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# Application of Second Zagreb Index on Bipolar Fuzzy Graphs

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**Abstract**

The Zagreb indices are important topological tools in mathematical chemistry and network analysis. This article presents a novel extension of the second Zagreb index to the domain of bipolar fuzzy graphs, a powerful structure for modeling systems with opposing or ambivalent interactions. We formally define the bipolar second Zagreb index and explore its theoretical foundations by calculating its values and bounds for standard bipolar fuzzy graphs like paths, cycles, and stars. We get the bounds for this index under complex graph operations-Cartesian product, composition, join, and union-which are essential for constructing sophisticated network models. This work not only enriches fuzzy graph theory but also provides a foundational tool for quantitatively analyzing bipolarity social. Finally, we present on Application of second Zagreb index to find the most suitable place to build a hospital.

*Keywords:* Fuzzy Graph, Bipolar Fuzzy Graph, Second Zagreb Index, Topological Index.
**2010 Mathematics Subject Classification:** 05C07, 05C09, 03E72,

## 1. Introduction

### 1.1. *Research background*

In 1965, fuzzy set was first introduced by Zadeh in [15] as a mathematical concept to represent uncertainty in the real world. Rosenfeld in 1975 introduced the concept of fuzzy graph which is a generalization of crisp graph [4], and due to various applications of fuzzy graph theory (FGT), nowadays a large number of researchers are working on topological indices. Many fuzzy analogues of graph theory concepts such as paths, circles, and connections have been explored by them. There are many problems that can be solved with the help of fuzzy graphs. Although it is very recent, it has been growing rapidly and has numerous applications in different fields.Investigated the bipolar fuzzy graph [16], generalized multipolar $FG$ [37], multipolar $FG$ [17], soft-fuzzy bipolar graph [33] and cyclic $CI$ [20] of an $FG$. Borzooei and Rashmanlou studied domination in vague graphs [10] and product of vague graph [25]. Single-valued neutrosophic graphs [9, 17, 3], Zhang in 1994 [41, 13], introduced the concept of bipolar fuzzy sets as a generalization of fuzzy

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sets. Bipolar fuzzy sets are an extension of fuzzy sets whose membership degree ranges from [-1, 1]. In a bipolar fuzzy set, a membership score of 0 for an element means that the element is irrelevant to the corresponding property, a membership score of (0, 1] for an element means that the element satisfies the property to some extent, and a membership score of [-1, 0) for an element means that the element satisfies the implied anti-property to some extent. Although bipolar fuzzy sets and intuitionistic fuzzy sets look similar to each other, they are fundamentally different sets [9]. In many fields, it is important to be able to handle bipolar information. Bipolar fuzzy graphs offer a significant advantage over ordinary fuzzy graphs by allowing simultaneous representation of positive and negative information within a single framework. While traditional fuzzy graphs assign only a degree of membership to vertices and edges, bipolar fuzzy graphs incorporate both a positive membership function- capturing supportive, favorable, or desirable relationships-and a negative membership function-representing conflicting, adverse, or undesirable relationships. This dual-valued structure enables more nuanced modeling of real-world systems where interactions are not purely positive but may include opposition, hesitation, or contradictory influences. As a result, bipolar fuzzy graphs provide a more expressive and realistic tool for analyzing complex networks such as decision-making systems, social opinion dynamics, and sentiment-laden interactions than ordinary fuzzy graphs.Topological indices are numerical quantities of a graph that describe its topology. The second $ZI$ for molecular graphs was studied by Kazemi [26], and Khalifa et al studied the second ZI for some graph operations [22]. Although Zagreb indices are classical tools in topological graph theory, their relevance to bipolar fuzzy graphs arises from their ability to quantify structural properties in networks that simultaneously carry positive and negative information. In bipolar fuzzy graphs, vertices and edges possess dual membership values, reflecting supportive and conflicting interactions within the same system. Zagreb indices, when adapted to this bipolar setting, aggregate these dual-valued degrees in a way that captures the overall connectivity, intensity, and balance of positive-negative relationships more effectively than simple degree-based measures. By incorporating both the magnitude and distribution of bipolar degrees, these indices provide deeper insight into how strongly a network is organized, how influence propagates, and where structural irregularities occur. Compared with other topological indices that may focus only on local measures or ignore polarity, bipolar Zagreb indices offer a more comprehensive structural descriptor, making them particularly advantageous for analyzing complex systems with intertwined positive and negative attributes. In this article, the second Zagreb index for bipolar fuzzy graphs is introduced and some graph operations, which is a more general explanation of the second Zagreb index for crisp graphs, and is widely used in spectral fuzzy graph theory, fuzzy graph theory, and many fields of fuzzy mathematics.

### *1.2. Motivation*

In crisp graphs, many results and applications of topological indices are available. However, many situations cannot be modeled using crisp graphs. In such cases, to deal with such a situation, it is necessary to introduce these topological indices to define a fuzzy graph. In 2019, Binu et al. Wiener index [21], introduced connectivity index [18], connectivity status [19] and cyclic connectivity index [20] for $FG$. Motivated by these articles, Islam and Pal recently studied the Wiener index [36] for $FG$ and introduced hype - Wiener index [35], first Zagreb index [32] hype - connectivity index [34], and $F$ - index [31, 33] for FGs. Studied the RSM index in fuzzy graph. They studied a specific index [26], the absolute Wiener index [29] and the Randek index [30] for bipolar fuzzy graphs. In [27] Kalathian et al. also presented several topological indices for fuzzy graphs. Based on those papers, in this paper we introduce second Zagreb index on bipolar fuzzy graphs, and Boundaries of the Second Zagreb Indices for Fuzzy Bipolar Graphs During Operations such as the Cartesian product $(\mathrm{G}_1 \times \mathrm{G}_2)$, the composition $(\mathrm{G}_1 [\mathrm{G}_2])$, the join $(\mathrm{G}_1 + \mathrm{G}_2)$, and union $(\mathrm{G}_1 \cup \mathrm{G}_2)$, of two bipolar fuzzy graphs.

### *1.3. Significance and objective of the article*

Topological indices play a vital role in wide and profound applications in everyday life and modern technology , spectral graph, molecular chemistry, network theory, $BFG$ theory, etc. Zagreb Index are degree-dependent topological descriptions, applies to the role of hospitals in improving public health by providing comprehensive healthcare services to individuals within communities.Previous work on fuzzy graphs has employed classical indices such as the Wiener index, the Randić index, and the first Zagreb index to capture distance-based properties, branching behavior, and degree-related structure. However, these measures were developed for set tings where relationships carry only a single (positive) membership value, limiting their ability to fully describe systems with dual polarity. Introducing the second Zagreb index for bipolar fuzzy graphs addresses this gap by incorporating both positive and negative degrees into a unified structural descriptor. Unlike the first Zagreb index, which focuses primarily on the squared degrees of individual vertices, the second Zagreb index emphasizes interactions between adjacent vertices, enabling a deeper examination of how positive and negative influences are distributed across connected pairs. This provides new analytical capabilities for understanding balance, conflict, and cohesion in networks where relationships may reinforce or oppose one another. Consequently, the second Zagreb index enhances structural analysis beyond what earlier fuzzy-graph indices can offer, making it a valuable tool for studying complex bipolar systems.

The second $ZI$ of $BFG$ is studied here. The main results are as follows:

(i) In Part 2, some basic notations and definitions are given. In Part 3, some theoretical developments of this indicator are investigated here, i.e., the limits of this indicator are calculated for several BFG: path, cycle, star, complete BFG, PBFSG, etc. For symmetric BFG, it is shown that the value of this indicator is the same. In Part 4, the bounds of this index for some bipolar fuzzy graph operations have been studied.

(ii) In the part 5, the application of the second Zagreb Index to select the most suitable site for building a hospital was studied. Four cities from Mazandaran province were selected to determine the most important city for building a hospital based on population density and the area of land suitable for building a hospital.

## 2. Preliminaries

Some essential definitions and theorems which are needed to develop our results are given, most of them one can be found in [32, 31].

For a universal set $X$, a fuzzy set is a pair $S = (X, \Omega)$ where $\Omega$ is called membership function of $S$ whose domain is $X$ and co-domain is [0,1].

Table 1: The list of abbreviation.

| Abbreviation | Meaning |
|---|---|
| $FS$ | Fuzzy set |
| FSS | Fuzzy subset |
| FG | Fuzzy graph |
| FSG | Fuzzy subgraph |
| PFSG | Partial fuzzy subgraph |
| PBFSG | partial Bipolar fuzzy subgraph |
| BFG | Bipolar fuzzy graph |
| BFSG | Bipolar fuzzy subgraph |
| CBFG | complete Bipolar fuzzy graph |
| TI | Topological index |
| ZI | Zagreb index |
| MV | Membership value |

**Definition 2.1.** An $FG$ is a triplet, $G=(X,\beta,\Omega)$, where X is called vertex set of $G$ with vertex membership function $\beta: X \rightarrow[0,1]$ and membership function of edges $\Omega: X \times X \rightarrow[0,1]$ satisfying $\beta(u, v) \leq \min \{\beta(u), \beta(v)\}$.The edge set is defined as $E=\{(u, v): \Omega(u, v)>0\}$.

**Definition 2.2.** Let $X$ be a nonempty set. Then we call a mapping $\beta=\left(\mu_\beta^P, \mu_\beta^N\right): X \rightarrow[-1,0] \times[0,1]$a bipolar fuzzy set on $X$ such that $\mu_\beta^P(u, v) \in[0,1]$ and $\mu_\beta^N(u, v) \in[-1,0]$.

**Definition 2.3.** A bipolar fuzzy graph $G$ is triplet $G=(X, \beta, \Omega)$ such that $X$ is a non-empty set, $\beta$ is a bipolar fuzzy set of $X$ and $\Omega$ is a bipolar fuzzy set of $X \times X$ satisfying $\mu_{\Omega}^P(u, v) \leq \mu_\beta^P(u) \wedge \mu_\beta^P(v)$ and $\mu_{\Omega}^N(u, v) \geq \mu_\beta^N(u) \vee \mu_\beta^N(v)$.

**Definition 2.4.** Let $G=(X, \beta, \Omega)$ be a $BFG$, For a vertex $u \in X$,the degree of $u$ is defined as

$$d(u)=\left(d^P(u), d^N(u)\right)=\left(\sum_{uv \in \mathrm{X}} \mu_{\Omega}^P(uv), \sum_{uv \in \mathrm{X}} \mu_{\Omega}^N(uv)\right).$$

Also, we define
$\Delta^P=\vee\left\{d^P(u) \mid u \in X\right\}, \delta^P=\wedge\left\{d^P(u) \mid u \in X\right\}, \Delta^N=\wedge\left\{d^N(u) \mid u \in X\right\}, \delta^N=\vee\left\{d^N(u) \mid u \in X\right\}$. Throughout this article, the $BFG, G_1=\left(X_1, \beta_1, \Omega_1\right)$ as $n_1-$ vertices, $m_1-edges$, edge set $E_1$ and $G_2=\left(X_2, \beta_2, \Omega_2\right)$ has $n_2-$ vertices, $m_2-$ edges, edge set $E_2$ and $\Delta_1^P=\vee\left\{d^P(u) \mid u \in X_1\right\}, \delta_1^P=\wedge\left\{d^P(u) \mid u \in X_1\right\}$, $\Delta_2^P=\vee\left\{d^P(u) \mid u \in X_2\right\}, \delta_2^P=\wedge\left\{d^P(u) \mid u \in X_2\right\}$ , $\Delta_1^N=$ $\wedge\left\{d^N(u) \mid u \in X_1\right\}, \delta_1^N=\vee\left\{d^N(u) \mid u \in X_1\right\}$, $\Delta_2^N=\wedge\left\{d^N(u) \mid u \in X_2\right\}, \delta_2^N=\vee\left\{d^N(u) \mid u \in X_2\right\}$.

Now, Cartesian product, composition, join and union of two $BFGs$ are defined below.

**Definition 2.5.** Two BFG $G_1=\left(X_1, \beta_1, \Omega_1\right)$ and BFG $G_2=\left(X_2, \beta_2, \Omega_2\right)$ are called isomorphic if there exist a bijective map $\alpha: X_1 \rightarrow X_2$, which satisfies the following conditions:

(i) $\mu_{\beta_1}^P\left(u_1\right)=\mu_{\beta_2}^P\left(\alpha\left(u_1\right)\right), \mu_{\beta_1}^N\left(u_1\right)=\mu_{\beta_2}^N\left(\alpha\left(u_1\right)\right)$

(ii) $\mu_{\Omega_1}^P,\left(u_1, v_1\right)=\mu_{\Omega_2}^P\left(\alpha\left(u_1\right), \alpha\left(v_1\right)\right), \mu_{\Omega_1}^N\left(u_1, v_1\right)=\mu_{\Omega_2}^N\left(\alpha\left(u_1\right), \alpha\left(v_1\right)\right)$

For all $u_1, v_1 \in X_1, u_1 v_1 \in \Omega_1$.

**Definition 2.6.** Let $\mathrm{G}_1=\left(X_1, \beta_1, \Omega_1\right), \mathrm{G}_2=\left(X_2, \beta_2, \Omega_2\right)$ be two $BFG$. Then the Cartesian product of $\mathrm{G}_1$ and $\mathrm{G}_2$ is an $BFG \mathrm{G}_1 \times \mathrm{G}_2=(X, \beta, \Omega)$, where $X=X_1 \times X_2, \mu_\beta^P(u, v)=\wedge\left\{\mu_{\beta_1}^P(u), \mu_{\beta_2}^P(v)\right\}, \mu_\beta^N(u, v)=$ $\vee\left\{\mu_{\beta_1}^N(u), \mu_{\beta_2}^N(v)\right\}$, for all $(u, v) \in X$ and

$$\mu_{\Omega}^P\left(\left(u_1, v_1\right),\left(u_2, v_2\right)\right)=\begin{cases}\wedge\left\{\mu_{\beta_1}^P\left(u_1\right), \mu_{\Omega_2}^P\left(v_1, v_2\right)\right\} & \text{if } u_1=u_2, v_1 v_2 \in E_2 \\ \wedge\left\{\mu_{\beta_2}^P\left(v_1\right), \mu_{\Omega_1}^P\left(u_1, u_2\right)\right\} & \text{if } u_1 u_2 \in E_1, v_1=v_2 \\ 0 \text{ otherwise.} & \end{cases}$$

$$\mu_{\Omega}^N\left(\left(u_1, v_1\right),\left(u_2, v_2\right)\right)=\begin{cases}\vee\left\{\mu_{\beta_1}^N\left(u_1\right), \mu_{\Omega_2}^N\left(v_1, v_2\right)\right\} & \text{if } u_1=u_2, v_1 v_2 \in E_2 \\ \vee\left\{\mu_{\beta_2}^N\left(v_1\right), \mu_{\Omega_1}^N\left(u_1, u_2\right)\right\} & \text{if } u_1 u_2 \in E_1, v_1=v_2 \\ 0 \text{ otherwise.} & \end{cases}$$

**Definition 2.7.** Let $\mathrm{G}_1=\left(X_1, \beta_1, \Omega_1\right), \mathrm{G}_2=\left(X_2, \beta_2, \Omega_2\right)$ be two $BFG$. Then the composition of $G_1$ and $G_2$ is an $BFG G_1\left[\mathrm{G}_2\right]=(X, \beta, \Omega)$ where $X=X_1 \times X_2, \mu_\beta^P(u, v)=\wedge\left\{\mu_{\beta_1}^P(u), \mu_{\beta_2}^P(v)\right\}, \mu_\beta^N(u, v)=$ $\vee\left\{\mu_{\beta_1}^N(u), \mu_{\beta_2}^N(v)\right\}$, all for $(u, v) \in X$ and

$$\mu_{\Omega}^P\left(\left(u_1, v_1\right),\left(u_2, v_2\right)\right)=\begin{cases}\wedge\left\{\mu_{\beta_1}^P\left(u_1\right), \mu_{\Omega_2}^P\left(v_1, v_2\right)\right\} & \text{if } u_1=u_2, v_1 v_2 \in E_2 \\ \wedge\left\{\mu_{\beta_2}^P\left(v_1\right), \mu_{\Omega_1}^P\left(u_1, u_2\right)\right\} & \text{if } u_1 u_2 \in E_1 \\ 0 \text{ otherwise .} & \end{cases}$$

$$\mu_{\Omega}^{N}\left(\left(u_1,v_1\right),\left(u_2,v_2\right)\right)=\begin{cases}\vee\left\{\mu_{\beta_1}^{N}\left(u_1\right),\mu_{\Omega_2}^{N}\left(v_1,v_2\right)\right\} \text{ if } u_1=u_2, v_1v_2\in E_2\\ \vee\left\{\mu_{\beta_2}^{N}\left(v_1\right),\mu_{\Omega_1}^{N}\left(u_1,u_2\right)\right\} \text{ if } u_1u_2\in E_1\\ 0 \text{ otherwise.}\end{cases}$$

**Definition 2.8.** Let $\mathrm{G}_1=(X_1,\beta_1,\Omega_1),\mathrm{G}_2=(X_2,\beta_2,\Omega_2)$ be two $BFG$ such that $X_1\cap X_2=\emptyset$. The Join of $\mathrm{G}_1$ and $\mathrm{G}_2$ the $BFG$ is $\mathrm{G}_1+\mathrm{G}_2=(X,\beta,\Omega)$, where $X=X_1\cup X_2$,

$$\left(\mu_{\beta}^{P}(u),\mu_{\beta}^{N}(u)\right)=\begin{cases}\left(\mu_{\beta_1}^{P}(u),\mu_{\beta_1}^{N}(u)\right), \text{ if } u\in X_1\\ \left(\mu_{\beta_2}^{P}(u),\mu_{\beta_2}^{N}(u)\right), \text{ if } u\in X_2\end{cases}$$

$$\mu_{\Omega}^{P}(u,v)=\begin{cases}\wedge\left\{\mu_{\beta_1}^{P}(u),\mu_{\beta_2}^{P}(v)\right\} \text{ if } u\in X_1 \text{ and } v\in X_2\\ \mu_{\Omega_1}^{P}(u,v) \text{ if } uv\in E_1\\ \mu_{\Omega_2}^{P}(u,v) \text{ if } uv\in E_2\\ 0 \text{ otherwise.}\end{cases},$$

$$\mu_{\Omega}^{N}(u,v)=\begin{cases}\vee\left\{\mu_{\beta_1}^{N}(u),\mu_{\beta_2}^{N}(v)\right\} \text{ if } u\in X_1 \text{ and } v\in X_2\\ \mu_{\Omega_1}^{N}(u,v) \text{ if } uv\in E_1\\ \mu_{\Omega_2}^{N}(u,v) \text{ if } uv\in E_2\\ 0 \text{ otherwise.}\end{cases}$$

**Definition 2.9.** Let $\mathrm{G}_1=(X_1,\beta_1,\Omega_1)$ and $\mathrm{G}_2=(X_2,\beta_2,\Omega_2)$ be two $BFGs$.The union of $\mathrm{G}_1$ and $\mathrm{G}_2$ is $\mathrm{G}_1\cup\mathrm{G}_2=(X,\beta,\Omega)$, where $X=X_1\cup X_2$,

$$\mu_{\beta}^{P}(u)=\begin{cases}\mu_{\beta_1}^{P}(u), \text{ if } u\in X_1\backslash X_2\\ \mu_{\beta_2}^{P}(u), \text{ if } u\in X_2\backslash X_1\\ \vee\left\{\mu_{\beta_1}^{P}(u),\mu_{\beta_2}^{P}(u)\right\}, \text{ if } u\in X_1\cap X_2\end{cases}$$

$$\mu_{\beta}^{N}(u)=\begin{cases}\mu_{\beta_1}^{N}(u), \text{ if } u\in X_1\backslash X_2\\ \mu_{\beta_2}^{N}(u), \text{ if } u\in X_2\backslash X_1\\ \wedge\left\{\mu_{\beta_1}^{N}(u),\mu_{\beta_2}^{N}(u)\right\}, \text{ if } u\in X_1\cap X_2\end{cases}$$

And

$$\mu_{\Omega}^{P}(u,v)=\begin{cases}\mu_{\Omega_1}^{P}(u,v) \text{ if } uv\in E_1\backslash E_2\\ \mu_{\Omega_2}^{P}(u,v) \text{ if } uv\in E_2\backslash E_1\\ \vee\left\{\mu_{\beta_1}^{P}(u,v),\mu_{\beta_2}^{P}(u,v)\right\} \text{ if } (u,v)\in E_1\cap E_2\\ 0 \text{ otherwise.}\end{cases}$$

$$\mu_{\Omega}^{N}(u,v)=\begin{cases}\mu_{\Omega_1}^{N}(u,v) \text{ if } uv\in E_1\backslash E_2\\ \mu_{\Omega_2}^{N}(u,v) \text{ if } uv\in E_2\backslash E_1\\ \wedge\left\{\mu_{\beta_1}^{N}(u,v),\mu_{\beta_2}^{N}(u,v)\right\}, \text{ if } (u,v)\in E_1\cap E_2\\ 0 \text{ otherwise.}\end{cases}$$

## 3. Second Zagreb index of Bipolar fuzzy graphs

Topological indices have an important character in chemical graph, spectral graph theory, network theory, molecular chemistry, BFG theory, etc. In 1972, Gutman and Trinajstic [12] introduced the second $ZI$ of a crisp graph as:

**Definition 3.1.** [12] Suppose $G = (X, E)$ is a crisp graph. The second $ZI$ of $G$ is defined by:

$$ZI_2(G) = \sum_{uv \in E} d(u)d(v)$$

Recently, Kalathian et al. [27] introduced second $ZI$ for a $FG$.

**Definition 3.2.** [27] Suppose $G = (X, \beta, \Omega)$ be an $FG$. The second $ZI$ of $G$ is defined by:

$$M^*(G) = \sum_{uv \in \mathrm{E}} \mu_\beta(u)d(u)\mu_\beta(v)d(v).$$

**Definition 3.3.** Suppose $G = (X, \beta, \Omega)$ is $BFG$.The second $ZI$ of of the $BFGG$ is defined by:

$$M_2(G) = \left(M_2^P(G), M_2^N(G)\right) = \left(\sum_{uv \in \mathrm{E}} \mu_\beta^P(u)d^P(u)\mu_\beta^P(v)d^P(v), \sum_{uv \in \mathrm{E}} \mu_\beta^N(u)d^N(u)\mu_\beta^N(v)d^N(v)\right).$$

**Example 3.4.** Suppose $G$ is a crisp graph, fuzzy graph and bipolar fuzzy graph shown in Figure 1 with vertex set $X = \{u_1, u_2, u_3\}$. We find $M_2(G), M_2^*(G), M^*(G)$

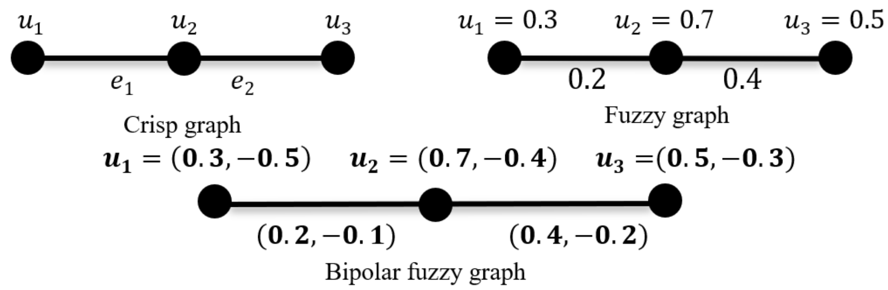

Figure 1: Fuzzy graph and bipolar fuzzy graph $G = P$

$$M_2(G) = \sum_{uv \in E} d(u)d(v) = [1 \times 2] + [2 \times 1] = 4$$

$$M^*(G) = \sum_{uv \in \mathrm{E}} \mu_\beta(u)d(u)\mu_\beta(v)d(v) = [0.3 \times 0.2 \times 0.7 \times 0.6] + [0.7 \times 0.6 \times 0.5 \times 0.4] = 0.1092$$

$$M_2(G) = \left(M_2^P(G), M_2^N(G)\right) = \left(\sum_{uv \in \mathrm{E}} \mu_\beta^P(u)d^P(u)\mu_\beta^P(v)d^P(v), \sum_{uv \in \mathrm{E}} \mu_\beta^N(u)d^N(u)\mu_\beta^N(v)d^N(v)\right)$$

$$M_2^P(G) = [0.3 \times 0.2 \times 0.7 \times 0.6] + [0.7 \times 0.6 \times 0.5 \times 0.4] = 0.1092$$

$$M_2^N(G) = [-0.5 \times -0.1 \times -0.4 \times -0.3] + [-0.4 \times -0.3 \times -0.3 \times -0.2] = 0.0132.$$

Table 2: Values second ZI for three types of graphs.

| Type of Graphs | crisp graph | Fuzzy graph | Bipolar fuzzy graph |
|---|---|---|---|
| The second $ZI$ | 4 | 0.1092 | (0.1092,0.0132) |

In the next theorem, an upper bound of this index is provided.

**Theorem 3.5.** *Suppose $G = (X, \beta, \Omega)$ is a BFG having $n$ vertices and $m$ edges. Then*

$$M_2^P(G) \leq m(n-1)^2, M_2^N(G) \leq m(n-1)^2$$

*Proof.* As $0 \leq \mu_\beta^P(v) \leq 1$ and $-1 \leq \mu_\beta^N(v) \leq 0$, then

$$M_2^P(G) = \sum_{uv \in \mathrm{E}} \mu_\beta^P(u) d^P(u) \mu_\beta^P(v) d^P(v) \leq \sum_{uv \in E} \left(\Delta^P\right)^2 = m\left(\Delta^P\right)^2 \leq m(n-1)^2,$$
$$M_2^N(G) = \sum_{uv \in \mathrm{E}} \mu_\beta^N(u) d^N(u) \mu_\beta^N(v) d^N(v) \leq \sum_{uv \in \mathrm{E}} d^N(u) d^N(v) \leq \sum_{uv \in E} \left(\Delta^N\right)^2 = m\left(\Delta^N\right)^2 \leq m(n-1)^2.$$

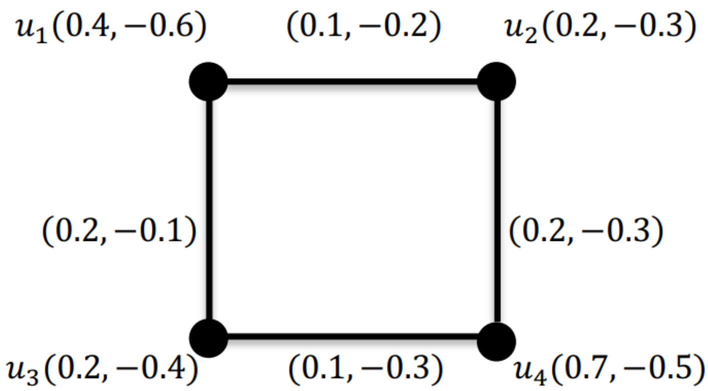


Figure 2: $ABFG$ with $\left(M_2^P(G), M_2^N(G)\right) = (0.0396, 0.1488)$.

□

**Example 3.6.** Suppose $G = (X, \beta, \Omega)$ is a $BFG$ shown in Figure 2 with vertex set $X = \{u_1, u_2, u_3, u_4\}$, and $E = \{u_1u_2, u_1u_3, u_2u_4, u_3u_4\}$ such that

| $X$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ |
|---|---|---|---|---|
| $\mu_\beta^P$ | 0.4 | 0.2 | 0.2 | 0.7 |
| $\mu_\beta^N$ | -0.6 | -0.3 | -0.4 | -0.5 |

| $E$ | $u_1u_2$ | $u_1u_3$ | $u_2u_4$ | $u_3u_4$ |
|---|---|---|---|---|
| $\mu_{\Omega_1}^P$ | 0.1 | 0.2 | 0.2 | 0.1 |
| $\mu_{\Omega_1}^N$ | -0.2 | -0.1 | -0.3 | -0.3 |

The degree of the vertices $X$

| $X$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ |
|---|---|---|---|---|
| $d^P$ | 0.3 | 0.3 | 0.3 | 0.3 |
| $d^N$ | -0.3 | -0.5 | -0.4 | -0.6 |

Then

$$M_2(G) = \left(M_2^P(G), M_2^N(G)\right) = \left(\sum_{uv \in \mathrm{E}} \mu_\beta^P(u) d^P(u) \mu_\beta^P(v) d^P(v), \sum_{uv \in \mathrm{E}} \mu_\beta^N(u) d^N(u) \mu_\beta^N(v) d^N(v)\right)$$
$$\begin{aligned} M_2^P(G) &= [0.4 \times 0.3 \times 0.2 \times 0.3] + [0.4 \times 0.3 \times 0.2 \times 0.3] \\ &+ [0.2 \times 0.3 \times 0.7 \times 0.3] + [0.2 \times 0.3 \times 0.7 \times 0.3] = 0.0396 \\ M_2^N &= [-0.6 \times -0.3 \times -0.3 \times -0.5] + [-0.6 \times -0.3 \times -0.4 \times -0.4] \\ &+ [-0.3 \times -0.5 \times -0.5 \times -0.6] + [-0.4 \times -0.4 \times -0.5 \times -0.6] = 0.1488 \end{aligned}$$

Example 3.6 shows Theorem 3.5 is satisfied for the $BFGG$ in Figure 2. Also, the value of this index for an $BFSG$ is less than the original $BFG$, as shown in the example below.

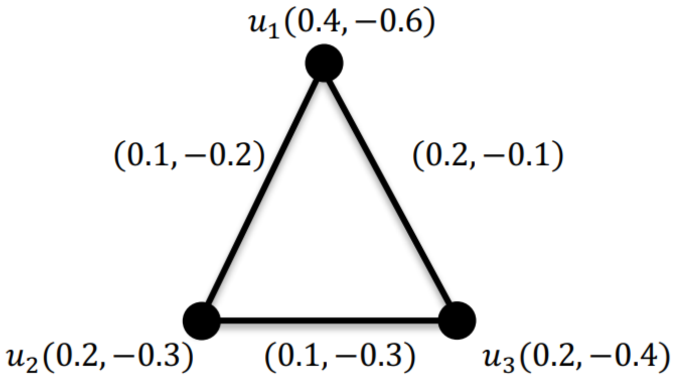


Figure 3: A BFG with $M_2(G') = (M_2^P(G'), M_2^N(G')) = (0.0168, 0.0798)$.

**Example 3.7.** Suppose, $G$ is a $BFG$ and $G'$ be its BFSG (Figure 3) gain by removing $u_4$.Then the degree of the vertices $X_{-u_4}$

| $X_{-u_4}$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|
| $d^P$ | 0.3 | 0.2 | 0.3 |
| $d^N$ | -0.3 | -0.5 | -0.4 |

$$M_2(G') = (M_2^P(G'), M_2^N(G')) = \left( \sum_{uv \in \mathrm{E}} \mu_\beta^P(u) d^P(u) \mu_\beta^P(v) d^P(v), \sum_{uv \in \mathrm{E}} \mu_\beta^N(u) d^N(u) \mu_\beta^N(v) d^N(v) \right)$$

$$M_2^P(G') = [0.4 \times 0.3 \times 0.2 \times 0.3] + [0.4 \times 0.3 \times 0.2 \times 0.3] + [0.2 \times 0.2 \times 0.2 \times 0.3] = 0.0168$$

$$M_2^N(G') = [-0.6 \times -0.3 \times -0.3 \times -0.5] + [-0.6 \times -0.3 \times -0.4 \times -0.4] + [-0.3 \times -0.5 \times -0.4 \times -0.4] = 0.0798.$$

Note that, from Examples 3.6 and 3.7, we get $M_2(G') \leq M_2(G)$. In the next proposition we proved the fact in general.

**Proposition 3.8.** *Let $G' = (X', \beta', \Omega')$ be a PBFSG of a BFG $G = (X, \beta, \Omega)$. Then*

$$M_2^P(G') \leq M_2^P(G), M_2^N(G') \leq M_2^N(G).$$

*Proof.* Since $G'$ is a PBFSG of $G$,then for any $u, v \in X', \mu_{\beta'}^P(u) \leq \mu_\beta^P(u), \mu_{\Omega'}^P(u,v) \leq \mu_\Omega^P(u,v)$ and $\mu_{\beta'}^N(u) \geq \mu_\beta^N(u), \mu_{\Omega'}^N(u,v) \geq \mu_\Omega^N(u,v)$. So,

$$\begin{aligned}
d_{G'}^P(u) &= \sum_{v \in X'} \mu_{\Omega'}^P(u,v) \leq \sum_{v \in X'} \mu_{\Omega'}^P(u,v) \leq \sum_{v \in X} \mu_\Omega^P(u,v) = d_G^P(u), \\
d_{G'}^N(u) &= \sum_{v \in X'} \mu_{\Omega'}^N(u,v) \geq \sum_{v \in X'} \mu_\Omega^N(u,v) \geq \sum_{v \in X} \mu_\Omega^N(u,v) = d_G^N(u). \quad \text{Therefore ,} \\
M_2^P(G') &= \sum_{uv \in E'} \mu_\beta^P(u) d_{G'}^P(u) \mu_\beta^P(v) d_{G'}^P(v) \leq \sum_{uv \in \mathrm{E}} \mu_\beta^P(u) d_G^P(u) \mu_\beta^P(v) d_G^P(v) = M_2^P(G), \\
M_2^N(G') &= \sum_{uv \in E'} \mu_\beta^N(u) d_{G'}^N(u) \mu_\beta^N(v) d_{G'}^N(v) \leq \sum_{uv \in E'} \mu_\beta^N(u) d_{G'}^N(u) \mu_\beta^N(v) d_{G'}^N(v) \\
&\leq \sum_{uv \in E} \mu_\beta^N(u) d_G^N(u) \mu_\beta^N(v) d_G^N(v) = M_2^N(G).
\end{aligned}$$

Let $G = (X, \beta, \Omega)$ be a $BFG$. For any $(r, t) \in [-1, 0] \times [0, 1]$, we define $BFG$ $G_{(r,t)} = (X', \beta', \Omega')$ such that, $X' = \left\{ u \in X \mid \mu_\beta^P(u) \leq t, \mu_\beta^N(u) \geq r \right\}$ and $\mu_{\beta'}^P(u) = \mu_\beta^P(u), \mu_{\beta'}^N(u) = \mu_\beta^N(u), \mu_{\Omega'}^P(uv) = \mu_\Omega^P(uv), \mu_{\Omega'}^N(uv) = \mu_\Omega^N(uv)$,for al $u, v \in X'$. □

**Corollary 3.9.** *Suppose $G=(X,\beta,\Omega)$ is a BFG and let $0\leq r_1\leq r_2\leq 1$ and $-1\leq r_1'\leq r_2'\leq 0$ Then*

$$M_2^P\left(G_{r_n}\right)\leq M_2^P\left(G_{r_{n-1}}\right)\leq\cdots\leq M_2^P\left(G_{r_2}\right)\leq M_2^P\left(G_{r_1}\right)$$
$$M_2^N\left(G_{r_n'}\right)\leq M_2^N\left(G_{r_{n-1}'}\right)\leq\cdots\leq M_2^N\left(G_{r_2'}\right)\leq M_2^N\left(G_{r_1'}\right)$$

*In the next theorem, second ZI of a bipolar path is discussed.*

**Theorem 3.10.** *Suppose $P=(X,\beta,\Omega)$ is a bipolar path having $n$ vertices. Then*

$$M_2^P(P)\leq 4(n-2), M_2^N(P)\leq 4(n-2).$$

*Proof.* Suppose $P\left(v_1,v_2,\cdots,v_n\right)$ is an $n$-vertex bipolar path. Then $d\left(v_1\right)=\left(d^P\left(v_1\right),d^N\left(v_1\right)\right)=\left(\mu_\Omega^P\left(v_1v_2\right),\right.$ $\left.\mu_\Omega^N\left(v_1v_2\right)\right),d\left(v_n\right)=\left(d^P\left(v_n\right),d^N\left(v_n\right)\right)=\left(\mu_\Omega^P\left(v_{n-1}v_n\right),\mu_\Omega^N\left(v_{n-1}v_n\right)\right)$ and $d\left(v_i\right)=\left(d^P\left(v_i\right),d^N\left(v_i\right)\right)=$ $\left(\mu_\Omega^P\left(v_{i-1}v_i\right),\mu_\Omega^N\left(v_{i-1}v_i\right)\right)+\left(\mu_\Omega^P\left(v_iv_{i-1}\right),\mu_\Omega^N\left(v_iv_{i-1}\right)\right)$ for $i=1,2,\cdots,n-1$. Therefore,

$$M_2(P)=\left(M_2^P(P),M_2^N(P)\right)=\left(\sum_{uv\in\mathrm{E}}\mu_\beta^P(u)d^P(u)\mu_\beta^P(v)d^P(v),\sum_{uv\in\mathrm{E}}\mu_\beta^N(u)d^N(u)\mu_\beta^N(v)d^N(v)\right)$$
$$\begin{aligned}M_2^P(P)=&\sum_{uv\in\mathrm{E}}\mu_\beta^P(u)d^P(u)\mu_\beta^P(v)d^P(v)=\mu_\beta^P\left(v_1\right)\mu_\beta^P\left(v_2\right)\mu_\Omega^P\left(v_1v_2\right)\left(\mu_\Omega^P\left(v_1v_2\right)+\mu_\Omega^P\left(v_2v_3\right)\right)\\&+\mu_\beta^P\left(v_{n-1}\right)\mu_\beta^P\left(v_n\right)\mu_\Omega^P\left(v_{n-1}v_n\right)\left(\mu_\Omega^P\left(v_{n-1}v_n\right)+\mu_\Omega^P\left(v_{n-2}v_n\right)\right)\\&+\sum_{i=2}^{n-2}\mu_\beta^P\left(v_i\right)\mu_\beta^P\left(v_{i-1}\right)\left(\mu_\Omega^P\left(v_{i-1}v_i\right)+\mu_\Omega^P\left(v_iv_{i+1}\right)\right)\\&\leq 2+2\sum_{i=2}^{n-2}2\left(\mu_\Omega^P\left(v_iv_{i+1}\right)+\mu_\Omega^P\left(v_{i+1}v_{i+2}\right)\right)\leq 2+2+\sum_{i=2}^{n-2}(2+2)=4(n-2)\end{aligned}$$

Similarly, we get $M_2^N(P)\leq 4(n-2)$.

Now second $ZI$ of a bipolar cycle is studied here. □

**Theorem 3.11.** *Suppose $C=(X,\beta,\Omega)$ is a bipolar cycle having $n$-vertices. Then*

$$M_2^P(C)\leq 4n, M_2^N(C)\leq 4n.$$

*Proof.* Suppose $C\left(v_1,v_2,\cdots,v_n\right)$ is $n$-vertex bipolar cycle, then $d\left(v_1\right)=\left(d^P\left(v_1\right),d^N\left(v_1\right)\right)=\left(\mu_\Omega^P\left(v_1v_2\right),\right.$ $\left.\mu_\Omega^N\left(v_1v_2\right)\right)+\left(\mu_\Omega^P\left(v_1v_n\right),\mu_\Omega^N\left(v_1v_n\right)\right),d\left(v_i\right)=\left(d^P\left(v_i\right),d^N\left(v_i\right)\right)=\left(\mu_\Omega^P\left(v_iv_{i-1}\right),\mu_\Omega^N\left(v_iv_{i-1}\right)\right)+\mu_\Omega^P\left(v_{i+1}v_i\right),$ $\left.\mu_\Omega^N\left(v_{i+1}v_i\right)\right)$ for $i=2,\cdots,n-1$ and $d\left(v_n\right)=\left(d^p\left(v_n\right),d^N\left(v_n\right)\right)=\left(\mu_\Omega^P\left(v_1v_n\right),\mu_\Omega^N\left(v_1v_n\right)\right)+\left(\mu_\Omega^P\left(v_{n-1}v_n\right),\right.$ $\left.\mu_\Omega^N\left(v_{n-1}v_n\right)\right)$. Therefore,

$$M_2(C)=\left(M_2^P(C),M_2^N(C)\right)=\left(\sum_{uv\in\mathrm{E}}\mu_\beta^P(u)d^P(u)\mu_\beta^P(v)d^P(v),\sum_{uv\in\mathrm{E}}\mu_\beta^N(u)d^N(u)\mu_\beta^N(v)d^N(v)\right)$$
$$\begin{aligned}M_2^P(C)=&\sum_{uv\in\mathrm{E}}\mu_\beta^P(u)d^P(u)\mu_\beta^P(v)d^P(v)\\=&\mu_\beta^P\left(v_1\right)\mu_\beta^P\left(v_2\right)\left(\mu_\Omega^P\left(v_1v_n\right)+\mu_\Omega^P\left(v_1v_2\right)\right)\left(\mu_\Omega^P\left(v_1v_2\right)+\mu_\Omega^P\left(v_2v_3\right)\right)\\&+\mu_\beta^P\left(v_1\right)\mu_\beta^P\left(v_n\right)\left(\mu_\Omega^P\left(v_1v_2\right)+\mu_\Omega^P\left(v_1v_n\right)\right)\left(\mu_\Omega^P\left(v_{n-1}v_n\right)+\mu_\Omega^P\left(v_1v_n\right)\right)\\&+\sum_{i=2}^{n-1}\mu_\beta^P\left(v_i\right)\mu_\beta^P\left(v_{i+1}\right)\left(\mu_\Omega^P\left(v_{i-1}v_i\right)+\mu_\Omega^P\left(v_iv_{i+1}\right)\right)\left(\mu_\Omega^P\left(v_iv_{i+1}\right)+\mu_\Omega^P\left(v_{i+1}v_{i+2}\right)\right)\\\leq&4+4+\sum_{i=2}^{n-1}4=4n\end{aligned}$$

Similarity, we obtain $M_2^N(C)\leq 4n$. □

**Theorem 3.12.** *Suppose $S$ is a bipolar star having $n+1$ vertices. Then*

$$M_2^P(S) \leq n^2, M_2^N(S) \leq n^2.$$

*Proof.* Suppose $S(v_0, v_1 : v_2, \cdots, v_n)$ is a bipolar star having $(n+1)$ vertices with center $v_0$, then

$$d(v_0) = \left(d^P(v_0), d^N(v_0)\right) = \left(\sum_{i=1}^{n} \mu_\Omega^P(v_0 v_i), \sum_{i=1}^{n} \mu_\Omega^N(v_0 v_i)\right)$$

and $d(v_i) = \left(d^P(v_i), d^N(v_i)\right) = \left(\mu_\Omega^P(v_0 v_i), \mu_\Omega^N(v_0 v_i)\right)$ for $i = 1, 2, \cdots, n$. Therefore,

$$M_2(S) = \left(M_2^P(S), M_2^N(S)\right) = \left(\sum_{uv\in \mathrm{E}} \mu_\beta^P(u) d^P(u) \mu_\beta^P(v) d^P(v), \sum_{uv\in \mathrm{E}} \mu_\beta^N(u) d^N(u) \mu_\beta^N(v) d^N(v)\right)$$

$$M_2^P(S) = \sum_{i=1}^{n} \mu_\beta^P(v_0) d^P(v_0) \mu_\beta^P(v_i) d^P(v_i) = \mu_\beta^P(v_0) \left(\sum_{i=1}^{n} \mu_\Omega^P(v_0 v_i)\right) \sum_{i=1}^{n} \mu_\beta^P(v_i) \mu_\Omega^P(v_0 v_i) \leq n^2.$$

$$\begin{aligned} M_2^N(S) &= \sum_{uv\in \mathrm{E}} \mu_\beta^N(v_0) d^N(v_0) \mu_\beta^N(v_i) d^N(v_i) = \mu_\beta^N(v_0) \left(\sum_{i=1}^{n} \mu_\Omega^N(v_0 v_i)\right) \sum_{i=1}^{n} \mu_\beta^N(v_i) \mu_\Omega^N(v_0 v_i) \\ &\leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} 1\right) = n^2. \end{aligned}$$

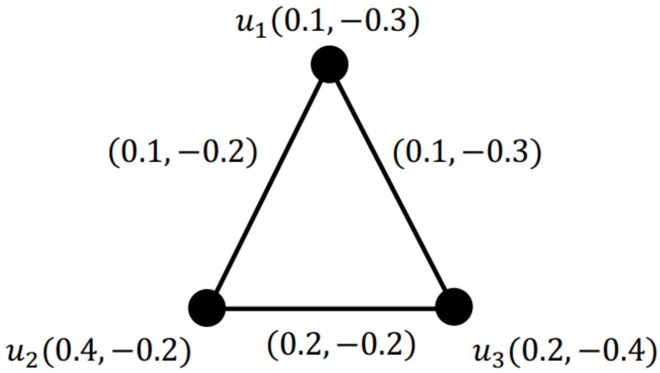


Figure 4: A complete $BFG$ with $M_2(G) = \left(M_2^P(G), M_2^N(G)\right) = (2.0024, 4.6)$.

Second $ZI$ of the complete $BFG$ depicted in Figure 4 is evaluated below. □

**Example 3.13.** Suppose $G$ is a complete bipolar fuzzy graph depicted in Figure 4 with $X = \{u_1, u_2, u_3\}$ is a vertex set and $E_1 = \{u_1u_2, u_1u_3, u_2u_3\}$ such that

| $X$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|
| $\mu_\beta^P$ | 0.1 | 0.4 | 0.2 |
| $\mu_\beta^P$ | -0.3 | -0.2 | -0.4 |

| $E$ | $u_1u_2$ | $u_1u_3$ | $u_2u_3$ |
|---|---|---|---|
| $\mu_{\Omega_1}^P$ | 0.1 | 0.1 | 0.2 |
| $\mu_{\Omega_1}^N$ | -0.2 | -0.3 | -0.2 |

Then the degree of the vertices $X$

| $X$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|
| $d^P$ | 0.2 | 0.3 | 0.3 |
| $d^N$ | -0.5 | -0.4 | -0.5 |

$$M_2(G)=\left(M_2^P(G),M_2^N(G)\right)=\left(\sum_{uv\in \mathrm{E}}\mu_\beta^P(u)d^P(u)\mu_\beta^P(v)d^P(v),\sum_{uv\in \mathrm{E}}\mu_\beta^N(u)d^N(u)\mu_\beta^N(v)d^N(v)\right)$$

$$M_2^P(G)=[0.1\times 0.2\times 0.4\times 0.3]+[0.1\times 0.2\times 0.2\times 0.3]+[0.4\times 0.3\times 0.2\times 0.3]=2.0024$$

$$M_2^N(G)=[-0.3\times -0.5\times -0.2\times -0.4]+[-0.3\times -0.5\times -0.4\times -0.5]$$
$$+[-0.2\times -0.4\times -0.4\times -0.5]=4.6.$$

**Theorem 3.14.** *Suppose $G=(X,\beta,\Omega)$ is a CBFG with vertex set $X=\{v_1,v_2,\cdots,v_n\}$. Then*

$$0\le \frac{n(n-1)^3}{2}\left(\mu_{\beta_1}^P\right)^4\le M_2^P(G)\le \frac{n(n-1)^3}{2}\left(\mu_{\beta_n}^P\right)^4\le \frac{n(n-1)^3}{2},$$
$$0\le \frac{n(n-1)^3}{2}\left(\mu_{\beta_1}^N\right)^4\le M_2^N(G)\le \frac{n(n-1)^3}{2}\left(\mu_{\beta_n}^N\right)^4\le \frac{n(n-1)^3}{2}.$$

*Where*

$\left(\mu_\beta^P,\mu_\beta^N\right)(v_i)=\left(\mu_{\beta_i}^P,\mu_{\beta_i}^N\right)$ *and* $\mu_{\beta_1}^P\le\mu_{\beta_2}^P\le\cdots\le\mu_{\beta_n}^P,\mu_{\beta_1}^N\le\mu_{\beta_2}^N\le\cdots\le$ $\mu_{\beta_n}^N$ *for* $i=1,2,\cdots,n$

*Proof.* As $G$ be a $CBFG$, and $\left(\mu_\beta^P,\mu_\beta^N\right)(v_i)=\left(\mu_{\beta_i}^P,\mu_{\beta_i}^N\right)$ and $\mu_{\beta_1}^P\le\mu_{\beta_2}^P\le\cdots\le\mu_{\beta_n}^P$ for $i=1,2,\cdots,n$, so $d(v_i)=(n-i)\mu_{\beta_i}^P+\sum_{j=1}^{i-1}\mu_{\beta_j}^P$. Therefore

$$\begin{aligned}M_2^P(G)&=\sum_{uv\in \mathrm{E}}\mu_\beta^P(u)d^P(u)\mu_\beta^P(v)d^P(v)=\sum_{1\le i<j\le n}\mu_{\beta_i}^P\mu_{\beta_j}^P\left\{\left[(n-i)\mu_{\beta_i}^P+\sum_{k=1}^{i-1}\mu_{\beta_k}^P\right]\left[(n-j)\mu_{\beta_j}^P+\sum_{k=1}^{j-1}\mu_{\beta_k}^P\right]\right\}\\&\le\sum_{1\le i<j\le n}\left(\mu_{\beta_n}^P\right)^2\left\{\left[(n-i)\mu_{\beta_n}^P+\sum_{k=1}^{i-1}\mu_{\beta_n}^P\right]\left[(n-j)\mu_{\beta_n}^P+\sum_{k=1}^{j-1}\mu_{\beta_n}^P\right]\right\}\\&=\left(\mu_{\beta_n}^P\right)^4\sum_{1\le i<j\le n}[(n-i)+i-1][(n-j)+j-1]=\frac{n(n-1)^3}{2}\left(\mu_{\beta_n}^P\right)^4,\end{aligned}$$

$$\begin{aligned}M_2^N(G)&=\sum_{uv\in \mathrm{E}}\mu_\beta^N(u)d^N(u)\mu_\beta^N(v)d^N(v)=\sum_{1\le i<j\le n}\mu_{\beta_i}^N\mu_{\beta_j}^N\left\{\left[(n-i)\mu_{\beta_i}^N+\sum_{k=1}^{i-1}\mu_{\beta_k}^N\right]\left[(n-j)\mu_{\beta_j}^N+\sum_{k=1}^{j-1}\mu_{\beta_k}^N\right]\right\}\\&\le\sum_{1\le i<j\le n}\left(\mu_{\beta_n}^N\right)^2\left\{\left[(n-i)\mu_{\beta_n}^N+\sum_{k=1}^{i-1}\mu_{\beta_n}^N\right]\left[(n-j)\mu_{\beta_n}^N+\sum_{k=1}^{j-1}\mu_{\beta_n}^N\right]\right\}\\&=\left(\mu_{\beta_n}^N\right)^4\sum_{1\le i<j\le n}[(n-i)+i-1][(n-j)+j-1]=\frac{n(n-1)^3}{2}\left(\mu_{\beta_n}^N\right)^4.\end{aligned}$$

We can prove other inequalities similarly.
Let $G=(X,\beta,\Omega)$ be a $BFG$. Now we construct the $BFGC[G]=(X,\beta,\Omega^c)$ with $\left(\mu_\Omega^P(uv)\right)^c=\wedge\left\{\mu_\beta^P(u),\mu_\beta^P(v)\right\}$ and $\left(\mu_\Omega^N(uv)\right)^c=\mathrm{v}\left\{\mu_\beta^N(u),\mu_\beta^N(v)\right\}$. We called the $BFG$ is completion bipolar fuzzy graph of the $BFGG$. □

**Theorem 3.15.** *Let $G=(X,\beta,\Omega)$ be a BFG. Then*

$$M_2^P(G)\le M_2^P(C[G]),M_2^N(G)\le M_2^N(C[G]).$$

*Proof.* As $C[G]$ is completion $BFG$ of $G$, then $\forall(u,v)\in E,\mu_\Omega^P(u,v)\le\mu_{\Omega^C}^P(u,v)$ and $\mu_\Omega^N(u,v)\ge\mu_{\Omega^C}^N(u,v)$. So, $G$ is bipolar partial fuzzy subgraph of $C[G]$. Therefore, the result follows by Proposition 3.8. □

**Corollary 3.16.** *Let $G = (X, \beta, \Omega)$ be a BFG with $n$ vertices. Then*
$$M_2^P(G) \leq \frac{n(n-1)^3}{2} \text{ and } M_2^N(G) \geq -\frac{n(n-1)^3}{2}.$$
*Second ZI is discussed for isomorphic BFGs in the next theorem.*

**Theorem 3.17.** *Suppose $G_1 = (X_1, \beta_1, \Omega_1)$ and $G_2 = (X_2, \beta_2, \Omega_2)$ are isomorphic BFGs. Then $M_2(G_1) = M_2(G_2)$.*

*Proof.* There exists a bijective mapping $\alpha : X_1 \to X_2$ such that , $\left(\mu^P_{\beta_1}(v), \mu^N_{\beta_1}(v)\right) = \left(\mu^P_{\beta_2}(\alpha(v)), \mu^N_{\beta_2}(\alpha(v))\right)$ and $\left(\mu^P_{\Omega_1}(u,v), \mu^N_{\Omega_1}(u,v)\right) = \left(\mu^P_{\Omega_2}(\alpha(u), \alpha(v)), \mu^N_{\Omega_2}(\alpha(u), \alpha(v))\right)$ for any $u, v \in X_1$. Then
$$d^P_{G_1}(u) = \sum_{u\in X_1} \left(\mu^P_{\Omega_1}(u,v)\right) = \sum_{u\in X_1} \mu^P_{\Omega_2}(\alpha(u), \alpha(v)) = \sum_{\alpha(u)\in X_2} \mu^P_{\Omega_2}(\alpha(u), \alpha(v)) = d^P_{G_2}(\alpha(u)).$$
$$d^N_{G_1}(u) = \sum_{u\in X_1} \left(\mu^N_{\Omega_1}(u,v)\right) = \sum_{u\in X_1} \mu^N_{\Omega_2}(\alpha(u), \alpha(v)) = \sum_{\alpha(u)\in X_2} \mu^N_{\Omega_2}(\alpha(u), \alpha(v)) = d^N_{G_2}(\alpha(u)).$$
Therefore,
$$\begin{aligned} M_2^P(G_1) &= \sum_{uv\in \mathrm{E}_1} \mu^P_{\beta_1}(u) d^P_{G_1}(u) \mu^P_{\beta_1}(v) d^P_{G_1}(v) = \sum_{uv\in \mathrm{E}_1} \mu^P_{\beta_2}(\alpha(u)) d^P_{G_2}(\alpha(u)) \mu^P_{\beta_2}(\alpha(v)) d^P_{G_2}(\alpha(v)) \\ &= \sum_{\alpha(u)\alpha(v)\in \mathrm{E}_2} \mu^P_{\beta_2}(\alpha(u)) d^P_{G_2}(\alpha(u)) \mu^P_{\beta_2}(\alpha(v)) d^P_{G_2}(\alpha(v)) = M_2^P(G_2). \\ M_2^N(G_1) &= \sum_{uv\in \mathrm{E}_1} \mu^N_{\beta_1}(u) d^N_{G_1}(u) \mu^N_{\beta_1}(v) d^N_{G_1}(v) = \sum_{uv\in \mathrm{E}_1} \mu^N_{\beta_2}(\alpha(u)) d^N_{G_2}(\alpha(u)) \mu^N_{\beta_2}(\alpha(v)) d^N_{G_2}(\alpha(v)) \\ &= \sum_{\alpha(u)\alpha(v)\in \mathrm{E}_2} \mu^N_{\beta_2}(\alpha(u)) d^N_{G_2}(\alpha(u)) \mu^N_{\beta_2}(\alpha(v)) d^N_{G_2}(\alpha(v)) = M_2^N(G_2). \end{aligned}$$
□

**Theorem 3.18.** *Let $G = (X, \beta, \Omega)$ be an BFG having $n$ vertices and $m$ edges. Then*
$$\frac{\left(\delta^P\right)^4}{m} \leq M_2^P(G) \leq m\left(\Delta^P\right)^2 \text{ and } \frac{\left(\delta^N\right)^4}{m} \leq M_2^N(G) \leq m\left(\Delta^N\right)^2.$$

*Proof.* We have $\delta^P \leq d^P(v) \leq m\left(\mu^P_\beta(v)\right)$ and $\delta^N \geq d^N(v) \geq m\left(\mu^N_\beta(v)\right)$ for any $v \in X$. Therefore, $\mu^P_\beta(v) \geq \frac{\delta^P}{m}$ and $\mu^N_\beta(v) \leq \frac{\delta^N}{m}$. Now,
$$M_2(G) = \left(M_2^P(G), M_2^N(G)\right) = \left(\sum_{uv\in \mathrm{E}} \mu^P_\beta(u) d^P(u) \mu^P_\beta(v) d^P(v), \sum_{uv\in \mathrm{E}} \mu^N_\beta(u) d^N(u) \mu^N_\beta(v) d^N(v)\right)$$
$$M_2^P(G) = \sum_{uv\in \mathrm{E}} \mu^P_\beta(u) d^P(u) \mu^P_\beta(v) d^P(v) \leq \left(\Delta^P\right)^2 \sum_{uv\in \mathrm{E}} \left[\mu^P_\beta(u) \mu^P_\beta(v)\right] \leq m\left(\Delta^P\right)^2.$$
$$M_2^N(G) = \sum_{uv\in \mathrm{E}} \mu^N_\beta(u) d^N(u) \mu^N_\beta(v) d^N(v) \leq \left(\Delta^N\right)^2 \sum_{uv\in \mathrm{E}} \left[\mu^N_\beta(u) \mu^N_\beta(v)\right] \leq m\left(\Delta^N\right)^2.$$
Again, $$M_2^P(G) = \sum_{uv\in \mathrm{E}} \mu^P_\beta(u) d^P(u) \mu^P_\beta(v) d^P(v) \geq \left(\delta^P\right)^2 \sum_{uv\in \mathrm{E}} \left[\mu^P_\beta(u) \mu^P_\beta(v)\right]$$
$$\geq \left(\delta^P\right)^2 \sum_1^m \left(\frac{\delta^P}{m}\right)^2 = \frac{\left(\delta^P\right)^4}{m}.$$
$$M_2^N(G) = \sum_{uv\in \mathrm{E}} \mu^N_\beta(u) d^N(u) \mu^N_\beta(v) d^N(v) \geq \left(\delta^N\right)^2 \sum_{uv\in \mathrm{E}} \left[\mu^N_\beta(u) \mu^N_\beta(v)\right] \geq \left(\delta^N\right)^2 \sum_1^m \left(\frac{\delta^N}{m}\right)^2 = \frac{\left(\delta^N\right)^4}{m}.$$
□

## 4. Bounds of second Zagreb indices for bipolar fuzzy graphs during operations

Some relations of second $ZI$ are discussed during some $BFG$ operations here.

**Theorem 4.1.** *For tow $BFG_sG_1 = (X_1, \beta_1, \Omega_1)$ and $G_2 = (X_2, \beta_2, \Omega_2)$ we have*

$$M_2^P(G_1 \times G_2) \leq m_2 M_1^P(G_1) + m_1 M_1^P(G_2) + n_2 M_1^P(G_1) + n_1 M_2^P(G_2) + 2\Delta_1^P \Delta_2^P(n_1 m_2 + n_2 m_1) \text{ and}$$
$$M_2^N(G_1 \times G_2) \geq m_2 M_1^N(G_1) + m_1 M_1^N(G_2) + n_2 M_1^N(G_1) + n_1 M_2^N(G_2) + 2\Delta_1^N \Delta_2^N(n_1 m_2 + n_2 m_1).$$

*Proof.* Since $G_1 \times G_2$ is the Cartesian product of $G_1$ and $G_2$, then for $(u,v),(u_1,v_1) \in X_1 \times X_2, \mu_\beta^P(u,v) = \wedge\left\{\mu_{\beta_1}^P(u), \mu_{\beta_2}^P(v)\right\}, \mu_\beta^N(u,v) = \vee\left\{\mu_{\beta_1}^N(u), \mu_{\beta_2}^N(v)\right\}$ and

$$\mu_\Omega^P((u,v),(u_1,v_1)) = \begin{cases} \wedge\left\{\mu_{\beta_1}^P(u), \mu_{\Omega_2}^P(v,v_1)\right\} & \text{if } u = u_1, vv_1 \in E_2 \\ \wedge\left\{\mu_{\beta_2}^P(v), \mu_{\Omega_1}^P(u,u_1)\right\} & \text{if } uu_1 \in E_1, v = v_1 \\ 0 \text{ otherwise.} \end{cases}$$
$$\mu_\Omega^N((u,v),(u_1,v_1)) = \begin{cases} \vee\left\{\mu_{\beta_1}^N(u), \mu_{\Omega_2}^N(v,v_1)\right\} & \text{if } u = u_1, vv_1 \in E_2 \\ \vee\left\{\mu_{\beta_2}^N(v), \mu_{\Omega_1}^N(u,u_1)\right\} & \text{if } uu_1 \in E_1, v = v_1 \\ 0 \text{ otherwise.} \end{cases}$$

Then

$$\begin{aligned} d_{G_1\times G_2}^P(u,v) &= \sum_{v_iv\in E_2} \mu_\Omega^P\{(u,v),(u,v_i)\} + \sum_{u_iu\in E_1} \mu_\Omega^P\{(u,v),(u_i,v)\} \\ &= \sum_{v_iv\in E_2} \wedge\{\mu_{\beta_1}^P(u), \mu_{\Omega_2}^P(v,v_i)\} + \sum_{u_iu\in E_1} \wedge\{\mu_{\beta_2}^P(v), \mu_{\Omega_1}^P(u,u_i)\} \\ &\leq \sum_{v_iv\in E_2} \mu_{\Omega_2}^P(v,v_i) + \sum_{u_iu\in E_1} \mu_{\Omega_1}^P(u,u_i) = d_{G_1}^P(u) + d_{G_2}^P(v), \\ d_{G_1\times G_2}^N(u,v) &= \sum_{v_iv\in E_2} \mu_\Omega^N\{(u,v),(u,v_i)\} + \sum_{u_iu\in E_1} \mu_\Omega^N\{(u,v),(u_i,v)\} \\ &= \sum_{v_iv\in E_2} \vee\{\mu_{\beta_1}^N(u), \mu_{\Omega_2}^N(v,v_i)\} + \sum_{u_iu\in E_1} \vee\{\mu_{\beta_2}^N(v), \mu_{\Omega_1}^N(u,u_i)\} \\ &\geq \sum_{v_iv\in E_2} \mu_{\Omega_2}^N(v,v_i) + \sum_{u_iu\in E_1} \mu_{\Omega_1}^N(u,u_i) = d_{G_1}^N(u) + d_{G_2}^N(v). \end{aligned}$$

Then second $ZI$ of $G_1 \times G_2$ is:

$$\begin{aligned} M_2^P(G_1\times G_2) &= \sum_{(u_1,v_1)(u_2,v_2)\in \mathrm{E}(G_1\times G_2)} \mu_\beta^P(u_1,v_1)\,\mu_\beta^P(u_2,v_2)\, d_{G_1\times G_2}^P(u_1,v_1)\, d_{G_1\times G_2}^P(u_2,v_2) \\ &= \sum_{u\in X_1, v_1v_2\in \mathrm{E}_2} \mu_\beta^P(u,v_1)\,\mu_\beta^P(u,v_2)\, d_{G_1\times G_2}^P(u,v_1)\, d_{G_1\times G_2}^P(u,v_2) \\ &+ \sum_{v\in X_2, u_1u_2\in \mathrm{E}_1} \mu_\beta^P(u_1,v)\,\mu_\beta^P(u_2,v)\, d_{G_1\times G_2}^P(u_1,v)\, d_{G_1\times G_2}^P(u_2,v) = K_1^P + K_2^P, \\ M_2^N(G_1\times G_2) &= \sum_{(u_1,v_1)(u_2,v_2)\in \mathrm{E}(G_1\times G_2)}^{N} \mu_\beta^N(u_1,v_1)\,\mu_\beta^N(u_2,v_2)\, d_{G_1\times G_2}^N(u_1,v_1)\, d_{G_1\times G_2}^N(u_2,v_2) \\ &= \sum_{u\in X_1, v_1v_2\in \mathrm{E}_2} \mu_\beta^N(u,v_1)\,\mu_\beta^N(u,v_2)\, d_{G_1\times G_2}^N(u,v_1)\, d_{G_1\times G_2}^N(u,v_2) \end{aligned}$$

$$+ \sum_{v\in X_2,u_1u_2\in \mathrm{E}_1} \mu_\beta^N(u_1,v)\,\mu_\beta^N(u_2,v)\,d^N_{G_1\times G_2}(u_1,v)\,d^N_{G_1\times G_2}(u_2,v) = K_1^N + K_2^N, \text{ where}$$

$$K_1{}^P = \sum_{u\in X_1,v_1v_2\in \mathrm{E}_2} \mu_\beta^P(u,v_1)\,\mu_\beta^P(u,v_2)\,d^P_{G_1\times G_2}(u,v_1)\,d^P_{G_1\times G_2}(u,v_2),$$

$$K_1{}^N = \sum_{u\in X_1,v_1v_2\in \mathrm{E}_2} \mu_\beta^N(u,v_1)\,\mu_\beta^N(u,v_2)\,d^N_{G_1\times G_2}(u,v_1)\,d^N_{G_1\times G_2}(u,v_2) \text{ and}$$

$$K_2{}^P = \sum_{v\in X_2,u_1u_2\in \mathrm{E}_1} \mu_\beta^P(u_1,v)\,\mu_\beta^P(u_2,v)\,d^P_{G_1\times G_2}(u_1,v)\,d^P_{G_1\times G_2}(u_2,v)$$

$$K_2{}^N = \sum_{v\in X_2,u_1u_2\in \mathrm{E}_1} \mu_\beta^N(u_1,v)\,\mu_\beta^N(u_2,v)\,d^N_{G_1\times G_2}(u_1,v)\,d^N_{G_1\times G_2}(u_2,v).$$

$$\begin{aligned}
K_1{}^P &= \sum_{u\in X_1,v_1v_2\in \mathrm{E}_2} \mu_\beta^P(u,v_1)\,\mu_\beta^P(u,v_2)\,d^P_{G_1\times G_2}(u,v_1)\,d^P_{G_1\times G_2}(u,v_2)\\
&\le \sum_{u\in X_1,v_1v_2\in \mathrm{E}_2} \left[\mu^P_{\beta_1}(u)d^P_{G_1}(u) + \mu^P_{\beta_2}(v_1)\,d^P_{G_2}(v_1)\right]\left[\mu^P_{\beta_1}(u)d^P_{G_1}(u)\right.\\
&\sum_{u\in X_1,v_1v_2\in \mathrm{E}_2} \left[\mu^P_{\beta_1}(u)d^P_{G_1}(u)\right]^2 + \sum_{u\in X_1,v_1v_2\in \mathrm{E}_2} \mu^P_{\beta_2}(v_1)\,d^P_{G_2}(v_1)\,\mu^P_{\beta_2}(v_2)\,d^P_{G_2}(v_2)\\
&+ \sum_{u\in X_1,v_1v_2\in \mathrm{E}_2} \mu^P_\beta(u)d^P_{G_1}(u)\left\{\mu^P_{\beta_2}(v_1)\,d^P_{G_2}(v_1) + \mu^P_{\beta_2}(v_2)\,d^P_{G_2}(v_2)\right\}\\
&\le \sum_{v_1v_2\in \mathrm{E}_2} M_1^P(G_1) + \sum_{u\in X_1} M_2^P(G_2) + 2\Delta_1^P\Delta_2^P \sum_{u\in X_1,v_1v_2\in \mathrm{E}_2} 1\\
&= m_2M_1^P(G_1) + n_1M_2^P(G_2) + 2n_1m_2\Delta_1^P\Delta_2^P.
\end{aligned}$$

So, $K_1{}^P \le m_2M_1^P(G_1) + n_1M_2^P(G_2) + 2n_1m_2\Delta_1^P\Delta_2^P$.
Similarly, one can get $K_1^P \le m_1M_1^P(G_2) + n_2M_2^P(G_1) + 2n_2m_1\Delta_1^P\Delta_2^P$.
Therefore, $M_2^P(G_1\times G_2) \le K_1{}^P + K_2{}^P \le m_2M_1^P(G_1) + m_1M_1^P(G_2) + n_2M_2^P(G_1) + n_1M_2^P(G_2) + 2\Delta_1^P\Delta_2^P(n_1m_2 + n_2m_1)$.

$$\begin{aligned}
K_1^N &= \sum_{u\in X_1,v_1v_2\in \mathrm{E}_2} \mu_\beta^N(u,v_1)\,\mu_\beta^N(u,v_2)\,d^N_{G_1\times G_2}(u,v_1)\,d^N_{G_1\times G_2}(u,v_2)\\
&\ge \sum_{u\in X_1,v_1v_2\in \mathrm{E}_2} \left[\mu^N_{\beta_1}(u)d^N_{G_1}(u) + \mu^N_{\beta_2}(v_1)\,d^N_{G_2}(v_1)\right]\left[\mu^N_{\beta_1}(u)d^N_{G_1}(u)\right.\\
&= \sum_{u\in X_1,v_1v_2\in \mathrm{E}_2} \left[\mu^N_{\beta_1}(u)d^N_{G_1}(u)\right]^2 + \sum_{u\in X_1,v_1v_2\in \mathrm{E}_2} \mu^N_{\beta_2}(v_1)\,d^N_{G_2}(v_1)\,\mu^N_{\beta_2}(v_2)\,d^N_{G_2}(v_2)\\
&= \sum_{u\in X_1,v_1v_2\in \mathrm{E}_2} \mu^N_\beta(u)d^N_{G_1}(u)\left\{\mu^N_{\beta_2}(v_1)\,d^N_{G_2}(v_1) + \mu^N_{\beta_2}(v_2)\,d^N_{G_2}(v_2)\right\}\\
&\ge \sum_{v_1v_2\in \mathrm{E}_2} M_1^N(G_1) + \sum_{u\in X_1} M_2^N(G_2) + 2\Delta_1^N\Delta_2^N \sum_{u\in X_1,v_1v_2\in \mathrm{E}_2} 1\\
&= m_2M_1^N(G_1) + n_1M_2^N(G_2) + 2n_1m_2\Delta_1^N\Delta_2^N.
\end{aligned}$$

So, $K_1^N \ge m_2M_1^N(G_1) + n_1M_2^N(G_2) + 2n_1m_2\Delta_1^N\Delta_2^N$.

Similarly, one can get $K_2{}^N \ge m_1M_1^N(G_2) + n_2M_2^N(G_1) + 2n_2m_1\Delta_1^N\Delta_2^N$.
Therefore, $M_2^N(G_1\times G_2) \ge K_1{}^N + K_2{}^N \ge m_2M_1^N(G_1) + m_1M_1^N(G_2) + n_2M_2^N(G_1) + n_1M_2^N(G_2) + 2\Delta_1^N\Delta_2^N(n_1m_2 + n_2m_1)$. □

**Example 4.2.** Suppose $G_1 = (X_1, \beta_1, \Omega_1)$ and $G_2 = (X_2, \beta_2, \Omega_2)$ are a $BFG$ shown in figure. 5 with vertex set $X_1 = \{u_1, u_2\}, E_1 = \{u_1u_2\}, X_2 = \{v_1, v_2, v_3\}, E_1 = \{v_1v_2, v_1v_3, v_2v_3\}$, Such that

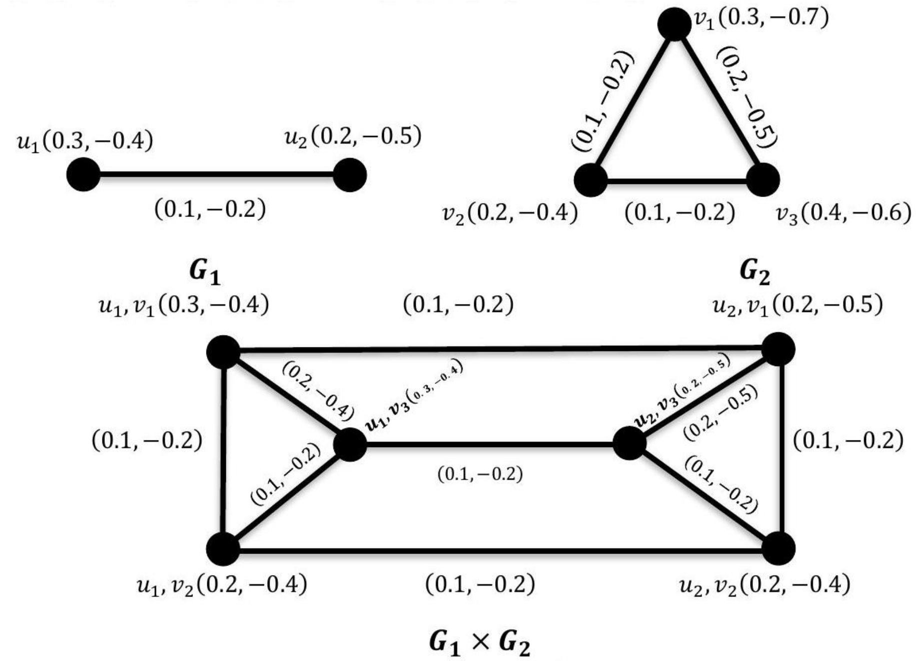


Figure 5: The Cartesian product of two BFG $\boldsymbol{G_1}$ and $\boldsymbol{G_2}$.

| $X_1$ | $u_1$ | $u_2$ |
|---|---|---|
| $\mu^P_{\beta_1}$ | 0.3 | 0.2 |
| $\mu^N_{\beta_1}$ | -0.4 | -0.5 |

| $X_2$ | $v_1$ | $v_2$ | $v_3$ |
|---|---|---|---|
| $\mu^P_{\beta_2}$ | 0.3 | 0.2 | 0.4 |
| $\mu^N_{\beta_2}$ | -0.7 | -0.4 | -0.6 |

| $E_1$ | $u_1u_2$ |
|---|---|
| $\mu^P_{\Omega_1}$ | 0.1 |
| $\mu^N_{\Omega_1}$ | -0.2 |

| $E_2$ | $v_1v_2$ | $v_1v_3$ | $v_2v_3$ |
|---|---|---|---|
| $\mu^P_{\Omega_2}$ | 0.1 | 0.2 | 0.1 |
| $\mu^N_{\Omega_2}$ | -0.2 | -0.5 | -0.2 |

Hence the following tables give $G_1 \times G_2$

| $X_1 \times X_2$ | $u_1, v_1$ | $u_1, v_2$ | $u_1, v_3$ | $u_2, v_1$ | $u_2, v_2$ | $u_2, v_3$ |
|---|---|---|---|---|---|---|
| $\mu^P_{\beta_1} \times \mu^P_{\beta_2}$ | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 | 0.2 |
| $\mu^N_{\beta_1} \times \mu^N_{\beta_2}$ | -0.4 | -0.4 | -0.4 | -0.5 | -0.2 | -0.5 |

| $E_1 \times E_2$ | $(u_1, v_1)$ $(u_2, v_1)$ | $(u_1, v_1)$ $(u_1, v_2)$ | $(u_1, v_1)$ $(u_1, v_3)$ | $(u_1, v_2)$ $(u_1, v_3)$ | $(u_1, v_3)$ $(u_2, v_3)$ | $(u_1, v_2)$ $(u_2, v_3)$ | $(u_2, v_3)$ $(u_2, v_1)$ | $(u_2, v_3)$ $(u_2, v_2)$ | $(u_2, v_1)$ $(u_2, v_2)$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mu^P_{\Omega_1} \times \mu^P_{\Omega_2}$ | 0.1 | 0.1 | 0.2 | 0.1 | 0.1 | 0.1 | 0.2 | 0.2 | 0.1 |
| $\mu^N_{\Omega_1} \times \mu^N_{\Omega_2}$ | -0.2 | -0.2 | -0.4 | -0.2 | -0.2 | -0.2 | -0.5 | -0.4 | -0.2 |

Therefore, the degree is all vertices $G_1 \times G_2$

| d $(X_1 \times X_2)$ | $u_1, v_1$ | $u_1, v_2$ | $u_1, v_3$ | $u_2, v_1$ | $u_2, v_2$ | $u_2, v_3$ |
|---|---|---|---|---|---|---|
| $d^P$ | 0.4 | 0.3 | 0.4 | 0.4 | 0.3 | 0.4 |
| $d^N$ | -0.8 | -0.6 | -0.8 | -0.9 | -0.6 | -0.9 |

Then

$$M_2\left(G_1 \times G_2\right) = \left(M_2^P\left(G_1 \times G_2\right), M_2^N\left(G_1 \times G_2\right)\right)$$
$$= \left(\sum_{uv \in \mathrm{E}} \mu^P_\beta(u) d^P(u) \mu^P_\beta(v) d^P(v), \sum_{uv \in \mathrm{E}} \mu^N_\beta(u) d^N(u) \mu^N_\beta(v) d^N(v)\right)$$

$$
\begin{aligned}
M_2^P\left(G_1 \times G_2\right) = & [0.3 \times 0.4 \times 0.2 \times 0.4]+[0.3 \times 0.4 \times 0.3 \times 0.4]+[0.3 \times 0.4 \times 0.2 \times 0.3] \\
& +[0.2 \times 0.3 \times 0.3 \times 0.4]+[0.2 \times 0.3 \times 0.2 \times 0.3]+[0.3 \times 0.4 \times 0.2 \times 0.4] \\
& +[0.2 \times 0.4 \times 0.2 \times 0.4]+[0.2 \times 0.4 \times 0.2 \times 0.3]+[0.2 \times 0.3 \times 0.2 \times 0.4]=0.0676 \\
M_2^N\left(G_1 \times G_2\right) = & [-0.4 \times -0.8 \times -0.5 \times -0.9]+[-0.4 \times -0.8 \times -0.4 \times -0.6] \\
& +[-0.4 \times -0.8 \times -0.3 \times -0.8]+[-0.4 \times -0.8 \times -0.4 \times -0.8] \\
& +[-0.4 \times -0.8 \times -0.5 \times -0.9]+[-0.4 \times -0.6 \times -0.4 \times -0.6] \\
& +[-0.5 \times -0.9 \times -0.5 \times -0.9]+[-0.5 \times -0.9 \times -0.4 \times -0.6] \\
& +[-0.4 \times -0.6 \times -0.5 \times -0.9]=1.0201
\end{aligned}
$$

Hence the result.

**Corollary 4.3.** *(i)* $M_2^P\left(G_1 \times G_2\right) \leq \left(n_1 m_2+n_2 m_1\right)\left(\Delta_1^P+\Delta_2^P\right)^2$, $M_2^N\left(G_1 \times G_2\right) \geq \left(n_1 m_2+n_2 m_1\right)$ $\left(\Delta_1^N+\Delta_2^N\right)^2$,

*(ii)* $M_2^P\left(G_1 \times G_2\right) \leq \left(n_1 m_2+n_2 m_1\right)\left(n_1+n_2-2\right)^2$ $M_2^N\left(G_1 \times G_2\right) \geq \left(n_1 m_2+n_2 m_1\right)\left(n_1+n_2-2\right)^2$ *and*

*(iii)* $M_2^P\left(G_1 \times G_2\right) \leq 4 m_1^2 n_1^2 m_2+4 m_2^2 n_2^2 m_1+m_1 n_2\left(n_1-1\right)^2+m_2 n_1\left(n_1-1\right)^2+2\left(n_1-1\right)\left(n_2-1\right)\left(n_1 m_2+n_2 m_1\right)$, $M_2^N\left(G_1 \times G_2\right) \geq 4 m_1^2 n_1^2 m_2+4 m_2^2 n_2^2 m_1+m_1 n_2\left(n_1-1\right)^2+m_2 n_1\left(n_1-1\right)^2+2\left(n_1-1\right)\left(n_2-1\right)\left(n_1 m_2+n_2 m_1\right)$.

**Theorem 4.4.** $M_2^P\left(G_1\left[G_2\right]\right) \leq n_2^2 m_2 M_1^P\left(G_1\right)+n_2^4 M_2^P\left(G_1\right)+\left(m_1+n_1\right) M_2^P\left(G_2\right)+2 n_2\left(n_1 m_2+m_1 n_2^2\right) \Delta_1^P \Delta_2^P+m_1\left(n_2^2-m_2\right)\left(\Delta_2^P\right)^2$, $M_2^N\left(G_1\left[G_2\right]\right) \geq n_2^2 m_2 M_1^N\left(G_1\right)+n_2^4 M_2^N\left(G_1\right)+\left(m_1+n_1\right) M_2^N\left(G_2\right)+2 n_2\left(n_1 m_2+m_1 n_2^2\right)$ $\Delta_1^N \Delta_2^N+m_1\left(n_2^2-m_2\right)\left(\Delta_2^N\right)^2$.

*Proof.* As $G_1\left[G_2\right]$ is the composition graph of $G_1$ and $G_2$, then for $(u, v),\left(u_1, v_1\right) \in X, \mu_\beta^P(u, v)$ $=\wedge\left\{\mu_{\beta_1}^P(u), \quad \mu_{\beta_2}^P \quad(v)\right\}$ and $\mu_\beta^N(u, v)=\mathrm{v}\left\{\mu_{\beta_1}^N(u), \mu_{\beta_2}^N(v)\right\}$ and

$$
\mu_{\Omega}^P\left((u, v),\left(u_1, v_1\right)\right)=\left\{\begin{array}{l}
\wedge\left\{\mu_{\beta_1}^P(u), \mu_{\Omega_2}^P\left(v v_1\right)\right\} \quad \text { if } u=u_1, v v_1 \in E_2 \\
\wedge\left\{\mu_{\beta_2}^P(v), \mu_{\beta_2}^P\left(v_1\right), \mu_{\Omega_1}^P\left(u u_1\right)\right\} \quad \text { if } u u_1 \in E_1, \\
0 \text { otherwise. }
\end{array}\right.
$$

$$
\mu_{\Omega}^N\left((u, v),\left(u_1, v_1\right)\right)=\left\{\begin{array}{c}
\vee\left\{\mu_{\beta_1}^N(u), \mu_{\Omega_2}^N\left(v v_1\right)\right\} \text { if } u=u_1, v v_1 \in E_2 \\
\vee\left\{\mu_{\beta_2}^N(v), \mu_{\beta_2}^P\left(v_1\right), \mu_{\Omega_1}^N\left(u u_1\right)\right\} \text { if } u u_1 \in E_1, \\
0 \text { otherwise. }
\end{array}\right.
$$

Then,

$$
\begin{aligned}
d_{G_1\left[G_2\right]}^P(u, v) & =\sum_{v_i v \in E_2} \mu_{\Omega}^P\left\{(u, v),\left(u, v_i\right)\right\}+\sum_{u_i u \in E_1, v_j \in X_2} \mu_{\Omega}^P\left\{(u, v),\left(u_i, v_i\right)\right\} \\
& =\sum_{v_i v \in E_2} \wedge\left\{\mu_{\beta_1}^P(u), \mu_{\Omega_2}^P\left(v v_i\right)\right\}+\sum_{u_i u \in E_1, v_j \in X_2} \wedge\left\{\mu_{\beta_2}^P(v), \mu_{\beta_2}^P\left(v_i\right), \mu_{\Omega_1}^P\left(u u_i\right)\right\} \\
& \leq \sum_{v_i v \in E_2} \mu_{\Omega_2}^P\left(v v_i\right)+\sum_{u_i u \in E_1, v_j \in X_2} \mu_{\Omega_1}^P\left(u u_i\right) \\
& =n_2 d_{G_1}^P(u)+d_{G_2}^P(v), \\
d_{G_1\left[G_2\right]}^N(u, v) & =\sum_{v_i v \in E_2} \mu_{\Omega}^N\left\{(u, v),\left(u, v_i\right)\right\}+\sum_{u_i u \in E_1, v_j \in X_2} \mu_{\Omega}^N\left\{(u, v),\left(u_i, v_i\right)\right\} \\
& =\sum_{v_i v \in E_2} \vee\left\{\mu_{\beta_1}^N(u), \mu_{\Omega_2}^N\left(v v_i\right)\right\}+\sum_{u_i u \in E_1, v_j \in X_2} \vee\left\{\mu_{\beta_2}^N(v), \mu_{\beta_2}^N\left(v_i\right), \mu_{\Omega_1}^N\left(u u_i\right)\right\}
\end{aligned}
$$

$$\geq \sum_{v_i v \in E_2} \mu_{\Omega_2}^{N}(vv_i) + \sum_{u_i u \in E_1, v_j \in X_2} \mu_{\Omega_1}^{N}(uu_i) = n_2 d_{G_1}^{N}(u) + d_{G_2}^{N}(v).$$

Then second $ZI$ of $G_1[G_2]$ is:

$$\begin{aligned}
M_2^P(G_1[G_2]) &= \sum_{(u_1,v_1)(u_2,v_2)\in \mathrm{E}(G_1[G_2])} \mu_\beta^P(u_1,v_1)\,\mu_\beta^P(u_2,v_2)\,d_{G_1[G_2]}^P(u_1,v_1)\,d_{G_1[G_2]}^P(u_2,v_2)\\
&= \sum_{u\in X_1, v_1v_2\in \mathrm{E}_2} \mu_\beta^P(u,v_1)\,\mu_\beta^P(u,v_2)\,d_{G_1[G_2]}^P(u,v_1)\,d_{G_1[G_2]}^P(u,v_2)\\
&+ \sum_{v_1,v_2\in X_2, u_1u_2\in \mathrm{E}_1} \mu_\beta^P(u_1,v_1)\,\mu_\beta^P(u_2,v_2)\,d_{G_1[G_2]}^P(u_1,v_1)\,d_{G_1[G_2]}^P(u_2,v_2) = K_1^P + K_2^P,\\
M_2^N(G_1[G_2]) &= \sum_{(u_1,v_1)(u_2,v_2)\in \mathrm{E}(G_1[G_2])} \mu_\beta^N(u_1,v_1)\,\mu_\beta^N(u_2,v_2)\,d_{G_1[G_2]}^N(u_1,v_1)\,d_{G_1[G_2]}^N(u_2,v_2)\\
&= \sum_{u\in X_1, v_1v_2\in \mathrm{E}_2} \mu_\beta^N(u,v_1)\,\mu_\beta^N(u,v_2)\,d_{G_1[G_2]}^N(u,v_1)\,d_{G_1[G_2]}^N(u,v_2)\\
&+ \sum_{v_1,v_2\in X_2, u_1u_2\in \mathrm{E}_2} \mu_\beta^N(u_1,v_1)\,\mu_\beta^N(u_2,v_2)\,d_{G_1[G_2]}^N(u_1,v_1)\,d_{G_1[G_2]}^N(u_2,v_2) = K_1^N + K_2^N, \text{ where}\\
K_1^P &= \sum_{u\in X_1, v_1v_2\in \mathrm{E}_2} \mu_\beta^P(u,v_1)\,\mu_\beta^P(u,v_2)\,d_{G_1[G_2]}^P(u,v_1)\,d_{G_1[G_2]}^P(u,v_2),\\
K_1^N &= \sum_{u\in X_1, v_1v_2\in \mathrm{E}_2} \mu_\beta^N(u,v_1)\,\mu_\beta^N(u,v_2)\,d_{G_1[G_2]}^N(u,v_1)\,d_{G_1[G_2]}^N(u,v_2) \text{ and}\\
K_2^P &= \sum_{v_1,v_2\in X_2, u_1u_2\in \mathrm{E}_2} \mu_\beta^P(u_1,v_1)\,\mu_\beta^P(u_2,v_2)\,d_{G_1[G_2]}^P(u_1,v_1)\,d_{G_1[G_2]}^P(u_2,v_2)\\
K_2^N &= \sum_{v_1,v_2\in X_2, u_1u_2\in \mathrm{E}_2} \mu_\beta^N(u_1,v_1)\,\mu_\beta^N(u_2,v_2)\,d_{G_1[G_2]}^N(u_1,v_1)\,d_{G_1[G_2]}^N(u_2,v_2).
\end{aligned}$$

Now

$$\begin{aligned}
K_1^P &= \sum_{u\in X_1, v_1v_2\in \mathrm{E}_2} \mu_\beta^P(u,v_1)\,\mu_\beta^P(u,v_2)\,d_{G_1[G_2]}^P(u,v_1)\,d_{G_1[G_2]}^P(u,v_2)\\
&\leq \sum_{u\in X_1, v_1v_2\in \mathrm{E}_2} \left[n_2\mu_{\beta_1}^P(u)d_{G_1}^P(u) + \mu_{\beta_2}^P(v_1)\,d_{G_2}^P(v_1)\right]\left[n_2\mu_{\beta_1}^P(u)d_{G_1}^P(u)\right.\\
&= n_2^2 \sum_{u\in X_1, v_1v_2\in \mathrm{E}_2} \left\{\mu_{\beta_1}^P(u)d_{G_1}^P(u)\right\}^2 + \sum_{u\in X_1, v_1v_2\in \mathrm{E}_2} \mu_{\beta_2}^P(v_1)\,d_{G_2}^P(v_1)\,\mu_{\beta_2}^P(v_2)\,d_{G_2}^P(v_2)\\
&+ n_2 \sum_{u\in X_1, v_1v_2\in \mathrm{E}_2} \mu_{\beta_1}^P(u)d_{G_1}^P(u)\left\{\mu_{\beta_2}^P(v_1)\,d_{G_2}^P(v_1) + \mu_{\beta_2}^P(v_2)\,d_{G_2}^P(v_2)\right\}\\
&\leq n_2^2 \sum_{v_1v_2\in \mathrm{E}_2} M_1^P(G_1) + \sum_{u\in X_1} M_2^P(G_2) + 2n_2\Delta_1^P\Delta_2^P \sum_{u\in X_1, v_1v_2\in \mathrm{E}_2} 1\\
&= n_2^2 m_2 M_1^P(G_1) + n_1 M_2^P(G_2) + 2n_1n_2m_2\Delta_1^P\Delta_2^P.\\
K_1^P &\leq n_2^2 m_2 M_1^P(G_1) + n_1 M_2^P(G_2) + 2n_1n_2m_2\Delta_1^P\Delta_2^P.\\
&= \sum_{u\in X_1, v_1v_2\in \mathrm{E}_2} \mu_\beta^N(u,v_1)\,\mu_\beta^N(u,v_2)\,d_{G_1}^N[G_2](u,v_1)\,d_{G_1}^N[G_2](u,v_2)\\
&\geq \sum_{u\in X_1, v_1v_2\in \mathrm{E}_2} \left[n_2\mu_{\beta_1}^N(u)d_{G_1}^N(u) + \mu_{\beta_2}^N(v_1)\,d_{G_2}^N(v_1)\right]\left[n_2\mu_{\beta_1}^N(u)d_{G_1}^N(u)\right.\\
&= n_2^2 \sum_{u\in X_1, v_1v_2\in \mathrm{E}_2} \left\{\mu_{\beta_1}^N(u)d_{G_1}^N(u)\right\}^2 + \sum_{u\in X_1, v_1v_2\in \mathrm{E}_2} \mu_{\beta_2}^N(v_1)\,d_{G_2}^N(v_1)\,\mu_{\beta_2}^N(v_2)\,d_{G_2}^N(v_2)
\end{aligned}$$

$$+ n_2 \sum_{u\in X_1, v_1v_2\in E_2} \mu^N_{\beta_1}(u) d^N_{G_1}(u) \left\{\mu^N_{\beta_2}(v_1) d^N_{G_2}(v_1) + \mu^N_{\beta_2}(v_2) d^N_{G_2}(v_2)\right\}$$
$$\geq n_2^2 \sum_{v_1v_2\in E_2} M^N_1(G_1) + \sum_{u\in X_1} M^N_2(G_2) + 2n_2\Delta^N_1\Delta^N_2 \sum_{u\in X_1, v_1v_2\in E_2} 1$$
$$= n_2^2 m_2 M^N_1(G_1) + n_1 M^N_2(G_2) + 2n_1n_2m_2\Delta^N_1\Delta^N_2.$$

So, $K^N_1 \geq n_2^2 m_2 M^N_1(G_1) + n_1 M^N_2(G_2) + 2n_1n_2m_2\Delta^N_1\Delta^N_2$.
Now

$$K_2{}^P = \sum_{v_1v_2\in X_2, u_1u_2\in E_1} \mu^P_\beta(u_1,v_1)\mu^P_\beta(u_2,v_2) d^P_{G_1}[G_2](u_1,v_1) d^P_{G_1[G_2]}(u_2,v_2)$$
$$\leq \sum_{v_1v_2\in X_2, u_1u_2\in E_1} \left[n_2\mu^P_{\beta_1}(u_1) d^P_{G_1}(u_1) + \mu^P_{\beta_2}(v_1) d^P_{G_2}(v_1)\right]\left[n_2\mu^P_{\beta_1}(u_2) d^P_{G_1}(u_2) + \mu^P_{\beta_2}(v_2) d^P_{G_2}(v_2)\right]$$
$$= n_2^2 \sum_{v_1v_2\in X_2, u_1u_2\in E_1} \mu^P_{\beta_1}(u_1) d^P_{G_1}(u_1) + \mu^P_{\beta_1}(u_2) d^P_{G_1}(u_2) + \sum_{v_1v_2\in X_2, u_1u_2\in E_1} \mu^P_{\beta_2}(v_1) d^P_{G_2}(v_1)\mu^P_{\beta_2}(v_2) d^P_{G_2}(v_2)$$
$$+ n_2 \sum_{v_1v_2\in X_2, u_1u_2\in E_1} \left[\mu^P_{\beta_1}(u_1) d^P_{G_1}(u_1)\mu^P_{\beta_2}(v_2) d^P_{G_2}(v_2)\right.$$
$$= n_2^2 \sum_{v_1v_2\in X_2, u_1u_2\in E_1} \mu^P_{\beta_1}(u_1) d^P_{G_1}(u_1)\mu^P_{\beta_1}(u_2) d^P_{G_1}(u_2) + \sum_{v_1v_2\in X_2, u_1u_2\in E_1} \mu^P_{\beta_2}(v_1) d^P_{G_2}(v_1)\mu^P_{\beta_2}(v_2) d^P_{G_2}(v_2)$$
$$+ \sum_{(v_1v_2)\in X_2\times X_2/E_2, u_1u_2\in E_1} \mu^P_{\beta_2}(v_1) d^P_{G_2}(v_1)\mu^P_{\beta_2}(v_2) d^P_{G_2}(v_2) + \sum_{v_1v_2\in X_2, u_1u_2\in E_1} \left[\mu^P_{\beta_1}(u_1) d^P_{G_1}(u_1)\mu^P_{\beta_2}(v_2) d^P_{G_2}(v_2)\right.$$
$$\left.+\mu^P_{\beta_1}(u_2) d^P_{G_1}(u_2)\mu^P_{\beta_2}(v_1) d^P_{G_2}(v_1)\right] + \sum_{v_1v_2\in X_2} \sum_{M^P_2(G_1)+} \sum_{u_1u_2\in E_1} \left(\Delta^P_2\right)^2 + n_2 + \sum_{v_1v_2\in X_2, u_1u_2\in E_1} 2\Delta^P_1\Delta^P_2$$
$$= n_2^4 M^P_2(G_1) + m_1 M^P_2(G_2) + m_1\left(n_2^2 - m_2\right)\Delta_2^2 + 2n_2^3 m_1\Delta^P_1\Delta^P_2.$$

So, $K_2{}^P \leq n_2^4 M^P_2(G_1) + m_1 M^P_2(G_2) + m_1\left(n_2^2 - m_2\right)\Delta_2^2 + 2n_2^3 m_1\Delta^P_1\Delta^P_2$.

$$K_2{}^N = \sum_{v_1v_2\in X_2, u_1u_2\in E_1} \mu^N_\beta(u_1,v_1)\mu^N_\beta(u_2,v_2) d^N_{G_1[G_2]}(u_1,v_1) d^N_{G_1[G_2]}(u_2,v_2)$$
$$\geq \sum_{v_1v_2\in X_2, u_1u_2\in E_1} \left[n_2\mu^N_{\beta_1}(u_1) d^N_{G_1}(u_1) + \mu^N_{\beta_2}(v_1) d^N_{G_2}(v_1)\right]\left[n_2\mu^N_{\beta_1}(u_2) d^N_{G_1}(u_2) + \mu^N_{\beta_2}(v_2) d^N_{G_2}(v_2)\right]$$
$$= n_2^2 \sum_{v_1v_2\in X_2, u_1u_2\in E_1} \mu^N_{\beta_1}(u_1) d^N_{G_1}(u_1) + \mu^N_{\beta_1}(u_2) d^N_{G_1}(u_2) + \sum_{v_1v_2\in X_2, u_1u_2\in E_1} \mu^N_{\beta_2}(v_1) d^N_{G_2}(v_1)\mu^N_{\beta_2}(v_2) d^N_{G_2}(v_2)$$
$$+ n_2 \sum_{v_1v_2\in X_2, u_1u_2\in E_1} \left[\mu^N_{\beta_1}(u_1) d^N_{G_1}(u_1)\mu^N_{\beta_2}(v_2) d^N_{G_2}(v_2) + \mu^N_{\beta_1}(u_2) d^N_{G_1}(u_2)\mu^N_{\beta_2}(v_1) d^N_{G_2}(v_1)\right]$$
$$= n_2^2 \sum_{v_1v_2\in X_2, u_1u_2\in E_1} \mu^N_{\beta_1}(u_1) d^N_{G_1}(u_1)\mu^N_{\beta_1}(u_2) d^N_{G_1}(u_2) + \sum_{v_1v_2\in X_2, u_1u_2\in E_1} \mu^N_{\beta_2}(v_1) d^N_{G_2}(v_1)\mu^N_{\beta_2}(v_2) d^N_{G_2}(v_2)$$
$$+ \sum_{(v_1v_2)\in X_2\times X_2/E_2, u_1u_2\in E_1} \mu^N_{\beta_2}(v_1) d^N_{G_2}(v_1)\mu^N_{\beta_2}(v_2) d^N_{G_2}(v_2) + \sum_{v_1v_2\in X_2, u_1u_2\in E_1} \left[\mu^N_{\beta_1}(u_1) d^N_{G_1}(u_1)\mu^N_{\beta_2}(v_2) d^N_{G_2}(v_2)\right.$$
$$\left.+\mu^N_{\beta_1}(u_2) d^N_{G_1}(u_2)\mu^N_{\beta_2}(v_1) d^N_{G_2}(v_1)\right]$$
$$\geq n_2^2 \sum_{v_1v_2\in X_2} M^N_2(G_1) + \sum_{u_1u_2\in E_1} M^N_2(G_2) + \sum_{(v_1v_2)\in X_2\times X_2/E_2, u_1u_2\in E_1} \left(\Delta^N_2\right)^2 + n_2 + \sum_{v_1v_2\in X_2, u_1u_2\in E_1} 2\Delta^N_1\Delta^N_2$$
$$= n_2^4 M^N_2(G_1) + m_1 M^{*N}(G_2) + m_1\left(n_2^2 - m_2\right)\left(\Delta^N_2\right)^2 + 2n_2^3 m_2\Delta^N_1\Delta^N_2.$$

So, $K_2{}^N = n_2^4 M^N_2(G_1) + m_1 M^N_2(G_2) + m_1\left(n_2^2 - m_2\right)\left(\Delta^N_2\right)^2 + 2n_2^3 m_1\Delta^N_1\Delta^N_2$.

Therefore,

$$\begin{aligned}M_2^P\left(G_1\left[G_2\right]\right) &= K_1^P + K_2^P \leq n_2^2 m_2 M_1^P\left(G_1\right) + n_2^4 M_2^P\left(G_1\right) + \left(m_1+n_1\right) M_2^P\left(G_2\right)\\ &+ 2n_2\left(n_1 m_2 + m_1 n_2^2\right)\Delta_1^P\Delta_2^P + m_1\left(n_2^2 - m_2\right)\left(\Delta_2^P\right)^2.\\ M_2^N\left(G_1\left[G_2\right]\right) &= K_1^P + K_2^P \geq n_2^2 m_2 M_1^N\left(G_1\right) + n_2^4 M_2^N\left(G_1\right) + \left(m_1+n_1\right) M_2^N\left(G_2\right)\\ &+ 2n_2\left(n_1 m_2 + m_1 n_2^2\right)\Delta_1^N\Delta_2^N + m_1\left(n_2^2 - m_2\right)\left(\Delta_2^N\right)^2.\end{aligned}$$

□

**Corollary 4.5.**

$$\begin{aligned}(i)\ M_2^P\left(G_1\left[G_2\right]\right) &\leq n_2^2\left(n_1-1\right)^2\left(n_1 m_2 + m_1 n_2^2\right),\\ M_2^N\left(G_1\left[G_2\right]\right) &\geq -n_2^2\left(n_1+1\right)2\left(n_1 m_2 + m_1 n_2^2\right),\end{aligned}$$

$$\begin{aligned}(ii)\ M_2^P\left(G_1\left[G_2\right]\right) &\leq 4m_1^2 n - m_1 n_2^2 + m_1 n_2^4\left(n_1-1\right)^2 + m_2\left(m_1+n_1\right)\left(n_2-1\right)^2 + 2n_2\left(n_2-1\right)\left(n_2-1\right)\\ &\left(n_1 m_2 + m_1 n_2^2\right) + m_1\left(n_2-1\right)^2\left(n_2^2 - m_2\right)\\ M_2^N\left(G_1\left[G_2\right]\right) &\geq -\Big(4m_1^2 n - m_1 n_2^2 + m_1 n_2^4\left(n_1-1\right)^2 + m_2\left(m_1+n_1\right)\\ &\left(n_2-1\right)^2 + 2n_2\left(n_2-1\right)\left(n_2-1\right)\left(n_1 m_2 + m_1 n_2^2\right) + m_1\left(n_2-1\right)^2\left(n_2^2 - m_2\right)\Big).\end{aligned}$$

**Theorem 4.6.**

$$\begin{aligned}&M_2^P\left(G_1\right) + M_2^P\left(G_2\right) \leq M_2^P\left(G_1+G_2\right) \leq M_2^P\left(G_1\right) + M_2^P\left(G_2\right)\\ &+ m_1 n_2\left(2\Delta_1^P + n_2\right) + n_1 m_2\left(2\Delta_2^P + n_1\right) + n_1 n_2\left(\Delta_1^P + n_2\right)\left(\Delta_2^P + n_1\right),\end{aligned}$$

*and*

$$\begin{aligned}&M_2^N\left(G_1\right) + M_2^N\left(G_2\right) \geq M_2^N\left(G_1+G_2\right) \geq M_2^N\left(G_1\right) + M_2^N\left(G_2\right)\\ &+ m_1 n_2\left(2\Delta_1^N + n_2\right) + n_1 m_2\left(2\Delta_2^N + n_1\right) + n_1 n_2\left(\Delta_1^N + n_2\right)\left(\Delta_2^N + n_1\right)\end{aligned}$$

*Proof.* As $G_1 + G_2$ is the join graph of $G_1$ and $G_2$, then for $u, v, u_1, v_1 \in X$,

$$\mu_\beta^P(u) = \begin{cases}\mu_{\beta_1}^P(u), & u \in X_1\\ \mu_{\beta_2}^P(u), & u \in X_2\end{cases}$$

$$\mu_\beta^N(u) = \begin{cases}\mu_{\beta_1}^N(u), & u \in X_1\\ \mu_{\beta_2}^N(u), & u \in X_2\end{cases}$$

and

$$\mu_\Omega^P(uv) = \begin{cases}\wedge\left\{\mu_{\beta_1}^P(u), \mu_{\beta_2}^P(u)\right\} & u \in X_1 \text{ and } v \in X_2\\ \mu_{\Omega_1}^P(uv) & uv \in E_1\\ \mu_{\Omega_2}^P(uv) & uv \in E_2\\ 0 \text{ otherwise.}\end{cases}$$

$$\mu_\Omega^N(uv) = \begin{cases}\vee\left\{\mu_{\beta_1}^N(u), \mu_{\beta_2}^N(u)\right\} & u \in X_1 \text{ and } v \in X_2\\ \mu_{\Omega_1}^N(uv) & uv \in E_1\\ \mu_{\Omega_2}^N(uv) & uv \in E_2\\ 0 \text{ otherwise.}\end{cases}$$

Then,

$$d^P_{G_1+G_2}(u)=\begin{cases} d^P_{G_1}(u)+\sum\limits_{v\in X_2}\wedge\left\{\mu^P_{\beta_1}(u),\mu^P_{\beta_2}(v)\right\}, & u\in X_1\\ d^P_{G_2}(u)+\sum\limits_{v\in X_1}\wedge\left\{\mu^P_{\beta_2}(u),\mu^P_{\beta_1}(v)\right\}, & u\in X_2\end{cases}$$
$$d^N_{G_1+G_2}(u)=\begin{cases} d^N_{G_1}(u)+\sum\limits_{v\in X_2}\vee\left\{\mu^N_{\beta_1}(u),\mu^N_{\beta_2}(v)\right\}, & u\in X_1\\ d^N_{G_2}(u)+\sum\limits_{v\in X_1}\vee\left\{\mu^N_{\beta_2}(u),\mu^N_{\beta_1}(v)\right\}, & u\in X_2\end{cases}, \tag{4.1}$$

Hence

$$d^P_{G_1+G_2}(u)\geq\begin{cases} d^P_{G_1}(u), & u\in X_1\\ d^P_{G_2}(u), & u\in X_2,\end{cases}$$
$$d^N_{G_1+G_2}(u)\leq\begin{cases} d^N_{G_1}(u), & u\in X_1\\ d^N_{G_2}(u), & u\in X_2.\end{cases} \tag{4.2}$$

Also, from Equation (4.1), we get

$$d^P_{G_1+G_2}(u)\leq\begin{cases} d^P_{G_1}(u)+\sum\limits_{v\in X_2}\mu^P_{\beta_1}(u), & u\in X_1\\ d^P_{G_2}(u)+\sum\limits_{v\in X_1}\mu^P_{\beta_2}(u), & u\in X_2\end{cases}$$
$$d^N_{G_1+G_2}(u)\geq\begin{cases} d^N_{G_1}(u)+\sum\limits_{v\in X_2}\mu^N_{\beta_1}(u), & u\in X_1\\ d^N_{G_2}(u)+\sum\limits_{v\in X_1}\mu^N_{\beta_2}(u), & u\in X_2\end{cases}, \tag{4.3}$$

So

$$d^P_{G_1+G_2}(u)\leq\begin{cases} d^P_{G_1}(u)+n_2\mu^P_{\beta_1}(u), & u\in X_1\\ d^P_{G_2}(u)+n_1\mu^P_{\beta_2}(u), & u\in X_2\end{cases}$$
$$d^N_{G_1+G_2}(u)\geq\begin{cases} d^N_{G_1}(u)+n_2\mu^N_{\beta_1}(u), & u\in X_1\\ d^N_{G_2}(u)+n_1\mu^N_{\beta_2}(u), & u\in X_2\end{cases}. \tag{4.4}$$

Then second $ZI$ of $G_1+G_2$ is:

$$\begin{aligned} M_2^P\left(G_1+G_2\right) &= \sum_{uv\in E(G_1+G_2)}\mu^P_\beta(u)\mu^P_\beta(v)d^P_{G_1+G_2}(u)d^P_{G_1+G_2}(v)=\sum_{uv\in E_1}\mu^P_\beta(u)\mu^P_\beta(v)d^P_{G_1+G_2}(u)d^P_{G_1+G_2}(v)\\ &+\sum_{uv\in E_2}\mu^P_\beta(u)\mu^P_\beta(v)d^P_{G_1+G_2}(u)d^P_{G_1+G_2}(v)+\sum_{u\in X_1,v\in X_2}\mu^P_\beta(u)\mu^P_\beta(v)d^P_{G_1+G_2}(u)d^P_{G_1+G_2}(v)\\ &\geq\sum_{uv\in E_1}\mu^P_{\beta_1}(u)\mu^P_{\beta_1}(v)d^P_{G_1}(u)d^P_{G_1}(v)+\sum_{uv\in E_2}\mu^P_{\beta_2}(u)\mu^P_{\beta_2}(v)d^P_{G_2}(u)d^P_{G_2}(v)\\ &\geq M_2^P\left(G_1\right)+M_2^P\left(G_2\right)\\ M_2^N\left(G_1+G_2\right) &= \sum_{uv\in E(G_1+G_2)}\mu^N_\beta(u)\mu^N_\beta(v)d^N_{G_1+G_2}(u)d^N_{G_1+G_2}(v)=\sum_{uv\in E_1}\mu^N_\beta(u)\mu^N_\beta(v)d^N_{G_1+G_2}(u)d^N_{G_1+G_2}(v)\\ &+\sum_{uv\in E_2}\mu^N_\beta(u)\mu^N_\beta(v)d^N_{G_1+G_2}(u)d^N_{G_1+G_2}(v)+\sum_{u\in X_1,v\in X_2}\mu^N_\beta(u)\mu^N_\beta(v)d^N_{G_1+G_2}(u)d^N_{G_1+G_2}(v)\end{aligned}$$

$$\leq \sum_{uv\in E_1} \mu^N_{\beta_1}(u)\mu^N_{\beta_1}(v)d^N_{G_1}(u)d^N_{G_1}(v) + \sum_{uv\in E_2} \mu^N_{\beta_2}(u)\mu^N_{\beta_2}(v)d^N_{G_2}(u)d^N_{G_2}(v) \leq M^N_2(G_1) + M^N_2(G_2)$$

Again,

$$\begin{aligned}
M^P_2(G_1+G_2) &= \sum_{uv\in \mathrm{E}(G_1+G_2)} \mu^P_\beta(u)\mu^P_\beta(v)d^P_{G_1+G_2}(u)d^P_{G_1+G_2}(v) = \sum_{uv\in E_1} \mu^P_\beta(u)\mu^P_\beta(v)d^P_{G_1+G_2}(u)d^P_{G_1+G_2}(v)\\
&+ \sum_{uv\in E_2} \mu^P_\beta(u)\mu^P_\beta(v)d^P_{G_1+G_2}(u)d^P_{G_1+G_2}(v) + \sum_{u\in X_1, v\in X_2} \mu^P_\beta(u)\mu^P_\beta(v)d^P_{G_1+G_2}(u)d^P_{G_1+G_2}(v)\\
&\leq \sum_{uv\in E_1} \mu^P_{\beta_1}(u)\mu^P_{\beta_1}(v)\left\{d^P_{G_1}(u) + n_2\mu^P_{\beta_1}(u)\right\}\left\{d^P_{G_1}(v) + n_2\mu^P_{\beta_1}(v)\right\}\\
&+ \sum_{uv\in E_2} \mu^P_{\beta_2}(u)\mu^P_{\beta_2}(v)\left\{d^P_{G_2}(u) + n_1\mu^P_{\beta_2}(u)\right\}\left\{d^P_{G_2}(v)\mu^P_{\beta_2}(v)\right\} \qquad (\text{by}(4.4))\\
&+ \sum_{u\in X_1, v\in X_2} \mu^P_{\beta_1}(u)\mu^P_{\beta_2}(v)\left\{d^P_{G_1}(u) + n_2\mu^P_{\beta_1}(u)\right\}\left\{d^P_{G_1}(v) + n_1\mu^P_{\beta_1}(v)\right\} = K^P_1 + K^P_2 + K^P_3,\\
M^N_2(G_1+G_2) &= \sum_{uv\in \mathrm{E}(G_1+G_2)} \mu^N_\beta(u)\mu^N_\beta(v)d^N_{G_1+G_2}(u)d^N_{G_1+G_2}(v) = \sum_{uv\in E_1} \mu^N_\beta(u)\mu^N_\beta(v)d^N_{G_1+G_2}(u)d^N_{G_1+G_2}(v)\\
&+ \sum_{uv\in E_2} \mu^N_\beta(u)\mu^N_\beta(v)d^N_{G_1+G_2}(u)d^N_{G_1+G_2}(v) + \sum_{u\in X_1, v\in X_2} \mu^N_\beta(u)\mu^N_\beta(v)d^N_{G_1+G_2}(u)d^N_{G_1+G_2}(v)\\
&\geq \sum_{uv\in E_1} \mu^N_{\beta_1}(u)\mu^N_{\beta_1}(v)\left\{d^N_{G_1}(u) + n_2\mu^N_{\beta_1}(u)\right\}\left\{d^N_{G_1}(v) + n_2\mu^N_{\beta_1}(v)\right\}\\
&+ \sum_{uv\in E_2} \mu^N_{\beta_2}(u)\mu^N_{\beta_2}(v)\left\{d^N_{G_2}(u) + n_1\mu^N_{\beta_2}(u)\right\}\left\{d^N_{G_2}(v) + n_1\mu^N_{\beta_2}(v)\right\}\\
&+ \sum_{u\in X_1, v\in X_2} \mu^N_{\beta_1}(u)\mu^N_{\beta_2}(v)\left\{d^N_{G_1}(u) + n_2\mu^N_{\beta_1}(u)\right\}\left\{d^N_{G_1}(v) + n_1\mu^N_{\beta_1}(v)\right\}\\
&= K^N_1 + K^N_2 + K^N_3, \text{where}\\
K^P_1 &= \sum_{uv\in E_1} \mu^P_{\beta_1}(u)\mu^P_{\beta_1}(v)\left\{d^P_{G_1}(u) + n_2\mu^P_{\beta_1}(u)\right\}\left\{d^P_{G_1}(v) + n_2\mu^P_{\beta_1}(v)\right\}\\
K^P_2 &= \sum_{uv\in E_2} \mu^P_{\beta_2}(u)\mu^P_{\beta_2}(v)\left\{d^P_{G_2}(u) + n_1\mu^P_{\beta_2}(u)\right\}\left\{d^P_{G_2}(v) + n_1\mu^P_{\beta_2}(v)\right\}\\
K^P_3 &= \sum_{u\in X_1, v\in X_2} \mu^P_{\beta_1}(u)\mu^P_{\beta_2}(v)\left\{d^P_{G_1}(u) + n_2\mu^P_{\beta_1}(u)\right\}\left\{d^P_{G_1}(v) + n_1\mu^P_{\beta_1}(v)\right\}
\end{aligned}$$

And

$$\begin{aligned}
K^N_1 &= \sum_{uv\in E_1} \mu^N_{\beta_1}(u)\mu^N_{\beta_1}(v)\left\{d^N_{G_1}(u) + n_2\mu^N_{\beta_1}(u)\right\}\left\{d^N_{G_1}(v) + n_2\mu^N_{\beta_1}(v)\right\}\\
K^N_2 &= \sum_{uv\in E_2} \mu^N_{\beta_2}(u)\mu^N_{\beta_2}(v)\left\{d^N_{G_2}(u) + n_1\mu^N_{\beta_2}(u)\right\}\left\{d^N_{G_2}(v) + n_1\mu^N_{\beta_2}(v)\right\}\\
K^N_3 &= \sum_{u\in X_1, v\in X_2} \mu^N_{\beta_1}(u)\mu^N_{G_2}(v)\left\{d^N_{G_1}(u) + n_2\mu^N_{G_1}(u)\right\}\left\{d^N_{G_1}(v) + n_1\mu^N_{\beta_1}(v)\right\}.
\end{aligned}$$

Now

$$K^P_1 = \sum_{uv\in E_1} \mu^P_{\beta_1}(u)\mu^P_{\beta_1}(v)\left\{d^P_{G_1}(u) + n_2\mu^P_{\beta_1}(u)\right\}\left\{d^P_{G_1}(v) + n_2\mu^P_{\beta_1}(v)\right\}$$

$$
\begin{aligned}
&= \sum_{uv\in E_1} \mu^P_{\beta_1}(u)\mu^P_{\beta_1}(v)\left[d^P_{G_1}(u)d^P_{G_1}(v) + n_2\left\{\mu^P_{\beta_1}(u)d^P_{G_1}(v) + \mu^P_{\beta_1}(v)d^P_{G_1}(u)\right\} + n_2^2\mu^P_{\beta_1}(u)\mu^P_{\beta_1}(v)\right] \\
&\leq M_2^P\left(G_1\right) + 2m_1n_2\Delta_1^P + m_1n_2^2. \\
K_1^N &= \sum_{uv\in E_1} \mu^N_{\beta_1}(u)\mu^N_{\beta_1}(v)\left\{d^N_{G_1}(u) + n_2\mu^N_{\beta_1}(u)\right\}\left\{d^N_{G_1}(v) + n_2\mu^N_{\beta_1}(v)\right\} \\
&= \sum_{uv\in E_1} \mu^N_{\beta_1}(u)\mu^N_{\beta_1}(v)\left[d^N_{G_1}(u)d^N_{G_1}(v) + n_2\left\{\mu^N_{\beta_1}(u)d^P_{G_1}(v) + \mu^N_{\beta_1}(v)d^N_{G_1}(u)\right\} + n_2^2\mu^N_{\beta_1}(u)\mu^N_{\beta_1}(v)\right] \\
&\geq M_2^N\left(G_1\right) + 2m_1n_2\Delta_1^N + m_1n_2^2.
\end{aligned}
$$

So $k_1^P \leq M_2^P\left(G_1\right) + 2m_1n_2\Delta_1^P + m_1n_2^2, k_1^N \geq M_2^N\left(G_1\right) + 2m_1n_2\Delta_1^N + m_1n_2^2$. Similarly, $k_2^P \leq M_2^P\left(G_2\right) + 2m_2n_2\Delta_2^P + m_2n_1^2, k_2^N \geq M_2^N\left(G_2\right) + 2m_2n_2\Delta_2^N + m_2n_1^2$, and $k_3^P \leq n_1n_2\left(\Delta_1^P + n_2\right)\left(\Delta_2^P + n_1\right), k_3^N \geq n_1n_2\left(\Delta_1^N + n_2\right)\left(\Delta_2^N + n_1\right)$. Therefore, $M_2^P\left(G_1 + G_2\right) \leq k_1^P + k_2^P + k_3^P \leq M_2^P\left(G_1\right) + M_2^P\left(G_2\right) + m_1n_2\left(2\Delta_1^P + n_2\right) + n_1m_2\left(2\Delta_2^P + n_1\right) + n_1n_2\left(\Delta_1^P + n_2\right)\left(\Delta_2^P + n_1\right), M_2^N\left(G_1 + G_2\right) \leq k_1^N + k_2^N + k_3^N \geq M_2^N\left(G_1\right) + M_2^N\left(G_2\right) + m_1n_2\left(2\Delta_1^N + n_2\right) + n_1m_2\left(2\Delta_2^N + n_1\right) + n_1n_2\left(\Delta_1^N + n_2\right)\left(\Delta_2^N + n_1\right)$. □

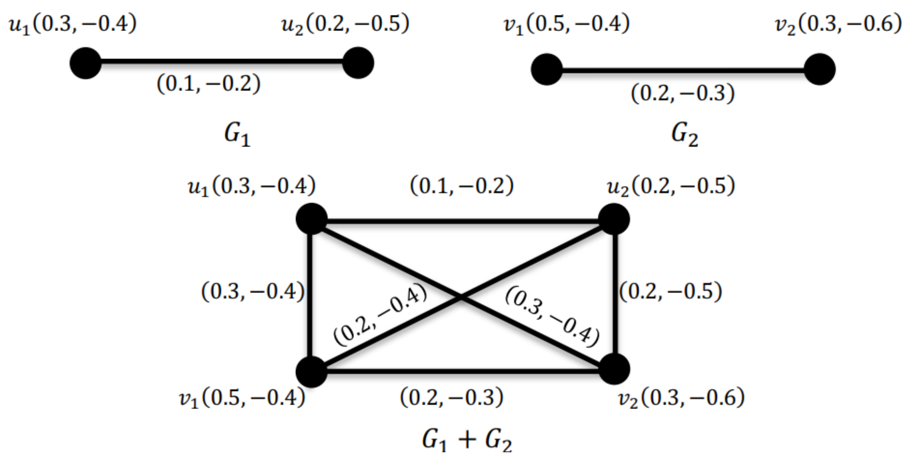


Figure 6: The join of two $BFGG_1$ and $G_2$.

**Example 4.7.** Suppose $G_1 = (X_1, \beta_1, \Omega_1)$ and $G_2 = (X_2, \beta_2, \Omega_2)$ are a $BFG$ shown in figure. 6 with vertex set $X_1 = \{u_1, u_2\}, E_1 = \{u_1u_2\}, X_2 = \{v_1, v_2\}, E_1 = \{v_1v_2\}, X_1 + X_2 = \{u_1, u_2, v_1, v_2\}$ Such that

| $X_1$ | $u_1$ | $u_2$ |
|---|---|---|
| $\mu^P_{\beta_1}$ | 0.3 | 0.2 |
| $\mu^N_{\beta_1}$ | -0.4 | -0.5 |

| $X_2$ | $v_1$ | $v_2$ |
|---|---|---|
| $\mu^P_{\beta_2}$ | 0.5 | 0.3 |
| $\mu^N_{\beta_2}$ | -0.4 | -0.6 |

Hence the following tables give $G_1 + G_2$.

| $X_1 + X_2$ | $u_1$ | $u_2$ | $v_1$ | $v_2$ |
|---|---|---|---|---|
| $\mu^P_{\beta_1} + \mu^P_{\beta_2}$ | 0.3 | 0.2 | 0.5 | 0.3 |
| $\mu^N_{\beta_1} + \mu^N_{\beta_2}$ | -0.4 | -0.5 | -0.4 | -0.6 |

| $E_1 + E_2$ | $u_1u_2$ | $u_1v_1$ | $u_1v_2$ | $u_2v_1$ | $u_2v_2$ | $v_1v_2$ |
|---|---|---|---|---|---|---|
| $\mu^P_{\Omega_1} + \mu^P_{\Omega_2}$ | 0.1 | 0.3 | 0.3 | 0.2 | 0.2 | 0.2 |
| $\mu^N_{\Omega_1} + \mu^N_{\Omega_2}$ | -0.2 | -0.4 | -0.4 | -0.4 | -0.5 | -0.3 |

| $E_1$ | $u_1u_2$ |
|---|---|
| $\mu^P_{\Omega_1}$ | 0.1 |
| $\mu^N_{\Omega_1}$ | -0.2 |

| $E_2$ | $v_1v_2$ |
|---|---|
| $\mu^P_{\Omega_2}$ | 0.2 |
| $\mu^N_{\Omega_2}$ | -0.3 |

Therefore, the degree is all vertices $G_1 + G_2$

| $d(X_1+X_2)$ | $u_1$ | $u_2$ | $v_1$ | $v_2$ |
|---|---|---|---|---|
| $d^P$ | 0.7 | 0.5 | 0.7 | 0.7 |
| $d^N$ | -1 | -1.1 | -1.1 | -1.2 |

$$\begin{aligned}
M_2(G_1+G_2) &= \left(M_2^P(G_1+G_2), M_2^N(G_1+G_2)\right)\\
&= \left(\sum_{uv\in \mathrm{E}} \mu_\beta^P(u)d^P(u)\mu_\beta^P(v)d^P(v), \sum_{uv\in \mathrm{E}} \mu_\beta^N(u)d^N(u)\mu_\beta^N(v)d^N(v)\right)\\
M_2^P(G_1+G_2) &= [0.3\times 0.7\times 0.2\times 0.5]+[0.3\times 0.7\times 0.5\times 0.7]+[0.3\times 0.7\times 0.3\times 0.7]+[0.2\times 0.5\times 0.5\times 0.7]\\
&\quad +[0.2\times 0.5\times 0.3\times 0.7]+[0.5\times 0.7\times 0.3\times 0.7]=4.978\\
M_2^N(G_1+G_2) &= [-0.4\times -1\times -0.5\times -1.1]+[-0.4\times -1\times -0.4\times -1.1]\\
&\quad +[-0.4\times -1\times -0.6\times -1.2]+[-0.5\times -1.1\times -0.4\times -1.1]\\
&\quad +[-0.5\times -1.1\times -0.6\times -1.2]+[-0.4\times -1.1\times -0.6\times -1.2]=1.6388.
\end{aligned}$$

**Corollary 4.8.**

$$\begin{aligned}
(i)\ & M_2^P(G_1+G_2) \le (m_1+m_2+n_1n_2)(n_1+n_2-1)^2,\\
& M_2^N(G_1+G_2) \ge (m_1+m_2+n_1n_2)(n_1+n_2-1)^2
\end{aligned}$$

*and*

$$\begin{aligned}
(ii)\ M_2^P(G_1+G_2) &\le m_1(n_1-1)^2+m_2(n_2-1)^2+m_1n_2(2n_1+n_2-1)\\
&\quad + m_2n_1(n_1+2n_2-1)+n_1n_2(n_1+n_2-1)^2,\\
M_2^N(G_1+G_2) &\ge m_1(n_1-1)^2+m_2(n_2-1)^2+m_1n_2(2n_1+n_2-1)\\
&\quad + m_2n_1(n_1+2n_2-1)+n_1n_2(n_1+n_2-1)^2.
\end{aligned}$$

**Theorem 4.9.** $M_2^P(G_1\cup G_2) \ge M_2^P(G_1)+M_2^P(G_2)-k^P\left(\Delta^P\right)^2$, $M_2^N(G_1\cup G_2) \le M_2^N(G_1)+M_2^N(G_2)-k^N\left(\Delta^N\right)^2$
*where* $r=|E_1\cap E_2|, \Delta^P=\max\left\{\Delta_1^P,\Delta_2^P\right\}, \Delta^N=\min\left\{\Delta_1^N,\Delta_2^N\right\}$.

*Proof.* As $G_1\cup G_2$ is union $BFG$ of $G_1$ and $G_2$, then

$$d_{G_1\cup G_2}^P(u)=\begin{cases} d_{G_1}^P(u) \text{ if } u\in V_1\backslash V_2\\ d_{G_2}^P(u) \text{ if } u\in V_2\backslash V_1\\ \max\left\{d_{G_1}^P(u), d_{G_1}^P(u)\right\} \text{ if } u\in V_1\cap V_2\end{cases}$$
$$d_{G_1\cup G_2}^N(u)=\begin{cases} d_{G_1}^N(u) \text{ if } u\in V_1\backslash V_2\\ d_{G_2}^N(u) \text{ if } u\in V_2\backslash V_1\\ \min\left\{d_{G_1}^N(u), d_{G_1}^N(u)\right\} \text{ if } u\in V_1\cap V_2\end{cases}$$

Then

$$\begin{aligned}
M_2^P(G_1\cup G_2) &= \sum_{uv\in E}\mu_\beta^P(u)\mu_\beta^P(v)d_{G_1\cup G_2}^P(u)d_{G_1\cup G_2}^P(v)=\sum_{uv\in E_1}\mu_\beta^P(u)\mu_\beta^P(v)d_{G_1\cup G_2}^P(u)d_{G_1\cup G_2}^P(v)\\
&\quad +\sum_{uv\in E_2}\mu_\beta^P(u)\mu_\beta^P(v)d_{G_1\cup G_2}^P(u)d_{G_1\cup G_2}^P(v)-\sum_{uv\in E_1\cap E_2}\mu_\beta^P(u)\mu_\beta^P(v)d_{G_1\cup G_2}^P(u)d_{G_1\cup G_2}^P(v)\\
&\quad \ge M_2^P(G_1)+M_2^P(G_2)-r\left(\Delta^P\right)^2,
\end{aligned}$$

$$M_2^N(G_1 \cup G_2) = \sum_{uv \in E} \mu_\beta^N(u)\mu_\beta^N(v) d_{G_1 \cup G_2}^N(u) d_{G_1 \cup G_2}^N(v) = \sum_{uv \in E_1} \mu_\beta^N(u)\mu_\beta^N(v) d_{G_1 \cup G_2}^N(u) d_{G_1 \cup G_2}^N(v)$$
$$+ \sum_{uv \in E_2} \mu_\beta^N(u)\mu_\beta^N(v) d_{G_1 \cup G_2}^N(u) d_{G_1 \cup G_2}^N(v) - \sum_{uv \in E_1 \cap E_2} \mu_\beta^N(u)\mu_\beta^N(v) d_{G_1 \cup G_2}^N(u) d_{G_1 \cup G_2}^N(v)$$
$$\geq M_2^N(G_1) + M_2^N(G_2) - r\left(\Delta^N\right)^2.$$

□

## 5. Application of second Zagreb index to find the most suitable place to build a hospital

Hospitals play a crucial role in improving public health by providing comprehensive healthcare services to individuals within communities. The importance of hospitals goes beyond treating illnesses and injuries. They also focus on preventive care, health education, and disease management to enhance the overall well-being of the public. Through their specialized health care professionals and advanced facilities, hospitals significantly contribute to disease prevention, early diagnosis, and timely treatment, thereby reducing the disease burden on society. In addition, hospitals serve as vital hubs for research, innovation, and public health initiatives, working in collaboration with various stakeholders to address healthcare challenges and promote healthy lifestyles. By offering a wide range of medical services, including primary care, specialized care, diagnostic procedures, and therapeutic treatments, hospitals meet the diverse healthcare needs of individuals across different age groups and backgrounds. Overall, the role of hospitals in advancing public heath cannot be overstated public health cannot be overstated, as they remain at the forefront of efforts to improve health outcomes and the quality of life for entire communities. Therefore,
considering the importance of building a hospital between four cities of Mazandaran province, namely Sari (AS), Neka (Ne), Behshahr (Be), and Qaimshahr (Qa), we went to determine the best place to build it.

(i) Sari is located in the center of the province and is at the highest level in terms of population and area.

(ii) Qaimshahr is superior to Neka both in terms of population and geographical area and is located in the southern part of the province.

(iii) Neka and Behshahr are located in the east of the prorince. Behshahr is superior to Neka. Consider the Bipolar fuzzy graph in Figure 7. The nodes shows the cities and the edges represent the amount of intercity traffic during a day. The population and area of each city are listed in Table **??**.

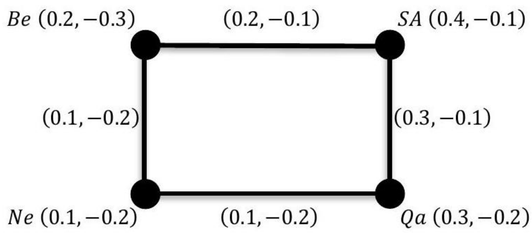


Figure 7: The join of two $BFGG_1$ and $G_2$.

Table 3: Name of cities and their population and area.

| City | Population | Area |
|---|---|---|
| Sari (SA) | 504298 | 30744 |
| Behshahr (Be) | 168769 | 141627 |
| Neka (Ne) | 119511 | 13588 |
| Qaomshahr (Qa) | 309199 | 54585 |

The Weight of nodes in $G$

| $X$ | SA | Be | Ne | Qa |
|---|---|---|---|---|
| $\mu_\beta^P$ | 0.4 | 0.2 | 0.1 | 0.3 |
| $\mu_\beta^N$ | -0.1 | -0.3 | - 0.2 | -0.2 |

Therefore, the degree is all vertices $G$

| $d_G$ | SA | Be | Ne | Qa |
|---|---|---|---|---|
| $d^P$ | 0.5 | 0.3 | 0.2 | 0.4 |
| $d^N$ | -0.2 | -0.3 | -0.4 | -0.3 |

$$M_2(G) = \left(M_2^P(G), M_2^N(G)\right) = \left(\sum_{uv\in \mathrm{E}} \mu_\beta^P(u)d^P(u)\mu_\beta^P(v)d^P(v), \sum_{uv\in \mathrm{E}} \mu_\beta^N(u)d^N(u)\mu_\beta^N(v)d^N(v)\right)$$

$$M_2^P(G) = [0.2\times 0.3\times 0.4\times 0.5] + [0.4\times 0.5\times 0.3\times 0.4] + [0.3\times 0.4\times 0.1\times 0.2] + [0.1\times 0.2\times 0.2\times 0.3] = 0.0396$$

$$\begin{aligned} M_2^N(G) &= [-0.3\times -0.3\times -0.1\times -0.2] + [-0.1\times -0.2\times -0.2\times -0.3] \\ &+ [-0.2\times -0.3\times -0.2\times -0.4] + [-0.2\times -0.4\times -0.3\times -0.3] \end{aligned}$$

Now we want to check the effect of removing each of the vertices on the amount of Zagreb index , so have

| $E$ | Be SA | SA Qa | Ne Qa | Be Ne |
|---|---|---|---|---|
| $\mu_\Omega^P$ | 0.2 | 0.3 | 0.1 | 0.1 |
| $\mu_\Omega^N$ | -0.1 | - 0.1 | -0.2 | - 0.2 |

$$\begin{aligned} M_2^P\left(G_{-Be}\right) &= [0.4\times 0.5\times 0.3\times 0.4] + [0.3\times 0.4\times 0.1\times 0.2] = 0.0264, \\ M_2^N\left(G_{-Be}\right) &= [-0.1\times -0.2\times -0.2\times -0.3] + [-0.2\times -0.3\times -0.2\times -0.4] = 0.006, \\ M_2^P\left(G_{-SA}\right) &= [0.3\times 0.4\times 0.1\times 0.2] + [0.1\times 0.2\times 0.2\times 0.3] = 0.0036, \\ M_2^N\left(G_{-SA}\right) &= [-0.2\times -0.3\times -0.2\times -0.4] + [-0.2\times -0.4\times -0.3\times -0.3] = 0.012, \\ M_2^P\left(G_{-Qa}\right) &= [0.2\times 0.3\times 0.4\times 0.5] + [0.1\times 0.2\times 0.2\times 0.3] = 0.0132 \\ M_2^N\left(G_{-Qa}\right) &= [-0.3\times -0.3\times -0.1\times -0.2] + [-0.2\times -0.4\times -0.3\times -0.3] = 0.009, \\ M_2^P\left(G_{-Ne}\right) &= [0.2\times 0.3\times 0.4\times 0.5] + [0.4\times 0.5\times 0.3\times 0.4] = 0.036, \\ M_2^N\left(G_{-Ne}\right) &= [-0.3\times -0.3\times -0.1\times -0.2] + [-0.1\times -0.2\times -0.2\times -0.3] = 0.003 \end{aligned}$$

As we can see in the calculation above, by removing the city of Sary the positive membership degree of the Zagreb index decreased drastically, so it can be concluded that the city of Sari is the most suitable place to build a hospital because, firstly, it has the largest population among other cities, secondly, the area of land suitable for building a hospital in the Sari city is more.

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