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Common Fixed Point Results in Fuzzy b-Metric Space Using the Integrals as Application

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Abstract

In recent years, the notion of fuzzy mathematics has become one of the interesting field of research. In this article, using the idea of fuzzy b-metric space and contraction mappings, we prove Banach contraction theorem. Also applying the idea of different compatible mappings, unique common fixed point results in fuzzy b-metric space are introduced and these results are connected in terms of the integral as the application. Some interesting instances to highlight the significance of our research are presented. Previous findings from the pertinent study are presented as lemmas and corollaries. Our findings complement and add to a large number of existing findings.

Keywords: Fuzzy b-metric space, common fixed point, convergent sequence, compatible mapping, integral equation.
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1. Introduction and Preliminaries

In the year 1965, Zadeh [19] introduced the pioneering concept of fuzzy sets, with the aim to provide an analytical framework to represent ambiguity and uncertainty. By using the membership function, fuzzy set is defined and denoted as $\mu : A \to [0, 1]$, which assigns to each element a in the set A a membership value $\mu(a)$ that lies within the unit interval $[0, 1]$, with reflecting the degree of membership of a in the fuzzy set. Subsequently, in 1975, Kramosil and Michálek [12] extended the classical notion of metric spaces by introducing the concept of fuzzy metric spaces, thereby drawing comparison between fuzzy metric spaces and probabilistic metric spaces, both serving as generalized forms of traditional metric spaces.
Advancing this theoretical framework, George and Veeramani [8] developed the topological ideas connecting with Hausdorff for fuzzy metric spaces. Then refined the ideas by incorporating more stringent conditions through the utilization of a continuous t-norm. In further developments, Gregori and Sapena [10] defined key

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concepts such as convergent sequences, Cauchy sequences, completeness, and compactness in fuzzy metric spaces. Additionally, Grabiec [9] applied fuzzy ideas to the Banach Contraction Principle. The foundations for the idea of b-metric spaces were laid due to Bakhtin and Bourbaki [3], and later, in 1993, Czerwik [7] formally introduced and defined b-metric space. Since then, numerous researchers have explored various examples and established fixed-point results within these areas. George and Veeramani [8] also formulated the Banach contraction theorem for fuzzy metric spaces. In 2012, Sedghi and Shobe [17] proposed the new concept for fuzzy metric space which is known as fuzzy b-metric space, by adopting a flexible formation of triangle inequality. This idea was further expanded in 2016 by Nădăban [14], leading to additional significant results by adding two extra properties than Sedghi and Shobe, as $\mathbb{F}(\mu,\nu,r)$ is non decreasing on r and $\lim_{r\to\infty} \mathbb{F}(\mu,\nu,r) = 1$. Also he established the property that open ball is open in fuzzy b-metric space and different topological properties are established, which was ignored by Sedghi and Shobe. Building upon this foundation, Öner and Šostak [15] introduced and described the concept of fuzzy strong b-metric spaces in 2020. Ali and Hassan [1] introduced the existence and uniqueness common fixed point theorem in fuzzy b-metric space providing an appropriate requirement for a sequence in fuzzy b-metric space to be a Cauchy. In 2024, Bhandari et al. [5] explored fixed-point properties in strong fuzzy b-metric spaces as the generalization of fuzzy b-metric space. Similarly, in 2025, Bhandari, Manandhar and Jha [4] introduced some common fixed point theorems in the context of fuzzy b-metric spaces leveraging the properties of convergence sequence and self-mappings.

Furthermore, in 2023, Gunaseelan Mani et al. [11] examined these spaces incorporating the property of b-triangular and explored for the applications to well known Fredholm integral equations and dynamic programming. In their work, they introduced the b-triangular property in fuzzy b-metric spaces and established new fixed-point results for non-continuous mappings. Their findings were also expressed in integral form, and illustrative examples were provided to demonstrate their practical applicability. In this paper, we adopt the framework of fuzzy b-metric spaces to analyze the existence of unique common fixed points under different contractive and compatible mappings. These results are connected to the integral equations as the application. The findings are also expressed in integral form as the application and illustrative examples are provided to demonstrate their practical applicability.

2. Auxiliary Facts and Preliminaries

Definition 2.1. [17] Suppose $*$ be an operation which defines a mapping as $* : [0,1]\times[0,1] \to [0,1]$ is called a continuous t-norm with the indicated properties.
(i) $l * m = m * l$, for $l, m \in [0,1]$;
(ii) $l * n \leq m * p$ whenever $l \leq m$ and $n \leq p$;
(iii) $(l * m) * n = l * (m * n)$, where $l, m, n \in [0,1]$;
(iv) $1 * l = l$, $\forall\ l \in [0,1]$.

Definition 2.2. [8] Suppose A is a universal set, $\mathbb{F}$ is a fuzzy set and $*$ is a triangular norm which is continuous and $\mathbb{F} : A \times A \times \mathbb{R}^+ \to [0,1]$ be a mapping. Then the three tuple $(A, \mathbb{F}, *)$ is known as the fuzzy metric space with the properties given below, where $\mu, \nu, \lambda \in A$ and $r, s > 0$
(i) $\mathbb{F}(\mu,\nu,r) > 0$,
(ii) $\mathbb{F}(\mu,\nu,r) = 1\ \forall\ r > 0 \iff \mu = \nu$
(iii) $\mathbb{F}(\mu,\nu,r) = \mathbb{F}(\nu,\mu,r)$,
(iv) $\mathbb{F}(\mu,\nu,\lambda) * \mathbb{F}(\nu,\lambda,s) \leq \mathbb{F}(\mu,\nu,r+s)\ \forall\ r,s > 0$,
(v) $\mathbb{F}(\mu,\nu,..) : \mathbb{R}^+ \to I$ is continuous

Where measure of closeness in μ and ν at any parameter $r > 0$ is indicated by $\mathbb{F}(\mu,\nu,r)$.

Example 2.3. [8] Suppose $A = \mathbb{N}$, the t-norm is defined as $l * m = l.m$ and

$$\mathbb{F}(\mu,\nu,r) = \begin{cases} \dfrac{\mu}{\nu} & \text{if } \mu \leq \nu, \\ \dfrac{\nu}{\mu} & \text{if } \nu \leq \mu \end{cases}$$

Then $\mathbb{F}(\mu, \nu, r)$ fulfills the properties of fuzzy metric space.

Sedghi and Shobe [17] introduced their ideas to explore the concept of fuzzy b-metric spaces, that is broader than fuzzy metric spaces, using a flexible form in the triangle inequality. Nadaban, in 2016, [14] extended this idea as generalize the notion of Sedghi and Shobe.

Definition 2.4. [14] Let $A \neq \phi$ and $b \geq 1$ be a given positive real number and $*$ be a binary operation defined on fuzzy set $\mathbb{F}$ on $A \times A \times \mathbb{R}^+ \to I$ where $I = [0, 1]$. If the conditions given bellow are full filled then the 3-tuple $(A, \mathbb{F}, *)$ is called the fuzzy b-metric space.
$\forall\ \mu, \nu, \lambda \in A$ where $r, s > 0$
(i) $\mathbb{F}(\mu, \nu, r) > 0$;
(ii) $\mathbb{F}(\mu, \nu, r) = 1\ \forall\ r > 0 \iff \mu = \nu$;
(iii) $\mathbb{F}(\mu, \nu, r) = \mathbb{F}(\nu, \mu, r)$;
(iv) $\mathbb{F}(\mu, \nu, r) * \mathbb{F}(\nu, \mu, s) \leq \mathbb{F}(\mu, \nu, b(r + s))\ \forall\ r, s > 0$;
(v) $\mathbb{F}(\mu, : (0, \infty) \to [0, 1]$ is continuous from the left;
(vi) $\lim\limits_{t\to\infty} \mathbb{F}(\mu, \nu, r) = 1$.
If $b = 1$ then we can say that every fuzzy b-metric space represents a fuzzy metric space.

Example 2.5. [14] Assume $A = I$, a unit interval, then a mapping $\mathbb{F} : A \times A \times [0, \infty) \to [0, 1]$ given as

$$(\mu, \nu, r) = \begin{cases} e^{-\dfrac{(\mu-\nu)^2}{r}} & \text{if } r > 0, \\ 0 & \text{if } r = 0. \end{cases}$$

Hence $(A, \mathbb{F}, *)$ is a fuzzy b-metric space with $b = 2$.

Definition 2.6. [8] Assume $(A, \mathbb{F}, *)$ be a fuzzy b-metric space. If we consider a sequence $\{\mu_n\} \in A$ which is known as to be convergent in A if
$\lim\limits_{n\to\infty} \mathbb{F}(\mu_n, \mu, r) = 1$ for positive number r.

Definition 2.7. [8] A sequence $\{\mu_n\}$ is known as the Cauchy sequence in A if
$\lim\limits_{n\to\infty} \mathbb{F}(\mu_n, \mu_m, r) = 1$ where $r > 0$ and $m, n > 0$.

A fuzzy b-metric space is known as the complete, if every Cauchy sequence of it is convergent in it.

Definition 2.8. [6] Two self mappings f and g of a fuzzy metric space $(A, \mathbb{F}, *)$ are said to be compatible (a) if, $\forall\ r > 0$, $\lim\limits_{n\to\infty} \mathbb{F}(fg\mu_n, gf\mu_n, r) = 1$ whenever $\{\mu_n\}$ is a sequence in A such that, $\lim\limits_{n\to\infty} f\mu_n = \lim\limits_{n\to\infty} g\mu_n = \nu$, where $\nu \in A$.
(b) Which are known as weakly compatible when they make a sense at their coincidence point, that is $f\mu = g\mu$ which gives that $fg\mu = gf\mu$.
(c) and are known as semi-compatible if, $\forall\ r > 0$, $\lim\limits_{n\to\infty} \mathbb{F}(f\mu_n, g\mu_n, r) = 1$ where $\{\mu_n\}$ is a sequence in A such that, $\lim\limits_{n\to\infty} f\mu_n = \lim\limits_{n\to\infty} g\mu_n = \nu$, for some $\nu \in A$.

Definition 2.9. [9] Let us consider a space as a fuzzy b-metric $(A, \mathbb{F}, *)$ where $b \in [1, \infty)$, define a map $f : A \to A$ is called a fuzzy b-contraction for $k \in \left(0, \dfrac{1}{b}\right)$ where

$$\frac{1}{\mathbb{F}(f\nu_1, f\nu_2, r)} - 1 \leq k \left(\frac{1}{\mathbb{F}(\nu_1, \nu_2, r)} - 1 \right), \tag{1}$$

for all $\nu_1, \nu_2 \in A$ and $r > 0$.

Definition 2.10. [16] Suppose $(A, \mathbb{F}, *)$ be a fuzzy b-metric space where $b \geq 1$ and define a mapping $f : A \to A$, then it is called as a rational type fuzzy b-contraction if there exists $k_1 \in \left(0, \frac{1}{b}\right)$ and $k_2 \geq 0$ where

$$\frac{1}{\mathbb{F}(f\nu_1, f\nu_2, r)} - 1 \leq k_1 \left(\frac{1}{\mathbb{F}(\nu_1, \nu_2, r)} - 1\right) + k_2 \left(\frac{\mathbb{F}(\nu_1, \nu_2, r)}{\mathbb{F}(\nu_1, f\nu_1, r) * \mathbb{F}(\nu_2, f\nu_1, 2r)} - 1\right)$$

for all $\nu_1, \nu_2 \in A$, $r > 0$.

Definition 2.11. [13] If we take a space $(A, \mathbb{F}, *)$ as a fuzzy b-metric space, where $b \in [1, \infty)$. Then the fuzzy b-metric $\mathbb{F}$ is known as the b-triangular with the following property.

$$\frac{1}{\mathbb{F}(\nu_1, \nu_2, r)} - 1 \leq s \left(\frac{1}{\mathbb{F}(\nu, \lambda, r)} - 1 + \frac{1}{\mathbb{F}(\lambda, \nu, r)} - 1\right)$$

Example 2.12 ([13])**.** . Let $\mathbb{F} : A^2 \times (0, \infty) \to [0, 1]$ be defined by:

$$\mathbb{F}(\nu_1, \nu_2, r) = \frac{r}{r + |\nu_1 - \nu_2|^2},$$

for all $\nu_1, \nu_2 \in A$, $r > 0$ then $\mathbb{F}$ is b-triangular.

3. Main Result

Lemma 3.1. *If a metric d on A induced a fuzzy b-metric $(A, \mathbb{F}, *)$. Then $\{\mu_n\}$ lying in A is known as contractive in (A, d) if and only if $\{\mu_n\}$ is contractive as fuzzy in $(A, \mathbb{F}, *)$.*

Corollary 3.2. *[2] If we take a sequence $\{\mu_n\}$ from $(A, \mathbb{F}, *)$ which converges to μ iff $\mathbb{F}(\mu_n, \mu, r) \to 1$ as $n \to \infty$.*

Theorem 3.3. *(Banach Contraction Theorem in Fuzzy b-Metric Space) Assume a fuzzy b-metric space $(A, \mathbb{F}, *)$ that is complete with $b \geq 1$ in which the fuzzy b-contractive sequences are obviously Cauchy sequences. Define $f : A \to A$ be a mapping with contractive constant $d \in (0, 1]$ and $k \in \left(0, \frac{1}{b}\right)$ and the mapping f has a fixed point which is unique.*

Proof. Assume that $\mu \in A$, if $\mu_n = f^n(\mu)$ for each $n > 0$. and, $\forall\ r > 0$,

$$\frac{1}{F(f(\mu), f^2(\mu), kr)} - 1 \leq d \left(\frac{1}{\mathbb{F}(\mu_1, \mu_2, r)} - 1\right),$$

then, by using the induction method

$$\frac{1}{\mathbb{F}(\mu_{n+1}, \mu_{n+2}, kr)} - 1 \leq d \left(\frac{1}{\mathbb{F}(\mu_n, \mu_{n+1}, r)} - 1\right)$$

$\forall\ n > 0$. Thus $\{\mu_n\}$ is fuzzy b-contractive sequence, then we can say that it is a Cauchy sequence, and so $\mu_n \to \nu$ for some $\nu \in A$.
Again, we have to show ν be a fixed point of f. As we have

$$\frac{1}{\mathbb{F}(f(\nu), f(\mu_n), kr)} - 1 \leq d \left(\frac{1}{\mathbb{F}(\nu, \mu_n, r)} - 1\right) \to 0 \quad \text{as } n \to \infty.$$

So

$$\lim_{n \to \infty} \mathbb{F}(f(\nu), f(\mu_n), kr) = 1 \quad \text{for all } r > 0,$$

this gives $\lim_{n\to\infty} f(\mu_n) = f(\nu)$. Hence,

$$\lim_{n\to\infty} \mu_{n+1} = f(\nu) \quad \text{and hence } f(\nu) = \nu$$

Now for uniqueness:
assume that $f(\lambda) = \lambda$ for some $\lambda \in A$. Then, $\forall\ r > 0$,

$$\begin{aligned}\frac{1}{\mathbb{F}(\nu,\lambda,t)} - 1 =& \frac{1}{\mathbb{F}(f(\nu), f(\lambda), kr)} - 1\\ &\leq d\left(\frac{1}{\mathbb{F}(\nu,\lambda,r)} - 1\right)\\ &\leq \cdots\\ &\leq d^n\left(\frac{1}{\mathbb{F}(\nu,\lambda,r)} - 1\right) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence, $\mathbb{F}(\nu, \lambda, r) = 1$, then $\nu = \lambda$.

Lemma 3.4. *[5] Assume $(A, \mathbb{F}, *)$ be a fuzzy b-metric space, and given a positive number $b \geq 1$. If there is $k \in \left(0, \frac{1}{b}\right)$ which gives $\mathbb{F}(\mu, \nu, kr) \geq \mathbb{F}(\mu, \nu, r)$ then $\mu = \nu$.*

Corollary 3.5. *Assume that $(A, \mathbb{F}*)$ be a fuzzy b-metric space which is complete and there exists a real number $b \geq 1$ where*

$$\lim_{r\to\infty} \mathbb{F}(\mu, \nu, r) = 1$$

and $k \in \left(0, \frac{1}{b}\right)$. If $f : A \to A$ be a function with $\forall\ \mu, \nu \in A$, then

$$\mathbb{F}(f\mu, f\nu, kr) \geq \mathbb{F}(\mu, \nu, r). \tag{3.1}$$

Then f contains a fixed point in A which is unique.

Theorem 3.6. *Let us consider a fuzzy b-metric space $(A, \mathbb{F}, *)$ which is complete and for a given $b \geq 1$. If g and f are any two mappings which are weakly compatible in A with the given properties*
(i) $\mathbb{F}(f\mu, f\nu, kr) \geq \mathbb{F}(g\mu, g\nu, r)$ where $k \in \left(0, \frac{1}{b}\right)$;
(ii) $f(A) \subseteq g(A)$.
If $f(A)$ or $g(A)$ is complete, then both f and g contain a common fixed point which is unique.

Proof. Assume $\mu_1 \in A$. As $f(A) \subseteq g(A)$, take $\mu_2 \in A$ where $f(\mu_1) = g(\mu_2)$. Let us take μ_{n+1} in A such that $\nu_n = g\mu_{n+1} = f\mu_n$. Then,

$$\begin{aligned}\mathbb{F}(g\mu_n, g\mu_{n+1}, r) =& \mathbb{F}(f\mu_{n-1}, f\mu_n, r)\\ &\geq \mathbb{F}\left(g\mu_{n-1}, g\mu_n, \frac{r}{k}\right)\\ &\geq \mathbb{F}\left(f\mu_{n-2}, f\mu_{n-1}, \frac{r}{k^2}\right)\\ &\geq \dots\\ &\geq \mathbb{F}\left(g\mu_1, g\mu_2, \frac{r}{k^n}\right).\end{aligned}$$

Then we have $\{g\mu_n\} = \{\nu_n\}$ as a Cauchy sequence.
Now by using the property of completeness, $\{\nu_n\} = \{g\mu_n\}$ is convergent, say at a point λ. Then

$$\lim_{n\to\infty} g\mu_n = \lim_{n\to\infty} f\mu_n = \lambda.$$

As $g(A)$ is complete, so we can find a point $\nu \in A$ with $g\nu = \lambda$.
Now,

$$\mathbb{F}(f\nu, f\mu_n, kr) \geq \mathbb{F}(g\nu, g\mu_n, r).$$

As $n \to \infty$,

$$\mathbb{F}(f\nu, \lambda, kr) \geq \mathbb{F}(g\nu, \lambda, r).$$

Therefore,

$$\mathbb{F}(f\nu, \lambda, kr) \geq \mathbb{F}(\lambda, \lambda, r) = 1,$$

Hence $\mathbb{F}(f\nu, \lambda, kr) = 1$, which implies $f\nu = \lambda = g\nu$.
Since g and f are weakly compatible, we have $gf(\nu) = fg(\nu)$, that is, $g\lambda = f\lambda$.
Now we will show that λ is a fixed point which is common on both g and f, for this assume

$$\mathbb{F}(f\lambda, f\mu_n, kr) \geq \mathbb{F}(g\lambda, g\mu_n, r).$$

As $n \to \infty$,

$$\mathbb{F}(f\lambda, \lambda, kr) \geq \mathbb{F}(g\lambda, \lambda, r),$$

This gives $f\lambda = \lambda = g\lambda$.
Then, λ is a fixed point which is common in both f and g.
To show for uniqueness, choose t be the next fixed point of f and g as common. Then

$$\mathbb{F}(f\lambda, ft, kr) \geq \mathbb{F}(g\lambda, gt, r),$$

and since $\lambda = t$, then we have λ is the unique fixed point of f and g which is also common.
This theorem can be verified by using th following numerical example.

Example 3.7. Let us assume that $I = [0, 1]$ and and define a fuzzy b-metric $\mathbb{F} : A \times A \times \mathbb{R}^+ \to I$ by

$$\mathbb{F}(\mu, \nu, r) = \begin{cases} 1 & \text{if } \mu = \nu \\ \frac{r}{r+|\mu-\nu|} & \text{if } \mu \neq \nu. \end{cases}$$

where $r > 0$, take $b = 2$.

Then we to show that $(A, \mathbb{F}, *)$ is a fuzzy b-metric space as well as complete.
If $f, g : A \to A$ defined as:

$$f(\mu) = \frac{\mu}{4}, \quad g(\mu) = \frac{\mu}{2}.$$

(i) $(\mathbb{F}(\mu, f\nu, kr) \geq \mathbb{F}(\mu, g\nu, r)$ for $k \in \left(0, \frac{1}{b}\right)$ for any $\mu, \nu \in A$ and $r > 0$:

$$\mathbb{F}(f(\mu), f(\nu), kr) = \frac{kr}{kr + \frac{|\mu-\nu|}{4}},$$

$$\mathbb{F}(g(\mu), g(\nu), r) = \frac{r}{r + \frac{|\mu-\nu|}{2}}.$$

Since $0 < k < \frac{1}{b} = \frac{1}{2}$, we have $kr < \frac{r}{2}$.
Hence

$$\frac{kr}{kr + \frac{|\mu-\nu|}{4}} \geq \frac{r}{r + \frac{|\mu-\nu|}{2}}.$$

This inequality holds because $kr + \frac{|\mu-\nu|}{4} \leq r + \frac{|\mu-\nu|}{2}$, and

$$\frac{kr}{kr + \frac{|\mu-\nu|}{4}} \geq \frac{r}{r + \frac{|\mu-\nu|}{2}}.$$

Hence $\mathbb{F}(f\mu, f\nu, r) \geq \mathbb{F}(g\mu, g\nu, r)$ for $k \in \left(0, \frac{1}{b}\right)$.
(ii) $f(A) \subseteq g(A)$. For any $\mu \in A$, $f(\mu) = \frac{\mu}{4} \in \left[0, \frac{1}{4}\right]$. Since $g(\mu) = \frac{\mu}{2} \in \left[0, \frac{1}{2}\right]$, it is clear that $f(A) \subseteq g(A)$.
Now we will show that $f(A)$ is complete:
As the set $f(A) = \left[0, \frac{1}{4}\right]$ is a closed and bounded subset of A. To show that $f(A)$ is complete, consider any Cauchy sequence $\{\mu_n\}$ in $f(A)$. Since $f(A) \subseteq \mathbb{R}$ and $\mathbb{R}$ is complete, $\{\mu_n\}$ converges to some limit $K \in \mathbb{R}$. Given that $f(A)$ is closed, $K \in f(A)$. Therefore, $f(A)$ is complete.
Since the required properties of Theorem 3.6 are satisfied for the maps f and g on the set A, it gives both f and g have as required fixed point in A.
Now we will show that it is common
Since $f(\mu) = \mu$ as well as $g(\mu) = \mu$

$$f(\mu) = \mu \implies \frac{\mu}{4} = \mu \implies 3\mu = 0,$$

$$g(\mu) = \mu \implies \frac{\mu}{2} = \mu \implies \mu = 0.$$

Hence, μ is as the required fixed point of f and g in A .

4. Fixed Point Theorems in-terms of Integral

Theorem 4.1. ***Theorem 4.1*** *Let us consider a complete fuzzy b-metric space $(A, \mathbb{F}, *)$, then there is a real number $b \geq 1$. Assume the compatible self mappings f and g stated on A which fulfill the given properties*

(i) $g(A) \subseteq f(A)$;
(ii) g *is continuous;*
(iii)
$$\int_0^{\mathbb{F}(f\mu, f\nu, kw)} J(u)du \ \geq \int_0^{\mathbb{F}(g\mu, g\nu, w)} J(u)du \tag{4.1}$$
and $\mu, \nu \in A, k \in (0, \frac{1}{b}), u > 0$

Then they have a unique common fixed point in A.
Where the function $J : \mathbb{R}^+ \to [0, 1]$ is defined as for each $\epsilon > 0$, $\int_0^{\epsilon} h(t)dt > 0$.

Proof. Let us choose an arbitrary point $\mu_0 \in A$. Define a sequence $\{\mu_n\}$ in A by setting $\mu_{2n+1} = g(\mu_{2n})$ and $\mu_{2n+2} = f(\mu_{2n+1})$ for all $n \geq 0$.
We will show $\{\mu_n\}$ is a sequence in A as a Cauchy. Choosing a positive number u and $n \geq 0$, consider the inequality
$$\int_0^{\mathbb{F}(f(\mu_{2n+1}), f(\mu_{2n+3}), ku)} J(u)\, du \geq \int_0^{\mathbb{F}(g(\mu_{2n+1}), g(\mu_{2n+3}), u)} J(u)\, du$$
Since $\mu_{2n+2} = f(\mu_{2n+1})$ and $\mu_{2n+4} = f(\mu_{2n+3})$, this becomes

$$\int_0^{\mathbb{F}(\mu_{2n+2}, \mu_{2n+4}, ku)} J(u)\, dr \geq \int_0^{\mathbb{F}(\mu_{2n+1}, \mu_{2n+3}, u)} J(u)\, du$$

Let $I_n(p) = \int_0^{\mathbb{F}(\mu_{2n}, \mu_{2n+2}, u)} J(u)\, du$. Then

$$I_{n+1}(u) \geq I_n\left(\frac{u}{k}\right)$$

By using induction hypothesis, we obtain

$$I_{n+m}(u) \geq I_n\left(\frac{u}{k^m}\right)$$

As $m \to \infty$ and noting that the right-hand side of above expression becomes $\int_0^1 J(u)\,du$, which is always positive.
Therefore, $I_n(p$ is bounded by a positive constant, which gives that $\mathbb{F}(\mu_{2n}, \mu_{2n+2}, u) \to 1$ as $n \to \infty$.
Since $\mathbb{F}(\mu_{2n}, \mu_{2n+2}, u) \to 1$, so there is N with $m, n \geq N$, $\mathbb{F}(\mu_m, \mu_n, u) \to 1$.
This shows that $\{\mu_n\}$ is a Cauchy sequence.
But $(A, \mathbb{F}, *)$ is complete, so we have $\mu^* \in A$ with μ_n converges to μ^* as $n \to \infty$.
But g is continuous, and $\mu_{2n+1} = g(\mu_{2n})$, taking the limit as $n \to \infty$ which gives $g(\mu^*) = \mu^*$.
From (i), we have $\nu \in A$ satisfying $g(\mu^*) = f(\nu)$. Since $g(\mu^*) = \mu^*$, we have $\mu^* = f(\nu)$.
Now by Using the commutative property of of both functions, we get $f(g(\nu)) = g(f(\nu)) = g(\nu^*) = \nu^*$.
Thus, $f(g(\nu)) = \mu^*$.
Assume there exists another point $c \in A$ such that $f(\lambda) = g(\lambda) = \lambda$.
Applying the contractive condition to μ^* and λ, we have

$$\int_0^{\mathbb{F}(f(\mu^*), f(\lambda), ku)} J(u\,du \geq \int_0^{\mathbb{F}(g(\mu^*), g(\lambda), u)} J(u)\,du \tag{4.2}$$

Since $f(\mu^*) = \mu^*$ and $g(\mu^*) = \mu^*$, this gives

$$\int_0^{\mathbb{F}(\mu^*, \lambda, ku)} J(u)\,du \geq \int_0^{\mathbb{F}(\mu^*, \lambda, u)} J(u)\,du$$

This inequality holds is if $\mathbb{F}(\mu^*, \lambda, u) = 1$ for all $u > 0$, using $\mu^* = \lambda$.

Above theorem can be verified numerically as:

Example 4.2. Let $A = [0, 1] = I$ be the unit internal and $b \geq 1$.
Define a fuzzy b-metric space $\mathbb{F} : A \times A \times [0, \infty) \to I$ by

$$\mathbb{F}(\mu, \nu, u) = \frac{u}{u + |\mu - \nu|} \tag{4.3}$$

where μ and ν are in A and $u > 0$.

This expression satisfies the definition of a fuzzy b-metric space taking $b = 2$. Define two self-maps $f, g : A \to A$ as follows

$$f(\mu) = \frac{\mu}{3}$$
$$g(\mu) = \frac{\mu}{6}$$

$g(A) \subseteq f(A)$ For any $\mu \in A$, $g(\mu) = \frac{\mu}{6}$ and $f(A) = \left[0, \frac{1}{3}\right]$. Since $\frac{\mu}{6} \leq \frac{\mu}{3}$ for all $\mu \in A$, it follows that $g(A) \subseteq f(A)$. Being the function $g(\mu) = \frac{\mu}{6}$ is continuous on A. Now we will show that that

$$\int_0^{\mathbb{F}(f\mu, f\mu, ku)} J(u)\,du \geq \int_0^{\mathbb{F}(g\mu, g\nu, u)} J(u)\,du \tag{4.4}$$

for $b \geq 1$ and $\mu, nu \in A$. Let's choose $J(u) = e^{-u}$, which satisfies $\int_0^{\epsilon} J(u)\,du > 0$ for any $\epsilon > 0$.

Now we Calculate $\mathbb{F}(f\mu, f\nu, ku)$

$$\mathbb{F}(f\mu, f\nu, ku) = \frac{ku}{ku + |f(\mu) - f(\nu)|} = \frac{ku}{ku + \left|\frac{\mu}{3} - \frac{\nu}{3}\right|} = \frac{br}{br + \frac{|\mu - \nu|}{3}}$$

Similarly, calculate $\mathbb{F}(g\mu, g\nu, u)$

$$\mathbb{F}(g\mu, g\nu, u) = \frac{u}{u + |g(\mu) - g(\nu)|} = \frac{u}{u + \left|\frac{\mu}{6} - \frac{\nu}{6}\right|} = \frac{u}{u + \frac{|\mu - \nu|}{6}}$$

To satisfy condition (iii), we need

$$\frac{ku}{ku + \frac{|\mu-\nu|}{3}} \geq \frac{u}{u + \frac{|\mu-\nu|}{6}}$$

Taking LHS

$$\frac{ku}{ku + \frac{|\mu-\nu|}{3}} = \frac{1}{1 + \frac{|\mu-\nu|}{3u}}$$

Taking RHS

$$\frac{u}{u + \frac{|\mu-\nu|}{6}} = \frac{1}{1 + \frac{|\mu-\nu|}{6u}}$$

To hold the inequality we need to show

$$\frac{1}{1 + \frac{|\mu-\nu|}{3ku}} \geq \frac{1}{1 + \frac{|\mu-\nu|}{6u}}$$

This gives

$$\frac{|\mu-\nu|}{3ku} \leq \frac{|\mu-\nu|}{6u}$$

Canceling $|\mu - \nu|$ (assuming $\mu \neq \nu$) and u (since $u > 0$

$$\frac{1}{3k} \leq \frac{1}{6}$$

Which gives

$$6 \leq 3k$$

$$k \geq 2.$$

Hence we conclude that $f(\mu) = \frac{\mu}{3}$ and $g(\mu) = \frac{\mu}{6}$ satisfy conditions (i), (ii) and (iii) of the theorem.

Theorem 4.3. *Let us consider a fuzzy b-metric space$(A, \mathbb{F}, *)$ which is complete and there exists a positive number b which is greater than or equal to 1. Let the weakly compatible mappings f and g on A satisfying:*
(a) $\int_0^{\mathbb{F}(f\mu,f\nu,ku)} J(u)du \geq \int_0^{\mathbb{F}(g\mu,g\nu,u)} J(u)du$, where $0 < k < \dfrac{1}{b}$;
(b) $f(A) \subseteq g(A)$.
If either $f(a)$ or $g(a$ is complete, then g and f have a unique common fixed point.
Where the function $h : (0, \infty) \to [0, 1]$ is defined such that for each $\epsilon > 0$, $\int_0^{\epsilon} J(u)du > 0$.

Proof. Let $\mu_0 \in A$. Since $f(A) \subseteq g(A)$, choose $\mu_1 \in A$ such that $f(\mu_0) = g(\mu_1)$. In general, choose μ_{n+1} such that $\nu_n = g(\mu_{n+1}) = f(\mu_n)$. Then,

$$\begin{aligned}\int_0^{\mathbb{F}(g\mu_n,g\mu_{n+1},u)} J(u)du &= \int_0^{\mathbb{F}(f\mu_{n-1},f\mu_n,u)} J(u)du \\ &\geq \int_0^{\mathbb{F}(g\mu_{n-1},g\mu_n,u/k)} J(u)du \\ &\geq \dots \\ &\geq \int_0^{\mathbb{F}(g\mu_0,g\mu_1,u/k^n)} J(u)du.\end{aligned}$$

Then we have, $\{g(\mu_n)\} = \{\nu_n\}$ as a Cauchy sequence, so by using the property of completeness of A, $\{\nu_n\} = \{g(\mu_n)\}$ is convergent.
And assume that it converges to the point λ, then

$$\lim_{n\to\infty} g(\mu_n) = \lim_{n\to\infty} f(\mu_n) = \lambda.$$

As $g(A)$ is complete, so there is a point $u \in A$ where $g(u) = \lambda$.
Now,

$$\int_0^{\mathbb{F}(f\mu, f\mu_n, ku)} J(u)du \geq \int_0^{\mathbb{F}(g\mu, g\mu_n, u)} J(u)du.$$

As $n \to \infty$,

$$\int_0^{\mathbb{F}(f\mu, \lambda, ku)} J(u)du \geq \int_0^{\mathbb{F}(g\mu, \lambda, u)} J(u)du,$$

and

$$\int_0^{\mathbb{F}(f\mu, \lambda, ku)} J(u)du \geq \int_0^{\mathbb{F}(\lambda, \mu, u)} J(u)du.$$

Thus,

$$\int_0^{\mathbb{F}(f\mu, \lambda, ku)} J(u)du \geq 1,$$

which implies

$$\int_0^{\mathbb{F}(f\mu, \lambda, ku)} J(u)du = 1,$$

so $f\mu = \lambda = g\mu$. By the hypothesis of compatibility of g and f we have, $gf\mu = fg\mu$, which gives, $g\lambda = f\lambda$.
Again it required to show that λ is the required fixed point of both mappings.

$$\int_0^{\mathbb{F}(f\lambda, f\mu_n, uk)} J(u)du \geq \int_0^{\mathbb{F}(g\lambda, g\mu_n, u)} J(u)du.$$

As $n \to \infty$,

$$\int_0^{\mathbb{F}(f\lambda, \lambda, ku)} J(u)du \geq \int_0^{\mathbb{M}(gc, \lambda, u)} J(u)du,$$

and

$$\int_0^{\mathbb{M}(fc, \lambda, ku)} J(u)du \geq \int_0^{(f\lambda, \lambda, u)} J(u)du,$$

which implies $f\lambda = \lambda = g\lambda$. Hence, λ is a fixed point as required of both functions.
To show it is unique, if z be the next fixed point of g and f. Which gives,

$$\int_0^{(f\lambda, fz, ku)} J(u)du \geq \int_0^{(g\lambda, gz, u)} J(u)du,$$

and

$$\int_0^{\mathbb{F}(\lambda, z, ku)} J(u)du \geq \int_0^{\mathbb{F}(\lambda, z, u)} J(u)du.$$

Thus, $z = \lambda$. Therefore, λ is the fixed point of f and g as required.
This theorem can be illustrated for the image processing as below:

Example 4.4. Let

$$A = \{I_0, I_{1/2}, I_1\} \tag{4.5}$$

be three 2×2 images with all entries equal to the same scalar.

$$I_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad I_{1/2} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} \quad and \quad I_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

If we define a fuzzy b-metric as:

$$\mathbb{F} : A \times A \times (0, \infty) \to [0, 1]$$

by

$$\mathbb{F}(\mu,\nu,u) = \exp\Big(-\frac{\|\mu-\nu\|_1}{u}\Big), \tag{4.6}$$

where $\|\mu-\nu\|_1$ is the entry-wise L^1-norm. Take $b=1$ and Choose $J(u)=1$ for $u>0$. Then for any $\epsilon>0$,

$$\int_0^{\epsilon} J(u)\,du = \epsilon > 0,$$

so the hypothesis on J is satisfied. Let $k=0.5$ where $0<k<1/b=1$.
Define the two mappings $f,g : A \to A$.

(i) f sends every image to the middle image

$$f(I_0) = f(I_{1/2}) = f(I_1) = I_{1/2}.$$

(ii) g a simple identity

$$g(\mu) = \mu \quad \text{for all } \mu \in A.$$

(a) *Weak compatibility of f and g:*
A pair (f,g) is weakly compatible if whenever $f(\mu)=g(\mu)$ then $fg(\mu)=gf(\mu)$. The only possible coincidence $f(\mu)=g(\mu)$ occurs at $\mu=I_{1/2}$, because $f(I_{1/2})=I_{1/2}=g(I_{1/2})$ for that μ,

$$fg(I_{1/2}) = f(I_{1/2}) = I_{1/2} = g(I_{1/2}) = gf(I_{1/2}),$$

so f and g are weakly compatible.

(b) *Integral inequality:*
With $J\equiv 1$, each integral $\int_0^{\mathbb{F}(\mu,\nu,u)} J(u)\,du$ equals the upper limit value. So the condition

$$\int_0^{\mathbb{F}(f\mu,f\nu,ku)} J(u)\,du \geq \int_0^{\mathbb{F}(g\mu,g\nu,u)} J(u)\,du \tag{4.7}$$

reduces to

$$\mathbb{F}(f\mu,f\nu,ku) \;\geq\; \mathbb{F}(g\mu,g\nu,u) \tag{4.8}$$

for all $\mu,\nu\in A$ and $u>0$. Because f sends every element to the same image $I_{1/2}$,

$$\mathbb{F}(f\mu,f\nu,ku) = \mathbb{F}(I_{1/2},I_{1/2},ku) = \exp(0) = 1.$$

Where $\mathbb{F}(g\mu,g\nu,u)=\mathbb{F}(\mu,\nu,u)\leq 1$. Hence the inequality holds.
Since $f(A)=\{I_{1/2}\}$ and $g(A)=A$, so $f(A)\subseteq g(A)$.
The set A is finite, so every Cauchy sequence stabilizes and A is complete.

Since,

$$f(I_{1/2}) = I_{1/2}, \qquad g(I_{1/2}) = I_{1/2}.$$

So $I_{1/2}$ is a common fixed point.
For uniqueness: Suppose $z\in A$ is another common fixed point, so $f(z)=z$ and $g(z)=z$.
Because f maps every element to $I_{1/2}$, we must have $z=I_{1/2}$.
Hence the common fixed point is unique.
As I_1 and I_0 model two extreme corrupted inputs (bright or dark). The denoiser f collapses any corrupt image to the same image $I_{1/2}$. The deblurring map g is identity.
The example guarantees that there is a unique image $I_{1/2}$ that is both denoised and deblurred, an analogue of a unique best restored image and iteratively applying these operators will lead to that unique image.
Numerically, if we take

$$\|I_1 - I_0\|_1 = 4.$$

Then for $u = 1$,

$$\mathbb{F}(I_1, I_0, 1) = \exp(-4/1) = e^{-4} \approx 0.0183.$$

Where,

$$\mathbb{F}(fI_1, fI_0, 0.5 \cdot 1) = \mathbb{F}(I_{1/2}, I_{1/2}, 0.5) = 1 \geq e^{-4},$$

confirming the integral-inequality condition numerically in this pair.

5. Conclusion

By using the notion of fuzzy b-metric space and applying its terminologies in terms of contraction mappings and compatible mappings, we proved the Banach contraction theorem. Stating some of the lemmas and corollaries as the previous results we proved unique common fixed point theorems in fuzzy b-metric space. Also theorem 4.1 and theorem 4.2 are introduced and proved in terms of the integral form as an application. Examples 3.7, 4.2 and 4.4 verified the related results numerically. We conclude that the integral equations in fuzzy b-metric space can be used to compare the area of irregular objects, symptoms of the different diseases, prices of the different commodities, image processing for similar objectives and many more of same kinds, using the same parameter. Using these results, further works can be done in fuzzy integral equations as well as fuzzy differential equations.

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