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Structures on wp-semiring

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Abstract

This paper introduces the concept of wp-semiring and investigates its algebraic properties. Closed weakly Stone and dense elements according to wp-complementation are defined, and their properties are proved. Connections between wp-semiring and other ring-like structures are verified. Structures of closed and dense elements are shown as orthopseudoring and presemiring, respectively.

Keywords: Boolean ring, Closed Element, Dense Element, Ordered Semiring, Orthopseudoring.
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1. Introduction

Golan J. S. introduced the concept of semiring in [6]. A semiring is a very important algebraic structure in both theoretical and applied fields. It is considered a generalization of a distributive lattice and ring. So, it has common properties with lattices and rings. In addition to semirings and their derivative structures, which have many applications in computer science, as referenced in [1], [7], [4], [3],[10], and [12]. An ordered semiring is a semiring equipped with an ordered relation connected with the operations of semiring with some conditions, see [6], [7]. In this work, we define an ordered semiring in which each element has wp-complemented (wp-semiring) as a generalization of a pseudocomplemented semiring introduced by P. Nasehpour in [11]. The concept of weak pseudocomplementation is introduced in [9], by Kwuida under the name of dual weakly complementation on a bounded lattice. This operation and its dual are raised from the formal concept analysis (FCA) theory and its applications, as a generalization of operations of concept algebra. In the case of a distributive lattice the dual weak complementation considered as a generalization of pseudocomplementation. In this situation, the pseudocomplementation is the largest one of dual weak complementation elements. Here we extend this concept to an ordered semiring and then we get pseudocomplemented semiring of P. Nasehpour as a special case. This generalization get more application in broad fields of data analysis, information and computer science in integration with (FCA) theory. The algebraic properties of the wp-semiring structure are proved. Closed and dense elements with respect to

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weak pseudocomplementation are defined. The algebraic structures of sets of all closed and dense elements represent relations between a wp-semiring, orthopseudoring, Boolean ring, and presemiring. The reader is supposed to know the basics of lattice theory, see [2].

Section 2 lists the most important previous definitions and terminology on which the rest of the paper is built. In section 3, wp-semiring is defined, and its important properties are shown. Definitions of closed, weak Stone, and dense elements are introduced. In Section 4, the algebraic structure of the sets of all closed is investigated. Section 5 proved properties of weakly Stone and dense elements. The results are supported by illustrative examples throughout the article.

2. Preliminaries

In this section, we list some primary concepts and terms used in the remaining parts.

Definition 2.1. [6] A structure $\mathfrak{O} = (O; +, \cdot)$ of nonempty set O and two binary operations "+" and "$\cdot$" is called a semiring if $(O; +)$and $(O; \cdot)$ are commutative monoids with two different identities 0 and 1, respectively and it satisfies that:

i) $w \cdot (u + v) = (w \cdot u) + (w \cdot v)$, for any $u, v, w \in O$
ii) $u \cdot 0 = 0$, for any $u \in O$

A semiring $\mathfrak{O} = (O; +, \cdot)$ is simple if any $u \in O$, has that $u + 1 = 1$. It is multiplicatively idempotent if any $u \in O$ has $u \cdot u = u^2 = u$.

An ordered semiring $\mathfrak{O} = (O; +, \cdot, \leq)$ is a semiring $\mathfrak{O} = (O; +, \cdot)$ equipped with partial ordered relation "$\leq$" satisfies that:

i) If $u \leq v$, then $w + u \leq w + v$;
ii) If $u \leq v$ and $0 \leq w$, then $u \cdot w \leq v \cdot w$;

for any $u, v, w \in O$.

A positive ordered semiring is an ordered semiring with $0 \leq u$ for all $u \in O$. In this case $u, v \leq u + v$ for all $u, v \in \mathfrak{O}$, see [7] and [4].

Definition 2.2. [5] A structure $\mathfrak{P} = (P; +, \cdot)$ of nonempty set P and two commutative binary operations "+" and "$\cdot$" is called an orthopseudoring if the operation"$\cdot$" is associative and idempotent,each element $u \in P$ is additive inverse of itself, e.i. $u + u = 0$ and it satisfies that for all $u, v \in P$:

i) $u + 0 = 0$, $u \cdot 0 = 0 \cdot u$ and $u \cdot 1 = u$;
ii) $u + (1 + v) = (u + 1) + v$;
iii) $(1 + (u \cdot v)) \cdot u = u + (u \cdot v \cdot u)$;
iv) $(1 + u) \cdot (1 + (u \cdot v)) = 1 + u$;
v) $(1 + u \cdot (1 + v)) \cdot (1 + v \cdot (1 + u)) = 1 + (u + v)$.

Boolean ring $\mathfrak{B} = (B; +, \cdot)$ is an orthopseudoring with the following more conditions:

i) $(u + (u \cdot v)) + (u \cdot v) = u$;
ii) $(u + v) + v = u + (v + v)$;
iii) $u \cdot (1 + v) = u + (u \cdot v)$.

A presemiring is an algebraic structure $\mathfrak{P} = (P; +, \cdot)$ of nonempty set P and two associative and commutative binary operations "+" and "$\cdot$" with $w \cdot (u + v) = (w \cdot u) + (w \cdot v)$ for all $u, v, w \in P$, refer to [8].

3. Definition and Basic Properties

Here, weakly p-semiring is introduced and its algebraic properties are verified.

Definition 3.1. Let $\mathfrak{O}$ be an ordered semiring and $u \in \mathfrak{O}$. Then u is a weak pseudocomplemented element (wp-complemented) iff there exists an element $u^{\omega_p} \in \mathfrak{O}$ satisfies that:

$W_p1)$ $u \leq u^{\omega_p \omega_p}$;
$W_p2)$ If $u \leq v$ and v^{ω_p} exists, then $u^{\omega_p} \geq v^{\omega_p}$;
$W_p3)$ $u \cdot u^{\omega_p} = 0$.

Consider an ordered semiring $\mathfrak{O}$, a set of all elements which have a wp-complemens is denoted by $W_P(\mathfrak{O})$. Since $0^{\omega_p} = 1$, then $0 \in \mathfrak{O}$ and $W_P(\mathfrak{O})$ is nonempty set. An element $u \in \mathfrak{O}$ is closed element, if $u = v^{\omega_p}$ for some $v \in \mathfrak{O}$. The set of all closed elements is denoted by $C^{\omega_p}(\mathfrak{O})$. A dense element $u \in \mathfrak{O}$ is an element with $u^{\omega_p} = 0$. The set of all dense elements is denoted by $D^{\omega_p}(\mathfrak{O})$. An ordered semiring $\mathfrak{O}$ is called weakly pseudocomplemented(Wp-semiring) iff $\mathfrak{O} = W_p(\mathfrak{O})$.

Example 3.2. Tables 1 and 2 describe an ordered semiring $\mathfrak{O} = (O; +, \cdot, \leq)$. The ordered relation $\leq$ on $\mathfrak{O}$ is given in Figure (1) (in the last page) and Table 3 gives wp-complemented elements of $\mathfrak{O}$. $C^{\omega_p}(\mathfrak{O}) = \{0, x, y, t\}$ and $D^{\omega_p}(\mathfrak{O}) = \{s, t, w\}$.

Table 1: + operation on $\mathfrak{O}$.

$+$	0	x	y	s	t	v	w	z	1
0	0	x	y	s	t	v	w	z	1
x	x	x	t	s	t	v	w	s	1
y	y	t	y	s	t	w	w	z	1
s	s	s	s	s	s	1	1	s	1
t	t	t	t	s	t	w	w	s	1
v	v	v	w	1	w	v	w	1	1
w	w	w	w	1	w	w	w	1	1
z	z	s	z	s	s	1	1	z	1
1	1	1	1	1	1	1	1	1	1

Table 2: $\cdot$ operation on $\mathfrak{O}$.

$\cdot$	0	x	y	s	t	v	w	z	1
0	0	0	0	0	0	0	0	0	0
x	0	0	0	x	x	x	x	0	x
y	0	0	0	y	y	0	y	y	y
s	0	x	y	s	t	x	t	z	s
t	0	x	y	t	t	x	t	y	t
v	0	x	0	x	x	v	v	0	v
w	0	x	y	t	t	v	w	y	w
z	0	0	y	z	y	0	y	z	z
1	0	x	y	s	t	v	w	z	1

Table 3: Wp-complemented elements of $\mathfrak{O}$

element	0	x	y	s	t	w
wp-comp.	t	y	x	0	0	0

Proposition 3.3. *Let $\mathfrak{O}$ be a positive ordered semiring. Then for every $u, v \in W_p(\mathfrak{O})$:*

i) $u^{\omega_p \omega_p \omega_p} = u^{\omega_p}$;
ii) $u^{\omega_p} \cdot u^{\omega_p \omega_p} = 0$;
iii) If $1 \in W_p(\mathfrak{O})$, *then* $1^{\omega_p} = 0$;
iv) If $v \leq u^{\omega_p}$, *then* $v \cdot u = 0$, *for all* $v \in \mathfrak{O}$;
v) $u \leq v^{\omega_p}$ *iff* $v \leq u^{\omega_p}$;
vi) If $0 \in W_p(\mathfrak{O})$, *then* $0 \leq u \leq 0^{\omega_p}$.

Proof. i) From Definition(3.1), $u^{\omega_p} \leq u^{\omega_p \omega_p \omega_p}$. Also, $u \leq u^{\omega_p \omega_p}$ implies $u^{\omega_p} \geq u^{\omega_p \omega_p \omega_p}$. Accordingly $u^{\omega_p} = u^{\omega_p \omega_p \omega_p}$.

ii) Set $u = u^{\omega_p}$ in Definition(3.1), axiom $W_p3)$ we get $u^{\omega_p} \cdot u^{\omega_p \omega_p} = 0$.

iii) Assume $1 \in W_p(\mathfrak{O})$. Since zero is the only element that has $1 \cdot 0 = 0$. Then $1^{\omega_p} = 0$.

iv) Let $v \leq u^{\omega_p}$ for some $v \in \mathfrak{O}$. Then using Definition (3.1), W_p3), $v \cdot u \leq u^{\omega_p} \cdot u = 0$.So, $v \cdot u = 0$.

v) $u \leq v^{\omega_p}$ iff $u^{\omega_p} \geq v^{\omega_p \omega_p} \geq v$.

vi) Since $\mathfrak{O}$ is positive, then $0 \leq u, u^{\omega_p}$. From v) $u \leq 0^{\omega_p}$. Hence, $0 \leq u \leq 0^{\omega_p}$.

□

A dual weakly complementation operation on a lattice $(L, +, \cdot)$ is a unary operation has W_p1 and W_p2 in Definition (3.1) and the condition: $(u + v) \cdot (u + v^{\omega_p}) = u$, for all $u, v \in L$. A dual weakly complemented distributive lattice is a bounded distributive lattice with dual weakly complementation operation, for more details refer to [13], [14], [9].

Theorem 3.4. *Let $\mathfrak{O}$ be a positive weakly p-semiring. Then the following conditions are equivalent:*

i) $0^{\omega_p} = 1$;

ii) 1 *is the largest element of* $\mathfrak{O}$;

iii) $u \cdot v \leq u, v$ *for all* $u, v \in \mathfrak{O}$;

iv) $\mathfrak{O}$ *is a simple ring.*

Moreover, if $\mathfrak{O}$ is multiplicatively idempotent, then it is equivalent that

v) $\mathfrak{O}$ *is a bounded distributive dual weakly complemented lattice.*

Proof. $i) \longleftrightarrow ii)$: Immediately from vi) in Propostion (3.3).
$ii) \longleftrightarrow iii)$: If 1 is the largest element of $\mathfrak{O}$, then $v \leq 1$, $u \leq 1$ and so $u \cdot v \leq u$ and $u \cdot v \leq v$. Conversely, if $u \cdot v \leq u, v$, then $u = u \cdot 1 \leq 1$ for all $u \in \mathfrak{O}$.
$iii) \longleftrightarrow iv)$: Let $u \cdot v \leq u, v$ for all $u, v \in \mathfrak{O}$. Then $1 \leq u + 1 = 1 \cdot (u + 1) \leq 1$, for all $u \in \mathfrak{O}$. So, $\mathfrak{O}$ is simple. Conversely, if $\mathfrak{O}$ is simple, then $u = u\cdot(v+1) = u\cdot v+u$, and $v = v\cdot(u+1) = v\cdot u+v$. Thus $u\cdot v \leq u, v$.

$iv) \longleftrightarrow v)$: It is enough to prove only the absorption laws $u + (u \cdot v) = u$ and $u \cdot (u + v) = u$, and the distributive law $(w + u) \cdot (w + v) = w + (u \cdot v)$ as follows:
$u + (u \cdot v) = u \cdot (1 + v) = u$, and $u \cdot (u + v) = u^2 + (u \cdot v) = u + (u \cdot v) = u \cdot (1 + v) = u$. Thus, the absorption laws hold.
$(w+u)\cdot(w+v) = w^2+(w\cdot v)+(u\cdot w)+(u\cdot v) = w+(w\cdot v)+(w\cdot u)+(u\cdot v) = w\cdot(1+v)+(w\cdot u)+(u\cdot v) = w + (w \cdot u) + (u \cdot v) = w \cdot (1 + u) + (u \cdot v) = w + (u \cdot v)$. Thus distributive law is hold. Also, we have, $(v + u) \cdot (v + u^{\omega_p}) = v + (u \cdot u^{\omega_p}) = v + 0 = v$. As a result ^{w_p} is a dual weak complementation on this lattice.

□

For brevity, we will refer to a positive wp-semiring as pwp-semiring and to simple positive wp-semiring as spwp-semiring. We will be denoted to the last condition in Theorem (3.4) as:

$\mathfrak{O}1$ $u.v$ is the largest lower bound of u and v, for any $u, v \in \mathfrak{O}$.

Example 3.5. Let $\mathfrak{O}' = (O; +, \cdot, \leq')$ be a spwp-semiring with operations " + "and " · " represented in Tables 1 and 2. The ordered relation $\leq'$ on $\mathfrak{O}'$ is given in Figure (2) (in the last page) and Table 4 gives wp-complements of elements in $\mathfrak{O}'$. $C^{\omega_p}(\mathfrak{O}) = \{0, x, y, v, z, 1\}$ and $D^{\omega_p}(\mathfrak{O}) = \{s, t, w, 1\}$.

Table 4: Wp-complementes of $\mathfrak{O}'$

element	0	x	y	s	t	v	w	z	1
wp-comp.	1	z	v	0	0	y	0	x	0

Corollary 3.6. *Let $\mathfrak{O}$ be a spwp-semiring and $u \cdot v$ is the largest lower bound of u and v, for any $u, v \in \mathfrak{O}$. Then :*

i) $u^{\omega_p} \cdot v^{\omega_p} = (u+v)^{\omega_p}$;

ii) $((u^{\omega_p} \cdot 1) + (u \cdot 1^{\omega_p})) = u^{\omega_p}$;

iii) $(u + u^{\omega_p})^{\omega_p} = 0$;

iv) $(u^{\omega_p\omega_p} + v^{\omega_p\omega_p})^{\omega_p} = (u+v)^{\omega_p}$;

Proof. i) Suppose $u, v \in \mathfrak{O}$ so $u, v \leq u+v$ and $(u+v)^{\omega_p} \leq u^{\omega_p}, v^{\omega_p}$. Thus $(u+v)^{\omega_p} \leq u^{\omega_p} \cdot v^{\omega_p}$. Now if $w \leq u^{\omega_p}, v^{\omega_p}$, then $w^{\omega_p} \geq u^{\omega_p\omega_p}, v^{\omega_p\omega_p}$. Accordingly $w^{\omega_p} \geq u^{\omega_p\omega_p} + v^{\omega_p\omega_p} \geq u+v$. Hence $w \leq w^{\omega_p\omega_p} \leq (u+v)^{\omega_p}$. Therefore, $u^{\omega_p} \cdot v^{\omega_p} = (u+v)^{\omega_p}$.

ii) It is trivial.

iii) Set $v = u^{\omega_p}$ in i), we get $(u + u^{\omega_p})^{\omega_p} = u^{\omega_p} \cdot u^{\omega_p\omega_p} = 0$.

iv) $(u^{\omega_p\omega_p} + v^{\omega_p\omega_p})^{\omega_p} = u^{\omega_p\omega_p\omega_p} \cdot v^{\omega_p\omega_p\omega_p} = u^{\omega_p} \cdot v^{\omega_p} = (u+v)^{\omega_p}$.

□

Example 3.7. Tables 5 and 6 describe a multiplicatively idempotent spwp-semiring $\mathfrak{O}'' = (O; +, \cdot, \leq'')$. The ordered relation " $\leq'$ " on $\mathfrak{O}''$ is represented Figure (3) (in the last page) and Table 7 gives wp-complements of elements.

Table 5: + operation on $\mathfrak{O}''$.

+	0	x	y	s	t	v	w	z	1
0	0	x	y	s	t	v	w	z	1
x	x	v	1	t	z	0	y	s	w
y	y	1	z	x	v	w	t	0	s
s	s	t	x	w	y	z	0	1	v
t	t	z	v	y	1	s	x	w	0
v	v	0	w	z	s	x	1	t	y
w	w	y	t	0	x	1	s	v	z
z	z	s	0	1	w	t	v	y	x
1	1	w	s	v	0	y	z	x	t

Table 6: · operation on $\mathfrak{O}''$.

·	0	x	y	s	t	v	w	z	1
0	0	0	0	0	0	0	0	0	0
x	0	x	0	x	v	v	v	0	x
y	0	0	y	z	z	0	y	z	y
s	0	x	z	1	w	v	t	y	s
t	0	v	z	w	1	x	s	y	t
v	0	v	0	v	x	x	x	0	v
w	0	v	y	t	s	x	1	z	w
z	0	0	z	y	y	0	z	y	z
1	0	x	y	s	t	v	w	z	1

Table 7: WP-complements of $\mathfrak{O}$

element	0	x	y	s	t	v	w	z	1
weak p-comp.	1	z	v	0	0	y	0	x	0

In this case the weakly p-semiring $\mathfrak{O}'' = (O; +, \cdot, ^{\omega_p})$ becomes a weakly distributive lattice under the new ordered relation " $\ll$ " represent in Figure(4) (in the last page)

Definition 3.8. [8] A kernel operator ρ on an ordered semiring $\mathfrak{O}$ is a monotone ($u \leq v \Rightarrow \kappa(u) \leq \kappa(v)$), contraction ($\kappa(u) \leq u$) and idempotent ($\kappa(\kappa(u)) = \kappa(u)$) operator.

Lemma 3.9. *Let κ be a kernel operator on a spwp-semiring $\mathfrak{O}$ and $u \cdot v$ is the largest lower bound of u and v, for any $u, v \in \mathfrak{O}$. Then:*

i) $\kappa(\kappa(u) \cdot \kappa(v)) = \kappa(u \cdot v)$;

ii) $\kappa(\kappa(u) + \kappa(v)) = \kappa(u) + \kappa(v)$.

Proof. i) We have $u \cdot v \leq u, v$, then $\kappa(u \cdot v) = \kappa(\kappa(u \cdot v)) \leq \kappa(\kappa(u) \cdot \kappa(v)) = \kappa(\kappa(\kappa(u) \cdot \kappa(v))) \leq \kappa(\kappa(u \cdot v)) = \kappa(u \cdot v)$.

ii) Since $\kappa(u), \kappa(v) \leq \kappa(u) + \kappa(v)$, then $\kappa(u) = \kappa(\kappa(\kappa(u))) \leq \kappa(\kappa(\kappa(u) + \kappa(v)))$ and $\kappa(v) = \kappa(\kappa(\kappa(u))) \leq \kappa(\kappa(\kappa(u) + \kappa(v)))$. Thus, $\kappa(u) + \kappa(v) \leq \kappa(\kappa(\kappa(u) + \kappa(v))) = \kappa(\kappa(u) + \kappa(v)) \leq \kappa(u) + \kappa(v)$.

□

Consider κ be a kernel operator on a spwp-semiring $\mathfrak{O}$ and $u \cdot v$ is the largest lower bound of u and v, for any $u, v \in \mathfrak{O}$. Define unary operation "$^{\omega_\kappa}$" on $\mathfrak{O}$ as:

$$u^{\omega_\kappa} = \kappa(u^{\omega_p}) \text{ for all } u \in \mathfrak{O}.$$

Moreover, define binary operation "$\cdot_{\omega_\kappa}$" as:

$$u \cdot_{\omega_\kappa} v = \kappa(u \cdot v) \text{ for all } u, v \in \mathfrak{O}.$$

Theorem 3.10. *Let κ be a kernel operator on a spwp-semiring $\mathfrak{O}$, with $\mathfrak{O}1$ and $\kappa(u \cdot v) = u \cdot v$ for all $u, v \in \kappa(\mathfrak{O})$.Then $(\kappa(\mathfrak{O}); +, \cdot_{\omega_\kappa})$ is a spwp-semiring.*

Proof. Suppose κ be a kernel operator on $\mathfrak{O}$. Clearly the operations $+$ and $\cdot_{\omega_\kappa}$ are associative, 0 is the additive identity and $0 \neq \kappa(1)$. Since $u \cdot_{\omega_\kappa} \kappa(1) = \kappa(u \cdot \kappa(1)) = \kappa(\kappa(u) \cdot \kappa(1)) = \kappa(u \cdot 1) = \kappa(u) = u$. Then $\kappa(1)$ is the multiplicative identity element.

$$\begin{aligned} u \cdot_{\omega_\kappa} (v + w) &= \kappa(u \cdot (v + w)) = \kappa((u \cdot v) + (u \cdot w)) \\ &= \kappa(\kappa(u \cdot v) + \kappa(u \cdot w)) \\ &= \kappa(u \cdot v) + \kappa(u \cdot w) \\ &= (u \cdot_{\omega_\kappa} v) + (u \cdot_{\omega_\kappa} w). \end{aligned}$$

For any $u \in \kappa(\mathfrak{O})$, $u \cdot_{\omega_\kappa} 0 = \kappa(u \cdot 0) = \kappa(0) = 0$. If $u \leq v$, then $u + w \leq v + w$ and $u \cdot w \leq v \cdot w$. Thus $\kappa(u \cdot w) \leq \kappa(v \cdot w)$ and hence $u \cdot_{\omega_\kappa} w \leq v \cdot_{\omega_\kappa} w$. To prove that u^{ω_κ} is wp-complement of u we get: $u^{\omega_\kappa \omega_\kappa} = \kappa(\kappa(u^{\omega_p})) = \kappa((\kappa(u^{\omega_p}))^{\omega_p}) \geq \kappa(u^{\omega_p \omega_p}) \geq \kappa(u) = u$. Assume $u \leq v$ so $u^{\omega_p} \geq v^{\omega_p}$. Then $u^{\omega_\kappa} = \kappa(u^{\omega_p}) \geq \kappa(v^{\omega_p}) = v^{\omega_\kappa}$. $(u \cdot_{\omega_\kappa} u^{\omega_\kappa}) = \rho((u \cdot u^{\omega_\kappa}) = \kappa(0) = 0$. Since $u \leq 1$ for any $u \in \mathfrak{O}$, then $\kappa(u) \leq \kappa(1)$ and $0^{\omega_\kappa} = \kappa(1)$. Therefore $(\kappa(\mathfrak{O}); +, \cdot_{\omega_\kappa})$ is spwp-semiring. □

Example 3.11. Consider the topological space (for more details about topological spaces refer to [15]) $\tau = \{\phi, \{1\}, \{3\}, \{1,3\}, \{1,4\}, \{2,3\}\{1,2,3\}, \{1,3,4\}, U\}$, where $U = \{1,2,3,4\}$. Define a kernel operator $\rho : P(U) \to P(U)$ as $\rho(A) = \bigcap\{B : B \text{is an open set}, B \subseteq A\}$ (for more details of topological spaces refer to [15]). We get the multiplicatively idempotent spwp-semiring $(\rho(U); \cup, \cdot_{\omega_\rho})$ with set inclusion as ordered relation. The operations "$\cup$" and "$\cdot_{\omega_\rho}$" are calculated in Tables 8 and 9. wp-complements are shown in Table 10.

Table 8: $\cup$ operation on $\rho(U)$

$\cup$	ϕ	{1}	{3}	{13}	{14}	{23}	{123}	{134}	U
ϕ	ϕ	{1}	{3}	{13}	{14}	{23}	{123}	{134}	U
{1}	{1}	{1}	{13}	{13}	{14}	{123}	{123}	{134}	U
{3}	{3}	{13}	{3}	{13}	{134}	{23}	{123}	U	U
{13}	{13}	{1,3}	{13}	{13}	{134}	{123}	{123}	{134}	U
{14}	{14}	{14}	{134}	{134}	{14}	U	U	{134}	U
{23}	{23}	{123}	{23}	{123}	U	{23}	{123}	U	U
{123}	{123}	{123}	{123}	{123}	U	{123}	{123}	U	U
{134}	{134}	{134}	{134}	{134}	{134}	1	1	{134}	U
U	U	U	U	U	U	U	U	U	U

Table 9: $\cdot_{\omega_\kappa}$ operation on $\rho(U)$

$\cdot_{\omega_\rho}$	ϕ	$\{1\}$	$\{3\}$	$\{13\}$	$\{14\}$	$\{23\}$	$\{123\}$	$\{134\}$	U
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{1\}$	ϕ	$\{1\}$	ϕ	$\{1\}$	$\{1\}$	ϕ	$\{1\}$	$\{1\}$	$\{1\}$
$\{3\}$	ϕ	ϕ	$\{3\}$	$\{3\}$	ϕ	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$
$\{13\}$	ϕ	$\{1\}$	$\{3\}$	$\{13\}$	$\{1\}$	$\{3\}$	$\{13\}$	$\{13\}$	$\{13\}$
$\{14\}$	ϕ	$\{1\}$	0	$\{1\}$	$\{14\}$	ϕ	$\{1\}$	$\{14\}$	$\{14\}$
$\{23\}$	ϕ	ϕ	$\{3\}$	$\{3\}$	ϕ	$\{23\}$	$\{23\}$	$\{3\}$	$\{23\}$
$\{123\}$	ϕ	$\{1\}$	$\{3\}$	$\{13\}$	$\{1\}$	$\{23\}$	$\{123\}$	$\{13\}$	$\{123\}$
$\{134\}$	ϕ	$\{1\}$	$\{3\}$	$\{13\}$	$\{14\}$	$\{3\}$	$\{13\}$	$\{134\}$	$\{134\}$
U	ϕ	$\{1\}$	$\{3\}$	$\{13\}$	$\{14\}$	$\{23\}$	$\{123\}$	$\{134\}$	U

Table 10: Wp-complemented elements of $(\rho(U); \cup, \cdot_{\omega_\rho})$

element	ϕ	$\{1\}$	$\{3\}$	$\{13\}$	$\{14\}$	$\{23\}$	$\{123\}$	$\{134\}$	U
wp-comp.	U	$\{23\}$	$\{14\}$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ

4. Closed Elements

In the present section, closed elements of weak pseudocomplementation (wp-complementation) are introduced. Essential properties and algebraic structures are proved.

Proposition 4.1. *Let $\mathfrak{O}$ be a spwp-semiring with $\mathfrak{O}1$. Then:*

i) $u \in C^{\omega_p}(\mathfrak{O})$ *iff* $u^{\omega_p\omega_p} = u$;

ii) If $u, v \in C^{\omega_p}(\mathfrak{O})$*, then* $u \cdot v \in C^{\omega_p}(\mathfrak{O})$;

iii) If $u + u^{\omega_p} = 1$*, then* $u \in C^{\omega_p}(\mathfrak{O})$

iv) If $u \in C^{\omega_p}(\mathfrak{O})$*, then* $((u^{\omega_p} \cdot u) + (u \cdot u^{\omega_p}))^{\omega_p\omega_p} = 1$;

v) If $u \le v$ *and* $u^{\omega_p} \cdot w \le v^{\omega_p} \cdot w$*, then*

$$((u^{\omega_p} \cdot w) + (u \cdot w^{\omega_p}))^{\omega_p\omega_p} \le ((v^{\omega_p} \cdot w) + (v \cdot w^{\omega_p}))^{\omega_p\omega_p},$$

for all $u, v, w \in C^{\omega_p}(\mathfrak{O})$.

Proof. i) Let $u \in C^{\omega_p}(\mathfrak{O})$. Then there exists $v \in \mathfrak{O}$ such that $u = v^{\omega_p}$. So, $u^{\omega_p} = v^{\omega_p\omega_p}$ and $u^{\omega_p\omega_p} = v^{\omega_p\omega_p\omega_p} = v^{\omega_p} = u$. In the opposite direction, if $u = u^{\omega_p\omega_p} = (u^{\omega_p})^{\omega_p}$, then $u \in C^{\omega_p}(\mathfrak{O})$.

ii) Let $u, v \in C^{\omega_p}(\mathfrak{O})$. Then $u = s^{\omega_p}$ and $v = r^{\omega_p}$ for some $s, r \in \mathfrak{O}$. We have $u \cdot v = s^{\omega_p} \cdot r^{\omega_p} = (s + r)^{\omega_p}$ and $s + r \in \mathfrak{O}$. Hence $u \cdot v \in C^{\omega_p}(\mathfrak{O})$.

iii) Assume $u + u^{\omega_p} = 1$, then $u^{\omega_p\omega_p} \cdot (u + u^{\omega_p}) = u^{\omega_p\omega_p} \cdot 1$. Thus $(u^{\omega_p\omega_p} \cdot u) + (u^{\omega_p\omega_p} \cdot u^{\omega_p}) = u^{\omega_p\omega_p}$. So, $u = u + 0 = u^{\omega_p\omega_p}$. Therefore, $u \in C^{\omega_p}(\mathfrak{O})$.

iv) Let $u \in C^{\omega_p}(\mathfrak{O})$. Then $((u^{\omega_p} \cdot u^{\omega_p}) + (u \cdot u^{\omega_p\omega_p}))^{\omega_p\omega_p} = (u^{\omega_p} + u)^{\omega_p\omega_p} = (u^{\omega_p} \cdot u^{\omega_p\omega_p})^{\omega_p} = 0^{\omega_p} = 1$.

v) Let $u \le v$ and $u^{\omega_p} \cdot w \le v^{\omega_p} \cdot w$. Then $u \cdot w^{\omega_p} \le v \cdot w^{\omega_p}$ and so $(u^{\omega_p} \cdot w) + (u \cdot w^{\omega_p}) \le (v^{\omega_p} \cdot w) + (v \cdot w^{\omega_p})$.

□

Theorem 4.2. *Let $\mathfrak{O}$ be a spwp-semiring with $\mathfrak{O}1$. Then $(C^{\omega_p}(\mathfrak{O}); \uplus, \cdot)$ is an orthopseudoring such that $u \uplus v = ((u^{\omega_p} \cdot v) + (u \cdot v^{\omega_p}))^{\omega_p\omega_p}$, for all $u, v \in C^{\omega_p}(\mathfrak{O})$.*

Proof. Suppose $\mathfrak{O}$ is a spwp-semiring with $\mathfrak{O}1$ and $u, v \in C^{\omega_p}(\mathfrak{O})$. It is enough to prove that:

i) $u \uplus (1 \uplus v) = u \uplus v^{\omega_p} = ((u^{\omega_p} \cdot v^{\omega_p}) + (u \cdot v^{\omega_p\omega_p}))^{\omega_p\omega_p}$
$= ((u^{\omega_p\omega_p} \cdot v) + ((u^{\omega_p} \cdot v^{\omega_p}))^{\omega_p\omega_p} = u^{\omega_p} \uplus v = (u \uplus 1) \uplus v$.

ii) $u \uplus u = ((u^{\omega_p} \cdot u) + (u \cdot u^{\omega_p})) = 0.$

iii) $u \uplus (u \cdot v \cdot u) = ((u^{\omega_p} \cdot (u \cdot v)) + (u \cdot (u \cdot v)^{\omega_p}))^{\omega_p\omega_p} = (u \cdot (u \cdot v)^{\omega_p})^{\omega_p\omega_p}$
$= ((u \cdot v)^{\omega_p} \cdot u) = (1 \uplus (u \cdot v)) \cdot u.$

iv) $(1 \uplus u) \cdot (1 \uplus (u \cdot v)) = u^{\omega_p} \cdot (u \cdot v)^{\omega_p} = u^{\omega_p} = 1 \uplus u.$

v) $(1 \uplus u \cdot (1 \uplus v)) \cdot (1 \uplus v \cdot (1 \uplus u)) = (1 \uplus (u \cdot v^{\omega_p}) \cdot (1 \uplus (v \cdot u^{\omega_p})))$
$= (u \cdot v^{\omega_p})^{\omega_p} \cdot (v \cdot u^{\omega_p})^{\omega_p} = ((u \cdot v^{\omega_p}) + (v \cdot u^{\omega_p}))^{\omega_p}$
$= ((u \cdot v^{\omega_p}) + (v \cdot u^{\omega_p}))^{\omega_p\omega_p\omega_p} = 1 \uplus (u \uplus v)).$

□

Example 4.3. Consider the spwp-semiring $\mathfrak{O}'' = (O; +, ., \leq'')$ in Example (3.7). Tables 11 and 12 represent the orthopseudoring $(C^{\omega_p}(\mathfrak{O}); \uplus, \cdot)$ of closed elements.

Table 11: $\uplus$ operation on $C^{\omega_p}(\mathfrak{O})$.

$\uplus$	0	x	y	v	z	1
0	0	x	y	v	z	1
x	x	x	1	v	1	1
y	y	1	y	1	z	1
v	z	v	1	v	1	1
z	z	1	z	1	z	1
1	1	1	1	1	1	1

Table 12: $\cdot$ operation on $C^{\omega_p}(\mathfrak{O})$.

$\cdot$	0	x	y	v	z	1
0	0	x	0	0	0	0
x	0	0	0	x	0	x
y	0	0	y	0	y	y
v	0	x	0	v	0	v
z	0	0	y	0	z	z
1	0	x	y	v	z	1

From Theorem (3.4) and the results in [9] we get the following corollary.

Corollary 4.4. *Let $\mathfrak{O}$ be a multiplicatively idempotent spwp-semiring. Then $(C^{\omega_p}(\mathfrak{O}); \vee, \cdot, ^{\omega_p})$ is an ortholattice such that $u \vee v = (u^{\omega_p} \cdot v^{\omega_p})^{\omega_p}$, for all $u, v \in C^{\omega_p}(\mathfrak{O})$.*

Theorem 4.5. *Let $\mathfrak{O}$ be a spwp-semiring, with $\mathfrak{O}1$ and conditions:*

i) $(u \cdot v) + (u \cdot v^{\omega_p}) = u;$

ii) $(((u^{\omega_p} \cdot v)^{\omega_p} + v) \cdot (u \cdot v^{\omega_p})^{\omega_p})^{\omega_p} = ((u^{\omega_p} \cdot v)^{\omega_p} + v)^{\omega_p} + (u \cdot v^{\omega_p});$

iii) $(u^{\omega_p} \cdot v)^{\omega_p} \cdot v + ((u^{\omega_p} \cdot v)^{\omega_p} + v)^{\omega_p} = u \cdot v;$

iv) $u \cdot (u \cdot v)^{\omega_p} = u \cdot v^{\omega_p}.$

for all $u, v \in \mathfrak{O}$. Then $(C^{\omega_p}(\mathfrak{O}); +, \cdot)$is a Boolean ring.

Proof. Let $u, v \in C^{\omega_p}(\mathfrak{O})$. Then

$$\begin{aligned}(u \uplus (u \cdot v)) \uplus (u \cdot v) &= ((u^{\omega_p} \cdot (u \cdot v)) + (u \cdot (u \cdot v)^{\omega_p}))^{\omega_p\omega_p} \uplus (u \cdot v)\\ &= (u \cdot (u \cdot v)^{\omega_p})^{\omega_p\omega_p} \uplus (u \cdot v) = (u \cdot (u \cdot v)^{\omega_p}) \uplus (u \cdot v)\\ &= (((u \cdot (u \cdot v)^{\omega_p})^{\omega_p}) \cdot (u \cdot v)) + ((u \cdot (u \cdot v)^{\omega_p}) \cdot (u \cdot v)^{\omega_p})^{\omega_p\omega_p}\\ &= (u \cdot v) + (u \cdot (u \cdot v)^{\omega_p}))^{\omega_p\omega_p}\\ &\qquad \text{since } (u \cdot (u \cdot v)^{\omega_p})^{\omega_p} \geq (u \cdot v)^{\omega_p\omega_p} \geq u \cdot v\\ &= ((u \cdot v)) + (u \cdot v^{\omega_p}))^{\omega_p\omega_p} = u.\end{aligned}$$

$$
\begin{aligned}
(u \uplus v) \uplus v &= ((u^{\omega_p} \cdot v) + (u \cdot v^{\omega_p}))^{\omega_p\omega_p} \uplus v \\
&= ((((u^{\omega_p} \cdot v) + (u \cdot v^{\omega_p}))^{\omega_p} \cdot v) + (((u^{\omega_p} \cdot v) + (u \cdot v^{\omega_p})^{\omega_p\omega_p}) \cdot v^{\omega_p}))^{\omega_p\omega_p} \\
&= (((u^{\omega_p} \cdot v)^{\omega_p} \cdot (u \cdot v^{\omega_p})^{\omega_p} \cdot v) + (((u^{\omega_p} \cdot v)^{\omega_p} \cdot (u \cdot v^{\omega_p})^{\omega_p}) + v)^{\omega_p})^{\omega_p\omega_p} \\
&\quad \text{since} (u \cdot v^{\omega_p})^{\omega_p} \geq v^{\omega_p\omega_p} = v \\
&= (((u^{\omega_p} \cdot v)^{\omega_p} \cdot v) + (((u^{\omega_p} \cdot v)^{\omega_p} + v) \cdot (u \cdot v^{\omega_p})^{\omega_p})^{\omega_p})^{\omega_p\omega_p} \\
&= (((u^{\omega_p} \cdot v)^{\omega_p} \cdot v) + (((u^{\omega_p} \cdot v)^{\omega_p} + v)^{\omega_p} + (u \cdot v^{\omega_p})))^{\omega_p\omega_p} \\
&= ((u \cdot v) + (u \cdot v^{\omega_p}))^{\omega_p\omega_p} \\
&= u = u \uplus 0 = u \uplus (v \uplus v)
\end{aligned}
$$

$$
\begin{aligned}
u \uplus (u \cdot v) &= ((u^{\omega_p} \cdot (u \cdot v)) + (u \cdot (u \cdot v)^{\omega_p}))^{\omega_p\omega_p} \\
&= (u \cdot (u \cdot v)^{\omega_p})^{\omega_p\omega_p} \\
&= u \cdot v^{\omega_p} = u \cdot (1 + v)
\end{aligned}
$$

□

Example 4.6. Let $\mathfrak{O}'' = (O; +, \cdot, \leq'')$ be a spwp-semiring with operations "+"and "·" represented in Tables 5 and 6. Its ordered relation " $\leq''$ " is given in Figure (3) represents and Table 13 gives wp-complements. Tables 14 and 15 represent the Boolean ring $(C^{\omega_p}(\mathfrak{O}); \uplus, \cdot)$ of closed elements.

Table 13: wP-complements $\mathfrak{O}$“

element	0	x	y	s	t	v	w	z	1
wp-comp.	1	y	x	0	0	0	0	0	0

Table 14: $\uplus$ operation on $C^{\omega_p}(\mathfrak{O})$.

$\uplus$	0	x	y	1
0	0	x	y	1
x	x	x	1	1
y	y	1	y	1
1	1	1	1	1

Table 15: $\cdot$ operation on $C^{\omega_p}(\mathfrak{O})$.

$\cdot$	0	x	y	1
0	0	0	0	0
x	0	x	0	x
y	0	0	y	y
1	0	x	y	1

Corollary 4.7. *Let $\mathfrak{O}$ be a multiplicatively idempotent spwp-semiring with conditions:*

i) $(u \cdot v) + (u \cdot v^{\omega_p}) = u$;

ii) $(((u^{\omega_p} \cdot v)^{\omega_p} + v) \cdot (u \cdot v^{\omega_p})^{\omega_p})^{\omega_p} = ((u^{\omega_p} \cdot v)^{\omega_p} + v)^{\omega_p} + (u \cdot v^{\omega_p}))$;

iii) $(u^{\omega_p} \cdot v)^{\omega_p} \cdot v + ((u^{\omega_p} \cdot v)^{\omega_p} + v)^{\omega_p} = u \cdot v$;

iv) $u \cdot (u \cdot v)^{\omega_p} = u \cdot v^{\omega_p}$.

for all $u, v \in \mathfrak{O}$. Then $(C^{\omega_p}(\mathfrak{O}); \vee, \cdot, ^{\omega_p})$ is a Boolean algebra.

5. Weakly Stone and Dense Elements

Here, weakly Stone and dense elements of wp-semiring are introduced. Weakly Stone semiring is defined. Properties of dense elements and algebraic structures are proved.

Lemma 5.1. *Let $\mathfrak{O}$ be a spwp-semiring with $\mathfrak{O}1$ and $(u \cdot v)^{\omega_p} = u^{\omega_p} + v^{\omega_p}$, for all $u, v \in \mathfrak{O}$. Then all conditions in Theorem (4.5) are hold.*

Proof. i) $(u \cdot v) + (u \cdot v^{\omega_p}) = u \cdot (v + v^{\omega_p}) = u \cdot (v^{\omega_p\omega_p} + v^{\omega_p}) = u \cdot (v^{\omega_p} \cdot v)^{\omega_p}$
$= u \cdot 1 = u$;

ii) Immediately, by putting $u = ((u^{\omega_p} \cdot v)^{\omega_p} + v)$ and $v = (u \cdot v^{\omega_p})$.

iii) $(u^{\omega_p} \cdot v)^{\omega_p} \cdot v + ((u^{\omega_p} \cdot v)^{\omega_p} + v)^{\omega_p} = (u^{\omega_p\omega_p} + v^{\omega_p}) \cdot v + ((u^{\omega_p} \cdot v).v^{\omega_p}) = u^{\omega_p\omega_p} \cdot v = u \cdot v$;

iv) $u \cdot (u \cdot v)^{\omega_p} = u \cdot (u^{\omega_p} + v^{\omega_p}) = u \cdot v^{\omega_p}$.

□

Example (5.2) shows that the converse is not true.

Example 5.2. Consider spwp-semiring $\mathfrak{O}'' = (O; +, \cdot, \leq'')$ in Example (4.6). It satisfies all conditions of Theorem (4.5). But, it does not satisfy that $(u \cdot v)^{\omega_p} = u^{\omega_p} + v^{\omega_p}$. E.g. $(x \cdot y)^{\omega_p} = 1 \neq x^{\omega_p} + y^{\omega_p}$

Definition 5.3. Let $\mathfrak{O}$ be a positive ordered semiring. Then $u \in \mathfrak{O}$ be a weak Stone element of $\mathfrak{O}$ if $u, u^{\omega_p} \in W_p(\mathfrak{O})$ and $u^{\omega_p} + u^{\omega_p\omega_p} = 1$.

The set of all weak Stone elements is denoted by $W_S(\mathfrak{O})$. If $\mathfrak{O} = W_S(\mathfrak{O})$, then $\mathfrak{O}$ is called weakly Stone semiring. So, Weakly Stone semiring is simple and positive.

Proposition 5.4. *Let $\mathfrak{O}$ be a wp-semiring. Then it is satisfying the following conditions:*

i) If $u \in W_S(\mathfrak{O})$, then $[u^{\omega_p}]^2 = u^{\omega_p}$;
ii) If $u \in W_S(\mathfrak{O})$, then $u^2 \leq u$;
iii) If $u \in W_S(\mathfrak{O}) \cap C^{\omega_p}(\mathfrak{O})$, then $u^2 = u$;
iv) $1 \in W_S(\mathfrak{O})$ iff $0^{\omega_p} = 1$.

Proof. i) suppose $u \in W_S(\mathfrak{O})$, then $u^{\omega_p} = u^{\omega_p} \cdot 1 = u^{\omega_p} \cdot [u^{\omega_p} + u^{\omega_p\omega_p}] = u^{\omega_p} \cdot u^{\omega_p} + u^{\omega_p} \cdot u^{\omega_p\omega_p} = u^{\omega_p} \cdot u^{\omega_p} = [u^{\omega_p}]^2$.

ii) Let $u \in W_S(\mathfrak{O})$. Then $u = u \cdot 1 = u \cdot [u^{\omega_p} + u^{\omega_p\omega_p}] = u \cdot u^{\omega_p} + u \cdot u^{\omega_p\omega_p} = u \cdot u^{\omega_p\omega_p}$. But we have $u \leq u^{\omega_p\omega_p}$, so $u^2 \leq u \cdot u^{\omega_p\omega_p} = u$.

iii) Suppose $u \in W_S(\mathfrak{O}) \cap C^{\omega_P}(\mathfrak{O})$, then $u = u \cdot 1 = u \cdot [u^{\omega_p} + u^{\omega_p\omega_p}] = u \cdot u^{\omega_p} + u \cdot u^{\omega_p\omega_p} = u \cdot u^{\omega_p\omega_p} = u \cdot u = u^2$.

iv) Suppose $1 \in W_S(\mathfrak{O})$ implies $1^{\omega_p} + 1^{\omega_p\omega_p} = 1$. But $1^{\omega_p} = 0$ and then $1^{\omega_p\omega_p} = 1$. So, $0^{\omega_p} = 1^{\omega_p\omega_p} = 1$. Conversely, if $0^{\omega_p} = 1$, then $1 \in C^{\omega_p}(\mathfrak{O})$. Thus $1 = 1^{\omega_P\omega_p} = 1^{\omega_p\omega_p} + 0 = 1^{\omega_p\omega_p} + 1^{\omega_p}$.

□

Theorem 5.5. *Let $\mathfrak{O}$ be a spwp-semiring. Then:*

i) If $(u \cdot v)^{\omega_p} = u^{\omega_p} + v^{\omega_P}$, for all $u, v \in \mathfrak{O}$, then $\mathfrak{O}$ is a weakly Stone semiring;
ii) If $\mathfrak{O}$ is a weakly Stone, then

$$u^{\omega_p} \cdot v = 0 \quad \textit{iff} \quad v \leq u^{\omega_p\omega_p} \quad \textit{for all} \quad u, v \in \mathfrak{O};$$

iii) $\mathfrak{O}$ is a multiplicatively idempotent weakly Stone iff

$$(u \cdot v)^{\omega_p} = u^{\omega_p} + v^{\omega_P}, \quad \textit{for all} \quad u, v \in \mathfrak{O}.$$

Proof. i) Suppose $(u \cdot v)^{\omega_p} = u^{\omega_P} + v^{\omega_p}$, for all $u, v \in \mathfrak{O}$. Thus $u^{\omega_P} + u^{\omega_p\omega_p} = (u \cdot u^{\omega_p})^{\omega_p} = 0^{\omega_p} = 1$, for any $u \in \mathfrak{O}$. Therefore $\mathfrak{O}$ is a weakly Stone.

ii) Let $u, v \in \mathfrak{O}$ and $u^{\omega_p} \cdot v = 0$. Then $v = v \cdot 1 = v \cdot (u^{\omega_p} + u^{\omega_p\omega_p}) = (v \cdot u^{\omega_p}) + (v \cdot u^{\omega_p\omega_p}) = v \cdot u^{\omega_p\omega_p}$. So, $v \leq u^{\omega_p\omega_p}$. The converse immediately from in Proposition (5.4).

iii) We have $(u \cdot v)^{\omega_p\omega_p} \cdot (u^{\omega_p} + v^{\omega_p}) = 0$. Now let $w \in \mathfrak{O}$ such that $(u \cdot v)^{\omega_p} \cdot w = 0$. Then $w \leq (u \cdot v)^{\omega_p\omega_p} \leq u^{\omega_p\omega_p}, v^{\omega_p\omega_p}$. Accordingly, $w = w^2 \leq u^{\omega_p\omega_p} \cdot v^{\omega_p\omega_p} = (u^{\omega_p} + v^{\omega_p})^{\omega_p}$. Thus $(u^{\omega_p} + v^{\omega_p})^{\omega_p} = (u \cdot v)^{\omega_p\omega_p}$. The second direction proved in i).

□

Corollary 5.6. *Let $\mathfrak{O}$ be a spwp-semiring with $\mathfrak{O}1$. Then:*

i) If $\mathfrak{O}$ is a weakly Stone, then $(C^{\omega_p}(\mathfrak{O}); +, \cdot)$is a Boolean ring
ii) If $\mathfrak{O}$ is a multiplicatively idempotent weakly Stone, then $(C^{\omega_p}(\mathfrak{O}); +, \cdot)$ is a Boolean algebra.

Example (4.6) shows that the converse is not true.

Proposition 5.7. *Let $\mathfrak{O}$ be a spwp-semiring with $\mathfrak{O}1$. Then:*

i) $1 \in D^{\omega_p}(\mathfrak{O})$;
ii) If $u \in \mathfrak{O}$, $v \in D^{\omega_p}(\mathfrak{O})$ and $v \le u$, then $u \in D^{\omega_p}(\mathfrak{O})$;
iii) If $u \in \mathfrak{O}$ and $v \in D^{\omega_p}(\mathfrak{O})$, then $u + v \in D^{\omega_p}(\mathfrak{O})$;
iv) $u + u^{\omega_p} \in D^{\omega_p}(\mathfrak{O})$;
v) $u^{\omega_p\omega_p} + v^{\omega_p\omega_p} \in D^{\omega_p}(\mathfrak{O})$ iff $u + v \in D^{\omega_p}(\mathfrak{O})$;
vi) If $(u \cdot v)^{\omega_p} = u^{\omega_p} + v^{\omega_p}$, for all $u, v \in D^{\omega_p}(\mathfrak{O})$, then $(u \cdot v) \in D^{\omega_p}(\mathfrak{O})$.

Proof. i) Immediately from Proposition (3.3).
ii) Assume $v \in \mathfrak{O}$ and $u \in D^{\omega_p}(\mathfrak{O})$ and $v \le u$. we get $0 = v^{\omega_p} \ge v^{\omega_p} = 0$, then $u \in D^{\omega_p}(\mathfrak{O})$.
iii) Suppose $v \in \mathfrak{O}$ and $u \in D^{\omega_p}(\mathfrak{O})$, so $(u + v)^{\omega_p} = u^{\omega_p} \cdot v^{\omega_p} = u^{\omega_p} \cdot 0 = 0$. Therefore $u + v \in D^{\omega_p}(\mathfrak{O})$.
iv) It follows immediately from (iii) in Corollary (3.6).
v) It follows immediately from (iv) in Corollary (3.6).
vi) Let $(u \cdot v)^{\omega_p} = u^{\omega_p} + v^{\omega_p}$, for all $u, v \in D^{\omega_p}(\mathfrak{O})$, then $(u \cdot v)^{\omega_p} = u^{\omega_p} + v^{\omega_p} = 0$. Hence $(u \cdot v) \in D^{\omega_p}(\mathfrak{O})$. □

From Proposition (5.7) we get the following result.

Corollary 5.8. *Let $\mathfrak{O}$ be a spwp-semiring with $\mathfrak{O}1$. Then $(D_{\omega_p}(\mathfrak{O}); +)$ is an commutative monoid. Moreover, If $(u \cdot v)^{\omega_p} = u^{\omega_P} + v^{\omega_p}$, for all $u, v \in D_{\omega_p}(\mathfrak{O})$, then $(D_{\omega_p}(\mathfrak{O}); +, \cdot)$ is a presemiring.*

Example 5.9. Consider the spwp-semiring $\mathfrak{O}'' = (O; +, ., \le'')$ in Example (4.6). Table 16 represent "+" operation of semigroup $D_{\omega_p}(\mathfrak{O}'')$.

Table 16: wP-complements $\mathfrak{O}$“

+	s	t	v	w	z	1
s	s	s	1	1	s	1
t	s	t	w	w	s	1
v	1	w	v	w	1	1
w	1	w	w	w	1	1
z	s	s	1	1	z	1
1	1	1	1	1	1	1

Table (16): + operation on $D_{\omega_p}(\mathfrak{O}'')$.

Example 5.10. Consider the spwp-semiring $\mathfrak{O}'' = (O; +, ., \le'')$ in Example (3.7). Tables 17 and 18 give the "+" and "·" operations of the presemiring $D'_{\omega_p}(\mathfrak{O}'')$.

Table 17: + operation on $D'_{\omega_p}(\mathfrak{O}'')$.

+	s	t	w	1
s	s	s	1	1
t	s	t	w	1
w	1	w	w	1
1	1	1	1	1

Table 18: · operation on $D'_{\omega_p}(\mathfrak{O}'')$.

·	s	t	w	1
s	s	t	t	s
t	t	t	t	t
w	t	t	w	w
1	s	t	w	1

From Theorem (3.4) and the results in [9] we get the following corollary.

Corollary 5.11. *Let $\mathfrak{D}$ be a multiplicatively idempotent simple positive weakly p-semiring. Then $(D_{\omega_p}(\mathfrak{D}); +, \cdot)$ is a semilattice with greatest element.*

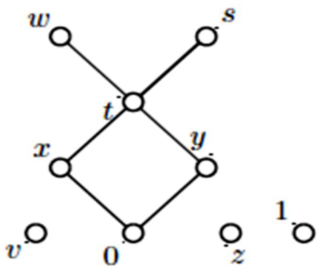

Figure 1: Ordered relation $\leq$ of $\mathfrak{D}$.

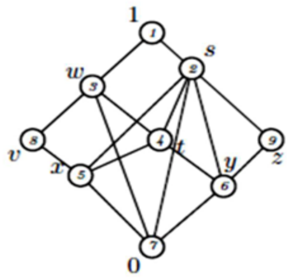

Figure 2: Ordered relation $\leq'$ of $\mathfrak{D}'$.

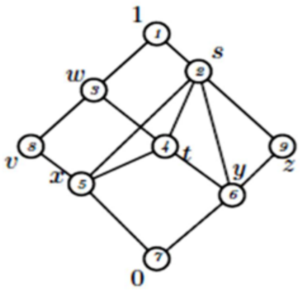

Figure 3: Ordered relation $\leq''$ of $\mathfrak{D}''$.

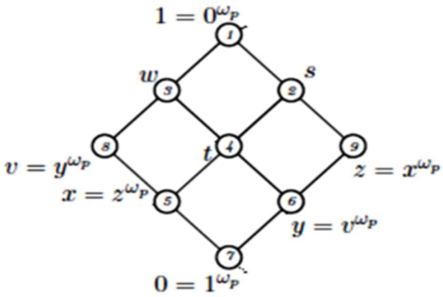

Figure 4: Ordered relation "$\ll$" of $\mathfrak{D}''$.

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