \mathbb{N}$. If

$$\lim_{t,s\to\infty} e(\varsigma_t, \varsigma_s) < \frac{1}{\lambda}, \tag{2.1}$$

then $\{\varsigma_t\}$ is a Cauchy sequence.

Mitrovic et al. [13] proved that instead of condition (2.1) in Lemma 6 can be used the weaker condition

$$\lim_{t,s\to\infty} \sup e(\varsigma_t, \varsigma_s) < \infty. \tag{2.2}$$

3. Main result

In this section, we present a sufficient condition for Cauchyness on sequences in *CMTS*.

The following lemma can be easily obtained from the proof of fixed point result in [9].

Lemma 3.1. *Let $\{\varsigma_t\}$ be a sequence in a CMTS (Ξ, ϖ_c). Suppose that there exists $\lambda \in [0, 1)$ satisfying*

$$\varpi_c(\varsigma_t, \varsigma_{t+1}) \leq \lambda \cdot \varpi_c(\varsigma_{t-1}, \varsigma_t),$$

for every $t \in \mathbb{N}$. If

$$\sup_{s\geq 1} \lim_{t\to\infty} \frac{\vartheta(\varsigma_{t+1}, \varsigma_{t+2})}{\vartheta(\varsigma_t, \varsigma_{t+1})} \vartheta(\varsigma_{t+1}, \varsigma_s) < \frac{1}{\lambda}, \tag{3.1}$$

then $\{\varsigma_t\}$ is Cauchy.

Lemma 3.2. *Let $\{\varsigma_t\}$ be a sequence in a CMTS (Ξ, ϖ_c). And let $\lambda \in [0, 1)$ and $K>0$ such that*

$$\varpi_c(\varsigma_t, \varsigma_{t+1}) < K \cdot \lambda^t,$$

for every $t \in \mathbb{N}$. If (3.1) is satisfied, then $\{\varsigma_t\}$ is a Cauchy sequence.

Proof. We can obtain directly from Lemma 7.

Instead of condition (3.1), we will use weaker condition

$$\sup_{s\geq 1} \lim_{t\to\infty} \frac{\vartheta(\varsigma_{t+1}, \varsigma_{t+2})}{\vartheta(\varsigma_t, \varsigma_{t+1})} \vartheta(\varsigma_{t+1}, \varsigma_s) < \infty. \tag{3.2}$$

Lemma 3.3. *Let $\{\varsigma_t\}$ be a sequence in a CMTS (Ξ, ϖ_c). If there exists $\lambda \in [0, 1)$ satisfying*

$$\varpi_c(\varsigma_t, \varsigma_{t+1}) \leq \lambda \cdot \varpi_c(\varsigma_{t-1}, \varsigma_t), \tag{3.3}$$

for every $t \in \mathbb{N}$, then $\{\varsigma_t\}$ is Cauchy.

Proof. Let $H = \sup_{s\geq 1}\lim_{t\to\infty} \frac{\vartheta(\varsigma_{t+1}, \varsigma_{t+2})}{\vartheta(\varsigma_t, \varsigma_{t+1})}\vartheta(\varsigma_{t+1}, \varsigma_s) < \infty$ and $L = \lim_{t,s\to\infty}\sup \vartheta(\varsigma_t, \varsigma_s) < \infty$. If $H \leq 1$, then the condition (3.1) is satisfied for $\lambda \in [0, 1)$, and it corresponds to Lemma 7. Therefore, we consider the case $H>1$. If $\lambda = 0$, the conclusion is immediate. So, we consider for $\lambda \in (0, 1)$. Repeating (3.3)

$$\varpi_c(\varsigma_t, \varsigma_{t+1}) \leq \lambda^t \varpi_c(\varsigma_0, \varsigma_1). \tag{3.4}$$

For all t, $r\in\mathbb{N}$, using the triangular inequality and $\vartheta(\varsigma_{t+r-1}, \varsigma_{t+r}) \geq 1$, we get:

$$\begin{aligned}\varpi_c(\varsigma_t, \varsigma_{t+r}) \leq & \vartheta(\varsigma_t, \varsigma_{t+1})\varpi_c(\varsigma_t, \varsigma_{t+1}) + \vartheta(\varsigma_{t+1}, \varsigma_{t+r})\varpi_c(\varsigma_{t+1}, \varsigma_{t+r})\\ \leq & \vartheta(\varsigma_t, \varsigma_{t+1})\varpi_c(\varsigma_t, \varsigma_{t+1}) + \vartheta(\varsigma_{t+1}, \varsigma_{t+r})\vartheta(\varsigma_{t+1}, \varsigma_{t+2})\varpi_c(\varsigma_{t+1}, \varsigma_{t+2})\\ & + \vartheta(\varsigma_{t+1}, \varsigma_{t+r})\vartheta(\varsigma_{t+2}, \varsigma_{t+r})\vartheta(\varsigma_{t+2}, \varsigma_{t+3})\varpi_c(\varsigma_{t+2}, \varsigma_{t+3})\\ & +\\ & \vdots\\ & + \vartheta(\varsigma_{t+1}, \varsigma_{t+r})\vartheta(\varsigma_{t+2}, \varsigma_{t+r})\cdots\vartheta(\varsigma_{t+r-2}, \varsigma_{t+r})\vartheta(\varsigma_{t+r-2}, \varsigma_{t+r-1})\varpi_c(\varsigma_{t+r-2}, \varsigma_{t+r-1})\\ & + \vartheta(\varsigma_{t+1}, \varsigma_{t+r})\vartheta(\varsigma_{t+2}, \varsigma_{t+r})\cdots\vartheta(\varsigma_{t+r-2}, \varsigma_{t+r})\vartheta(\varsigma_{t+r-2}, \varsigma_{t+r-1})\vartheta(\varsigma_{t+r-1}, \varsigma_{t+r})\varpi_c(\varsigma_{t+r-1}, \varsigma_{t+r}).\end{aligned}$$

The above inequality implies that:

$$\begin{aligned}\varpi_c(\varsigma_t, \varsigma_{t+r}) \leq & \vartheta(\varsigma_t, \varsigma_{t+1})\Bigg[\varpi_c(\varsigma_t, \varsigma_{t+1}) + \frac{\vartheta(\varsigma_{t+1}, \varsigma_{t+2})}{\vartheta(\varsigma_t, \varsigma_{t+1})}\vartheta(\varsigma_{t+1}, \varsigma_{t+r})\varpi_c(\varsigma_{t+1}, \varsigma_{t+2})\\ & + \frac{\vartheta(\varsigma_{t+1}, \varsigma_{t+2})}{\vartheta(\varsigma_t, \varsigma_{t+1})}\vartheta(\varsigma_{t+1}, \varsigma_{t+r})\cdot\frac{\vartheta(\varsigma_{t+2}, \varsigma_{t+3})}{\vartheta(\varsigma_{t+1}, \varsigma_{t+2})}\vartheta(\varsigma_{t+2}, \varsigma_{t+r})\varpi_c(\varsigma_{t+2}, \varsigma_{t+3})\\ & +\\ & \vdots\\ & + \frac{\vartheta(\varsigma_{t+1}, \varsigma_{t+2})}{\vartheta(\varsigma_t, \varsigma_{t+1})}\vartheta(\varsigma_{t+1}, \varsigma_{t+r})\cdots\frac{\vartheta(\varsigma_{t+r-1}, \varsigma_{t+r})}{\vartheta(\varsigma_{t+r-2}, \varsigma_{t+r-1})}\vartheta(\varsigma_{t+r-1}, \varsigma_{t+r})\varpi_c(\varsigma_{t+r-1}, \varsigma_{t+r})\Bigg].\end{aligned}$$

So, we obtain

$$\varpi_c(\varsigma_t, \varsigma_{t+r}) \leq \vartheta(\varsigma_t, \varsigma_{t+1})[\varpi_c(\varsigma_t, \varsigma_{t+1}) + H\cdot\varpi_c(\varsigma_{t+1}, \varsigma_{t+2}) + \cdots + H^{r-1}\cdot\varpi_c(\varsigma_{t+r-1}, \varsigma_{t+r})]. \tag{3.5}$$

Let $t_0\in\mathbb{N}$ such that

$$\lambda^{t_0} < \frac{1}{\sup_{s\geq 1}\lim_{t\to\infty}\frac{\vartheta(\varsigma_{t_0(t+1)}, \varsigma_{t_0(t+2)})}{\vartheta(\varsigma_{t_0 t}, \varsigma_{t_0(t+1)})}\vartheta(\varsigma_{t_0(t+1)}, \varsigma_{t_0 s})}. \tag{3.6}$$

Using (3.4), (3.5) and $H>1$, we have:

$$\begin{aligned}\varpi_c(\varsigma_{t_0 t}, \varsigma_{t_0(t+1)}) \leq & \vartheta(\varsigma_{t_0 t}, \varsigma_{t_0 t+1})[\varpi_c(\varsigma_{t_0 t}, \varsigma_{t_0 t+1}) + H\cdot\varpi_c(\varsigma_{t_0 t+1}, \varsigma_{t_0 t+2}) + \cdots + H^{t_0-1}\cdot\varpi_c(\varsigma_{t_0(t+1)-1}, \varsigma_{t_0(t+1)})]\\ \leq & \vartheta(\varsigma_{t_0 t}, \varsigma_{t_0 t+1})H^{t_0-1}[\varpi_c(\varsigma_{t_0 t}, \varsigma_{t_0 t+1}) + \varpi_c(\varsigma_{t_0 t+1}, \varsigma_{t_0 t+2}) + \cdots + \varpi_c(\varsigma_{t_0(t+1)-1}, \varsigma_{t_0(t+1)})]\\ \leq & \vartheta(\varsigma_{t_0 t}, \varsigma_{t_0 t+1})H^{t_0-1}\varpi_c(\varsigma_0, \varsigma_1)[\lambda^{t_0 t} + \lambda^{t_0 t+1} + \ldots \lambda^{t_0(t+1)-1}]\\ \leq & L\cdot H^{t_0-1}\varpi_c(\varsigma_0, \varsigma_1)\frac{\lambda^{t_0 t}}{1-\lambda}\\ \leq & K\cdot\mu^t,\end{aligned}$$

where $K = L\cdot H^{t_0-1}\frac{\varpi_c(\varsigma_0, \varsigma_1)}{1-\lambda}$ and $\mu = \lambda^{t_0}$. From (3.6), we have

$$\sup_{s\geq 1}\lim_{t\to\infty}\frac{\vartheta(\varsigma_{t_0(t+1)}, \varsigma_{t_0(t+2)})}{\vartheta(\varsigma_{t_0 t}, \varsigma_{t_0(t+1)})}\vartheta(\varsigma_{t_0(t+1)}, \varsigma_{t_0 s}) < \frac{1}{\mu}.$$

By Lemma 7, $\{\varsigma_{t_0 t}\}$ is Cauchy. We also get:

$$\begin{aligned}\varpi_c\left(\varsigma_{t_0\lfloor t/t_0\rfloor},\varsigma_t\right) &\le \vartheta(\varsigma_{t_0\lfloor t/t_0\rfloor},\, \varsigma_{t_0\lfloor t/t_0\rfloor+1})H^{t_0-1}\varpi_c(\varsigma_0,\, \varsigma_1)[\lambda^{t_0\lfloor t/t_0\rfloor}+\lambda^{t_0\lfloor t/t_0\rfloor+1}+\cdots+\lambda^{t-1}]\\ &\le L\cdot H^{t_0-1}\varpi_c(\varsigma_0,\, \varsigma_1)\frac{\lambda^{t_0\lfloor t/t_0\rfloor}}{1-\lambda}.\end{aligned}$$

Thus, we have

$$\lim_{t\to\infty}\varpi_c\left(\varsigma_{t_0\lfloor t/t_0\rfloor},\varsigma_t\right)=0. \tag{3.7}$$

And, by the triangle inequality (C3), we get:

$$\begin{aligned}\varpi_c\left(\varsigma_t,\varsigma_s\right) &\le \vartheta(\varsigma_t,\varsigma_{t_0\lfloor t/t_0\rfloor})\varpi_c(\varsigma_t,\varsigma_{t_0\lfloor t/t_0\rfloor})+\vartheta(\varsigma_{t_0\lfloor t/t_0\rfloor},\, \varsigma_s)\varpi_c(\varsigma_{t_0\lfloor t/t_0\rfloor},\, \varsigma_s)\\ &\le \vartheta(\varsigma_t,\varsigma_{t_0\lfloor t/t_0\rfloor})\varpi_c(\varsigma_t,\varsigma_{t_0\lfloor t/t_0\rfloor})+\vartheta(\varsigma_{t_0\lfloor t/t_0\rfloor},\, \varsigma_s)\vartheta(\varsigma_{t_0\lfloor t/t_0\rfloor},\, \varsigma_{t_0\lfloor s/t_0\rfloor})\varpi_c(\varsigma_{t_0\lfloor t/t_0\rfloor},\, \varsigma_{t_0\lfloor s/t_0\rfloor})\\ &\quad+\vartheta(\varsigma_{t_0\lfloor t/t_0\rfloor},\, \varsigma_s)\vartheta(\varsigma_{t_0\lfloor s/t_0\rfloor},\, \varsigma_s)\varpi_c(\varsigma_{t_0\lfloor s/t_0\rfloor},\, \varsigma_s).\end{aligned}$$

So, from (3.7) and the fact that $\{\varsigma_{t_0 t}\}$ is Cauchy, it follows that $\lim\limits_{t,s\to\infty}\varpi_c\left(\varsigma_t,\varsigma_s\right)=0$.

Example 3.4. Let $\Xi=[0,1]$. Define $\varpi_c:\Xi\times\Xi\to[0,\,\infty)$ and $\vartheta:\Xi\times\Xi\to[0,\,\infty)$ by
$\varpi_c(\varsigma,\,\rho)=(\varsigma-\rho)^2$, $\vartheta(\varsigma,\,\rho)=\varsigma+\rho+1$, for all $\varsigma,\,\rho\in\Xi$.
Then, $(\Xi,\,\varpi_c)$ is a *CMTS*. Consider a sequence $\{\varsigma_t\}$ defined by $\varsigma_t=(4/5)^t$ for all $t\ge 0$. Then, (3.3) is satisfied for all $\lambda\in[16/25,\,1)$ and $t\in\mathbb{N}$. And we obtain that

$$\sup_{s\ge1}\lim_{t\to\infty}\frac{\vartheta(\varsigma_{t+1},\,\varsigma_{t+2})}{\vartheta(\varsigma_t,\,\varsigma_{t+1})}\vartheta(\varsigma_{t+1},\,\varsigma_s)=\sup_{s\ge1}\lim_{t\to\infty}\left(\frac{\left(\frac{4}{5}\right)^{t+1}+\left(\frac{4}{5}\right)^{t+2}+1}{\left(\frac{4}{5}\right)^{t}+\left(\frac{4}{5}\right)^{t+1}+1}\right)\left(\left(\frac{4}{5}\right)^{t+1}+\left(\frac{4}{5}\right)^{s}+1\right)=\frac{9}{5}>\frac{1}{\lambda}.$$

So, Lemma 7 is not applicable. On the other hand, by Lemma 9, the sequence $\{\varsigma_l\}$ is Cauchy in Ξ.

4. Fixed point theorems

For a *CMTS* $(\Xi,\,\varpi_c)$ and $\Gamma:\Xi\to\Xi$, we will use the following classes of control functions.

$$A=\{\lambda:\Xi\to(0,\,1),\,\lambda(\Gamma\varsigma)\le\lambda(\varsigma)\ for\,each\,\varsigma\in\Xi\},$$

$$B=\{\lambda:\Xi\to(0,\,1/2),\ \lambda(\Gamma\varsigma)\le\lambda(\varsigma)\ for\,each\,\varsigma\in\Xi\}.$$

We propose the following nonlinear Ćirić type fixed point result via the above control functions in *CMTS*.

Theorem 4.1. *Let $(\Xi,\,\varpi_c)$ be a complete CMTS that ϖ_c is continuous. Let $\Gamma:\Xi\to\Xi$ such that*

$$\varpi_c(\Gamma\varsigma,\,\Gamma\rho)\le\lambda(\varsigma)\cdot M_c(\varsigma,\,\rho), forall \varsigma,\,\rho\in\Xi, \tag{4.1}$$

where $\lambda\in B$ and $M_c(\varsigma,\,\rho)=\max\{\varpi_c(\varsigma,\,\rho),\varpi_c(\varsigma,\,\Gamma\varsigma),\varpi_c(\rho,\,\Gamma\rho),\varpi_c(\varsigma,\,\Gamma\rho),\varpi_c(\rho,\,\Gamma\varsigma)\}$.
For $\varsigma_0\in\Xi$, we set $\varsigma_t=\Gamma^t\varsigma_0$. Suppose that

$$\sup_{t\ge1}\vartheta(\varsigma_{t-1},\varsigma_t)=\alpha\, exists\ and\ is\ finite, \tag{4.2}$$

$$0<\lambda(\varsigma_0)<\frac{1}{2\alpha}. \tag{4.3}$$

Then Γ has a unique fixed point.

Proof. Let $\varsigma_0 \in \Xi$. Consider the sequence $\{\varsigma_t\}$ defined by $\varsigma_t = \Gamma^t\varsigma_0 = \Gamma\varsigma_{t-1}$ for all $t \in \mathbb{N}$. If $\varsigma_t = \varsigma_{t-1}$, for some $t \in \mathbb{N}$, then ς_{t-1} is a fixed point of Γ. Suppose that $\varsigma_t \neq \varsigma_{t-1}$ for all $t \in \mathbb{N}$.
By (4.1), we obtain

$$\begin{aligned}\varpi_c(\varsigma_t, \varsigma_{t+1}) =& \varpi_c(\Gamma\varsigma_{t-1}, \Gamma\varsigma_t) \leq \lambda(\varsigma_{t-1}) \cdot M_c(\varsigma_{t-1}, \varsigma_t)\\ \leq& \lambda(\varsigma_0) \cdot M_c(\varsigma_{t-1}, \varsigma_t),\end{aligned}$$

where

$$\begin{aligned}M_c(\varsigma_{t-1}, \varsigma_t) =& \max\{\varpi_c(\varsigma_{t-1}, \varsigma_t), \varpi_c(\varsigma_{t-1}, \Gamma\varsigma_{t-1}), \varpi_c(\varsigma_t, \Gamma\varsigma_t), \varpi_c(\varsigma_{t-1}, \Gamma\varsigma_t), \varpi_c(\varsigma_l, \Gamma\varsigma_{t-1})\}\\ =& \max\{\varpi_c(\varsigma_{t-1}, \varsigma_t), \varpi_c(\varsigma_t, \varsigma_{t+1}), \varpi_c(\varsigma_{t-1}, \varsigma_{t+1})\}.\end{aligned}$$

Then we consider three cases:
Case 1. If $M_c(\varsigma_{t-1}, \varsigma_t) = \varpi_c(\varsigma_{t-1}, \varsigma_t)$, then $\varpi_c(\varsigma_t, \varsigma_{t+1}) \leq \lambda(\varsigma_0) \cdot \varpi_c(\varsigma_{t-1}, \varsigma_t)$. Since $\lambda(\varsigma_0) \in (0, 1/2)$, by Lemma 9, $\{\varsigma_t\}$ is Cauchy.
Case 2. If $M_c(\varsigma_{t-1}, \varsigma_t) = \varpi_c(\varsigma_t, \varsigma_{t+1})$, then $\varpi_c(\varsigma_t, \varsigma_{t+1}) \leq \lambda(\varsigma_0) \cdot \varpi_c(\varsigma_t, \varsigma_{t+1}) < \varpi_c(\varsigma_t, \varsigma_{t+1})$, which is a contradiction.
Case 3. If $M_c(\varsigma_{t-1}, \varsigma_t) = \varpi_c(\varsigma_{t-1}, \varsigma_{t+1})$, then we have

$$\varpi_c(\varsigma_t, \varsigma_{t+1}) \leq \lambda(\varsigma_0)\varpi_c(\varsigma_{t-1}, \varsigma_{t+1}) \leq \lambda(\varsigma_0)[\vartheta(\varsigma_{t-1}, \varsigma_t)\varpi_c(\varsigma_{t-1}, \varsigma_t) + \vartheta(\varsigma_t, \varsigma_{t+1})\varpi_c(\varsigma_t, \varsigma_{t+1})].$$

By (4.2), we get:

$$\varpi_c(\varsigma_t, \varsigma_{t+1}) \leq \alpha\lambda(\varsigma_0)[\varpi_c(\varsigma_{t-1}, \varsigma_t) + \varpi_c(\varsigma_t, \varsigma_{t+1})].$$

Thus,

$$\varpi_c(\varsigma_t, \varsigma_{t+1}) \leq \gamma \cdot \varpi_c(\varsigma_{t-1}, \varsigma_t),$$

where $\gamma = \frac{\alpha\lambda(\varsigma_0)}{1-\alpha\lambda(\varsigma_0)}$.
By (4.3), we have $\lambda(\varsigma_0) \cdot 2\alpha < 1$. This implies that $\alpha \cdot \lambda(\varsigma_0) < 1 - \alpha \cdot \lambda(\varsigma_0)$ and $\gamma < 1$. So, from Lemma 9, $\{\varsigma_t\}$ is Cauchy. Summarizing the three cases, we conclude that $\{\varsigma_t\}$ is Cauchy. Since Ξ is complete, there exists $\varsigma \in \Xi$ such that $\lim\limits_{t\to\infty} \varsigma_t = \varsigma$.
Now, we will show that $\varsigma = \Gamma\varsigma$. Assume that $\varsigma \neq \Gamma\varsigma$. By (4.1), we get:

$$\varpi_c(\varsigma_{t+1}, \Gamma\varsigma) = \varpi_c(\Gamma\varsigma_t, \Gamma\varsigma) \leq \lambda(\varsigma_t) \cdot M_c(\varsigma_t, \varsigma) \leq \lambda(\varsigma_0) \cdot M_c(\varsigma_t, \varsigma), \tag{4.4}$$

where

$$M_c(\varsigma_t, \varsigma) = \max\{\varpi_c(\varsigma_t, \varsigma), \varpi_c(\varsigma_t, \varsigma_{t+1}), \varpi_c(\varsigma, \Gamma\varsigma), \varpi_c(\varsigma_t, \Gamma\varsigma), \varpi_c(\varsigma, \varsigma_{t+1})\}.$$

Since ϖ_c is continuous, taking the limit as $t \to \infty$ in the both sides of (4.4), we have

$$\varpi_c(\varsigma, \Gamma\varsigma) \leq \lambda(\varsigma_0) \cdot \varpi_c(\varsigma, \Gamma\varsigma) < \varpi_c(\varsigma, \Gamma\varsigma).$$

This is a contradiction, and so $\varsigma = \Gamma\varsigma$.
For uniqueness, assume that $\Gamma\varsigma = \varsigma \neq \rho = \Gamma\rho$. Then

$$\begin{aligned}\varpi_c(\varsigma, \rho) =& \varpi_c(\Gamma\varsigma, \Gamma\rho) \leq \lambda(\varsigma) \cdot \max\{\varpi_c(\varsigma, \rho), \varpi_c(\varsigma, \Gamma\varsigma), \varpi_c(\rho, \Gamma\rho), \varpi_c(\varsigma, \Gamma\rho), \varpi_c(\rho, \Gamma\varsigma)\}\\ \leq& \lambda(\varsigma) \cdot \varpi_c(\varsigma, \rho) < \varpi_c(\varsigma, \rho),\end{aligned}$$

which is a contradiction.

Example 4.2. Let $\Xi = \{a, b, c\}$. $\varpi_c : \Xi \times \Xi \to [0, \infty)$ is symmetric and is defined by
$\varpi_c(a, b) = \frac{2}{5}$, $\varpi_c(a, c) = \frac{3}{10}$, $\varpi_c(b, c) = \frac{3}{20}$ and $\varpi_c(\varsigma, \varsigma) = 0$ for all $\varsigma \in \Xi$.
And $\vartheta : \Xi \times \Xi \to [1, \infty)$ is symmetric and is defined by

$$\vartheta(a, a) = \vartheta(b, b) = \vartheta(c, c) = \vartheta(b, c) = 1, \vartheta(a, c) = \frac{100}{99}, \vartheta(a, b) = \frac{3}{2}.$$

Given $\Gamma : \Xi \to \Xi$ as $\Gamma a = c$ and $\Gamma b = \Gamma c = b$.
Consider $\lambda : \Xi \to [0,\, 1/2)$as

$$\lambda(a) = \frac{49}{100}, \lambda(b) = \frac{45}{100}, \lambda(c) = \frac{47}{100}.$$

Then:

1. $(\Xi,\, \varpi_c)$ is a complete *CMTS*.
2. For $\varsigma = a$ and $\rho = b$, (4.1) is satisfied. In fact, it holds that

$$\varpi_c(\Gamma\varsigma,\, \Gamma\rho) = \varpi_c(c,\, b) = \frac{3}{20} \leq \frac{98}{500} = \frac{49}{100} \cdot \max\left\{\frac{2}{5}, \frac{3}{10}, 0, \frac{2}{5}, \frac{3}{20}\right\} = \lambda(a) \cdot M_c(a,\, b) = \lambda(\varsigma) \cdot M_c(\varsigma,\, \rho).$$

Similarly, we can be checked that (4.1) holds for all other $\varsigma,\, \rho \in \Xi$. It is obvious that $\lambda \in B$.

1. For $\varsigma_0 = a$, $\sup\limits_{t\geq 1}\vartheta(\varsigma_{t-1}, \varsigma_t) = \alpha = \frac{100}{99}$ and $0 < \lambda(\varsigma_0) = \lambda(a) = \frac{49}{100} < \frac{99}{200} = \frac{1}{2\alpha}$.

For $\varsigma_0 = b$, $\sup\limits_{t\geq 1}\vartheta(\varsigma_{t-1}, \varsigma_t) = \alpha = 1$ and $0 < \lambda(\varsigma_0) = \lambda(b) = \frac{45}{100} < \frac{1}{2} = \frac{1}{2\alpha}$.
For $\varsigma_0 = c$, $\sup\limits_{t\geq 1}\vartheta(\varsigma_{t-1}, \varsigma_t) = \alpha = 1$ and $0 < \lambda(\varsigma_0) = \lambda(c) = \frac{47}{100} < \frac{1}{2} = \frac{1}{2\alpha}$.
Therefore, (4.2) and (4.3) are satisfied.
So, all hypotheses of Theorem 11 are satisfied and the mapping Γ has a unique fixed point.

Theorem 4.3. *Let* $(\Xi,\, \varpi_c)$ *be a complete CMTS and* $\Gamma : \Xi \to \Xi$ *such that*

$$\varpi_c(\Gamma\varsigma,\, \Gamma\rho) \leq \lambda(\varsigma) \cdot \max\{\varpi_c(\varsigma,\, \rho), \varpi_c(\varsigma,\, \Gamma\varsigma), \varpi_c(\rho,\, \Gamma\rho)\}, \forall\varsigma,\, \rho \in \Xi \tag{4.5}$$

where $\lambda \in A$. *For* $\varsigma_0 \in \Xi$, *we set* $\varsigma_t = \Gamma^t\varsigma_0$. *Suppose that for all* $\varsigma \in \Xi$

$$\lim_{t\to\infty}\vartheta(\varsigma,\, \varsigma_t)\textit{exists and is finite, and}\lim_{t\to\infty}\vartheta(\varsigma_t,\, \varsigma) < \frac{1}{\lambda(\varsigma_0)}. \tag{4.6}$$

Then Γ *has a unique fixed point.*

Proof. By similar way to Theorem 11, we can get a sequence $\{\varsigma_t = \Gamma^t\varsigma_0\}$ such that $\lim\limits_{t\to\infty}\varsigma_t = \varsigma \in \Xi$.
Assume that $\varsigma \neq \Gamma\varsigma$. Then we have

$$\begin{aligned}
0 <&\varpi_c(\varsigma,\, \Gamma\varsigma) \leq \vartheta(\varsigma,\, \varsigma_{t+1})\varpi_c(\varsigma,\, \varsigma_{t+1}) + \vartheta(\varsigma_{t+1},\, \Gamma\varsigma)\varpi_c(\varsigma_{t+1},\, \Gamma\varsigma)\\
&\leq\vartheta(\varsigma,\, \varsigma_{t+1})\varpi_c(\varsigma,\, \varsigma_{t+1}) + \vartheta(\varsigma_{t+1},\, \Gamma\varsigma)\lambda(\varsigma_t)\max\{\varpi_c(\varsigma_t, \varsigma), \varpi_c(\varsigma_t, \Gamma\varsigma_t), \varpi_c(\varsigma, \Gamma\varsigma)\}\\
&\leq\vartheta(\varsigma,\, \varsigma_{t+1})\varpi_c(\varsigma,\, \varsigma_{t+1}) + \vartheta(\varsigma_{t+1},\, \Gamma\varsigma)\lambda(\varsigma_0)\max\{\varpi_c(\varsigma_t, \varsigma), \varpi_c(\varsigma_t, \varsigma_{t+1}), \varpi_c(\varsigma, \Gamma\varsigma)\}.
\end{aligned} \tag{4.7}$$

Taking the limit as $t \to \infty$ in (4.7) and using (4.6), it holds that $0 < \varpi_c(\varsigma,\, \Gamma\varsigma) < \varpi_c(\varsigma,\, \Gamma\varsigma)$, which is a contradiction. Thus, $\varsigma = \Gamma\varsigma$. The proof of uniqueness omits.

From Theorem 13 we get the nonlinear Banach type fixed point result under weaker hypotheses than Theorem 2.1 in [20] without the condition

$$\sup_{s\geq 1}\lim_{t\to\infty}\frac{\vartheta(\varsigma_{t+1},\, \varsigma_{t+2})}{\vartheta(\varsigma_t,\, \varsigma_{t+1})}\vartheta(\varsigma_{t+1},\, \varsigma_s) < \frac{1}{\lambda(\varsigma_0)}.$$

And we have the following nonlinear Kannan type fixed point result under weaker hypotheses than Theorem 2.3 in [20] without the condition

$$\sup_{s\geq 1}\lim_{t\to\infty}\frac{\vartheta(\varsigma_{t+1},\, \varsigma_{t+2})}{\vartheta(\varsigma_t,\, \varsigma_{t+1})}\vartheta(\varsigma_{t+1},\, \varsigma_s) < \frac{1-\lambda(\varsigma_0)}{\lambda(\varsigma_0)}.$$

Corollary 4.4. *Let* (Ξ, ϖ_c) *be a complete CMTS and* $\Gamma : \Xi \to \Xi$ *such that*

$$\varpi_c(\Gamma\varsigma, \Gamma\rho) \leq \lambda(\varsigma)[\varpi_c(\varsigma, \Gamma\varsigma) + \varpi_c(\rho, \Gamma\rho)], \forall \varsigma, \rho \in \Xi \tag{4.8}$$

where $\lambda \in B$*. For* $\varsigma_0 \in \Xi$*, we set* $\varsigma_t = \Gamma^t \varsigma_0$*. Suppose that for all* $\varsigma \in \Xi$

$$\lim_{t\to\infty} \vartheta(\varsigma, \varsigma_t) \textit{exists and is finite, and} \lim_{t\to\infty} \vartheta(\varsigma_t, \varsigma) < \frac{1}{\lambda(\varsigma_0)}. \tag{4.9}$$

Then Γ *has a unique fixed point.*

Proof. We have

$$\begin{aligned}\varpi_c(\Gamma\varsigma, \Gamma\rho) \leq& \lambda(\varsigma)[\varpi_c(\varsigma, \Gamma\varsigma) + \varpi_c(\rho, \Gamma\rho)] \leq 2\lambda(u) \max\{\varpi_c(\varsigma, \Gamma\varsigma), \varpi_c(\rho, \Gamma\rho)\} \\ \leq& 2\lambda(u) \max\{\varpi_c(\varsigma, \rho), \varpi_c(\varsigma, \Gamma\varsigma), \varpi_c(\rho, \Gamma\rho)\}.\end{aligned}$$

Since $\lambda \in B$ implies $2\lambda \in A$, Γ satisfies (4.5). By Theorem 13, Γ has a unique fixed point.

Example 4.5. Let $\Xi = \{a, b, c\}$. $\varpi_c : \Xi \times \Xi \to [0, \infty)$ is symmetric and is defined by $\varpi_c(a, b) = \frac{1}{2}$, $\varpi_c(a, c) = \frac{11}{20}$, $\varpi_c(b, c) = \frac{3}{20}$ and $\varpi_c(\varsigma, \varsigma) = 0$ for all $\varsigma \in \Xi$.
And $\vartheta : \Xi \times \Xi \to [1, \infty)$ is symmetric and is defined by

$$\vartheta(a, a) = \vartheta(b, b) = \vartheta(c, c) = \vartheta(b, c) = \frac{4}{3}, \vartheta(a, c) = 2, \vartheta(a, b) = \frac{3}{2}.$$

Given $\Gamma : \Xi \to \Xi$ as $\Gamma a = c$ and $\Gamma b = \Gamma c = b$.
Consider $\lambda : \Xi \to [0, 1/2)$ as

$$\lambda(a) = \frac{99}{200}, \lambda(b) = \frac{45}{100}, \lambda(c) = \frac{49}{100}.$$

Then, (Ξ, ϖ_c) is a complete *CMTS*. It is obvious that $\lambda \in B$.

For $\varsigma = a$ and $\rho = b$, (4.8) is satisfied. In fact, we have

$$\varpi_c(\Gamma\varsigma, \Gamma\rho) = \frac{3}{20} \leq \frac{99}{200} \cdot \left(\frac{11}{20} + 0\right) = \lambda(\varsigma)[\varpi_c(\varsigma, \Gamma\varsigma) + \varpi_c(\rho, \Gamma\rho)].$$

Similarly, we can be checked that (4.8) holds for all other $\varsigma, \rho \in \Xi$.

For $\varsigma_0 = a$, $\lim_{t\to\infty} \vartheta(\varsigma, \varsigma_t) = \vartheta(a, c) = 2$ and $\lim_{t\to\infty} \vartheta(\varsigma_t, \varsigma) = \vartheta(c, a) = 2 < \frac{200}{99} = \frac{1}{\lambda(a)} = \frac{1}{\lambda(\varsigma_0)}$.

For $\varsigma_0 = b$, $\lim_{t\to\infty} \vartheta(\varsigma, \varsigma_t) = \vartheta(b, b) = \frac{4}{3}$ and $\lim_{t\to\infty} \vartheta(\varsigma_t, \varsigma) = \vartheta(b, b) = \frac{4}{3} < \frac{100}{45} = \frac{1}{\lambda(b)} = \frac{1}{\lambda(\varsigma_0)}$.

For $\varsigma_0 = c$, $\lim_{t\to\infty} \vartheta(\varsigma, \varsigma_t) = \vartheta(c, b) = \frac{4}{3}$ and $\lim_{t\to\infty} \vartheta(\varsigma_t, \varsigma) = \vartheta(b, c) = \frac{4}{3} < \frac{100}{49} = \frac{1}{\lambda(c)} = \frac{1}{\lambda(\varsigma_0)}$.

So, (4.9) is satisfied.

On the other hand, we have

$\sup_{\substack{s\geq 1 \\ m\geq 1}} \lim_{t\to\infty} \frac{\vartheta(\varsigma_{t+1}, \varsigma_{t+2})}{\vartheta(\varsigma_t, \varsigma_{t+1})} \vartheta(\varsigma_{t+1}, \varsigma_s) = \frac{4}{3} > \frac{101}{99} = \frac{1-\lambda(\varsigma_0)}{\lambda(\varsigma_0)}$, for $\varsigma_0 = a$;

$\sup_{\substack{s\geq 1 \\ m\geq 1}} \lim_{t\to\infty} \frac{\vartheta(\varsigma_{t+1}, \varsigma_{t+2})}{\vartheta(\varsigma_t, \varsigma_{t+1})} \vartheta(\varsigma_{t+1}, \varsigma_s) = \frac{4}{3} > \frac{55}{45} = \frac{1-\lambda(\varsigma_0)}{\lambda(\varsigma_0)}$, for $\varsigma_0 = b$;

$\sup_{\substack{s\geq 1 \\ m\geq 1}} \lim_{t\to\infty} \frac{\vartheta(\varsigma_{t+1}, \varsigma_{t+2})}{\vartheta(\varsigma_t, \varsigma_{t+1})} \vartheta(\varsigma_{t+1}, \varsigma_s) = \frac{4}{3} > \frac{51}{49} = \frac{1-\lambda(\varsigma_0)}{\lambda(\varsigma_0)}$, for $\varsigma_0 = c$.

Therefore, Theorem 2.3 in [20] is not applicable. In other hand, by Corollary 14, Γ has unique fixed point $\varsigma = b$.

5. Conclusion

In this paper, I introduced a new sufficient condition for the Cauchyness on sequences in controlled metric spaces. This result can be effectively used to show the existence of fixed points in controlled metric spaces, and under the weaker assumption, it can improve the previous fixed point theorems. As application of obtained result, I presented some fixed point results involving nonlinear Ćirić type contractions via control functions, which give a partial answer to open questions in [20].

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