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CRITIC-MABAC Approach with Probabilistic Uncertain Linguistic q-rung Orthopair Fuzzy Sets for Selecting the Best Cloud Storage Service Alternative

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Abstract

Decision-making (DM) often experiences challenges because of ambiguity and uncertainty in practical life. It is normally not easy to give clear values to data in such conditions. Probabilistic uncertain linguistic q-rung orthopair fuzzy sets (PULq-ROFSs) offer a flexible and practical tool to handle this ambiguity and fuzziness. These sets provide a comprehensive framework for dealing with complex DM problems. A powerful solution for addressing such issues is the decomposition of the advantages of the multi-attributive border approximation area comparison (MABAC) method. Secondly criteria importance through inter-criteria correlation (CRITIC) method. CRITIC is efficient when distributing weights of criteria based on interrelations. Meanwhile MABAC is famous for its high rank towards options for their distance from the approximation area. A combination of these techniques leads to a thorough and systematic framework for the solution of uncertain DM problems. Based on this background, we extend the CRITIC-MABAC methodology to PULq-ROFSs. This methodology can tackle a real-world use case: choosing the best cloud storage service alternative. We use the MABAC technique to rank alternatives and the CRITIC method to weight criteria with the aim of making our results valid and meaningful. We confirm the practical aspect of the methodology and evaluate it is performance on a real-world DM situation. We also compare the results with those yielded by previous techniques. Our findings highlight the impressive performance and the applicability of the presented CRITIC-MABAC approach.

Keywords: Decision-making, CRITIC-MABAC, PULq-ROFSs, Cloud storage service.

1. Introduction

Dealing with uncertainty in data is essential for solving decision-making (DM) problems. In many cases, the original data is unclear, making it difficult to assign precise values. To address this challenge, Zadeh [1]

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introduced fuzzy set (FS) theory, where a membership function assigns values between 0 and 1 to represent uncertainty. Although useful, FSs rely on a single membership function (α), which is often insufficient for complex DM problems. To overcome this, Atanassov [2] proposed intuitionistic fuzzy sets (IFSs) by adding a non-membership function (β), with the condition $\alpha + \beta \leq 1$. Further extensions were developed to capture uncertainty more flexibly. Yager [3, 4] introduced Pythagorean fuzzy sets (PyFSs), where $\alpha^2 + \beta^2 \leq 1$, and generalized them to q-rung orthopair fuzzy sets (q-ROFSs) with $\alpha^q + \beta^q \leq 1$. Senapati and Yager [5] proposed Fermatean fuzzy sets (FFSs) ($\alpha^3 + \beta^3 \leq 1$), while Cuong [10] suggested picture fuzzy sets (PFSs) by including a neutral membership function δ satisfying $\alpha + \beta + \delta \leq 1$. These were later generalized into q-rung picture fuzzy sets (q-RPFSs) [11], with condition $\alpha^q + \beta^q + \delta^q \leq 1$. Recently, probabilistic uncertain linguistic q-ROFSs (PULq-ROFSs) have been developed to integrate probabilistic linguistic information with q-ROFSs, providing greater flexibility for managing ambiguity in complex DM environments. Numerous studies [6, 7, 8, 9] confirm their broad applicability across diverse domains.

Aggregation operators (AOs) have been developed by many scholars under different fuzzy frameworks. The works of [13, 14] discussed various AOs and their applicability. He et al. [15] introduced Q-RPF Dombi Hamy mean operators and validated their performance on numerical problems. Extensive coverage of AOs under different fuzzy settings is provided by Akram et al. [16, 17]. Akram and Shumaiza [18] applied VIseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) and Technique for the Order of Preference by Similarity to an Ideal Solution (TOPSIS) techniques within a q-RPF structure and validated their performance through two case studies. Later, Akram et al. [19] modified the TOPSIS approach for handling qFSs. Further studies on these fuzzy systems are given by Verma and Mittal [20] and Verma and Rohtagi [21]. Sitara et al. [22] proposed q-RPF graph topologies, creating new opportunities in DM. Progress by Pinar and Boran [23] and Akram et al. [24, 25] further increased the importance of q-RPFSs in handling uncertainty and improving DM. The MABAC method was first introduced by Pamucar and Cirovic [26] to deal with uncertainty in DM. Pamucar et al. [27] later improved its basic elements for more complex settings. A major strength of MABAC is its ability to manage conflicting criteria while focusing on border approximations. Xue et al. [28] extended MABAC to interval-valued IFSs, enhancing its use in complex conditions. Peng and Yang [29] applied a PyF Choquet integral in DM. Sun et al. [30] proposed a hesitant fuzzy linguistic projection-extension of MABAC to improve decisions with linguistic uncertainty. Liu et al. [31] combined MABAC with the Best Worst Method using q-ROF rough numbers. Mishra et al. [32] extended MABAC to interval-valued IFSs, and Gong et al. [33] improved MABAC for q-ROFSs, especially in evaluating teaching quality. Since multi-criteria group DM relies on accurate weight allocation, different methods have been developed. The CRITIC method was proposed by Diakoulaki et al. [34], while Hatefi [35] suggested Indifference Threshold-based Attribute Ratio Analysis (ITARA) for similar purposes.

Cloud storage services play a vital role in modern times due to their reliable and flexible data management services. Through cloud storage services, organizations and individuals carry out the protection of their data, easy access and operational efficiency. Identifying the best cloud storage service is a daunting task because it involves several attributes such as cost, reliability, security, and performance. Because of the difficulties that traditional DM methods have with uncertainties there is a critical need to further develop evaluation methods that are more robust. With the multiplication of cloud storage services, organizations are faced with an overwhelming number of options. Topping the list of the leading companies are Google Drive, Dropbox, and Microsoft OneDrive that have a strong market share. But some aspects beyond storage capacity are required to choose the best service. While making their decisions, decision-makers (DMs) are expected to examine data security measures, service reliability, and how well a service can be integrated into existing infrastructure. Since there are no standard guidelines for judging these factors. DM process becomes problematic and requires the use of systematic approaches towards selection of cloud storage. It has been shown that MCGDM techniques are widely used to estimate and prioritize cloud storage services with respect to multiple evaluation parameters. Therefore, using techniques such as Analytical Hierarchy Process (AHP), the DM process regarding choosing of cloud services has become easier due to the TOPSIS and CRITIC techniques. However, traditional methods frequently rely on specified attribute weights, not taking into consideration the existing uncertainties and personal preferences in an actual-life setting.

The PULq-ROFSs framework is a unique approach to MCGDM. This approach moves the management of uncertainty further by integrating probabilistic linguistic evaluations and enables the DMs to voice their opinions comparatively more freely. Compared to traditional models, PULq-ROFSs provide a higher degree of flexibility for accommodating divergence and uncertainty, thus better modeling subjective estimation. In situations where decisions have to be made, a panel of experts evaluates cloud storage services bearing in mind factors that are primary, such as affordability, dependability, cost, reliability, security, efficiency, and performance. Using linguistic expressions, modified with probability values, DMs explain how confident they are in each cloud storage service. By synthesizing the offered assessments, an ordered list of the cloud storage solutions is generated which facilitates open and non-biased choice mechanism. This approach is unique in the sense that it does not depend on the set of attribute weights to function making the evaluation procedure flexible to different situations. The CRITIC methodology is applied to assess the importance of each criterion. This approach subjectively identifies the relative importance of all attributes based on analysis of contrast intensity and correlations to ensure that the most important aspect are given a proper weighting. Using the CRITIC method helps to improve the evaluation process by minimizing one's initiative and increasing accuracy in ranking cloud storage solutions. The quality of cloud storage solution evaluation with PULq-ROFSs methodology is better than with standard approaches in several ways. Adoption of probabilistic linguistic terms provides DMs with the opportunity to create a clear picture of their confidence and reservations about their judgments. By departing from pre-specified weight allocation, the CRITIC process calculates attribute relevance from analytical findings, which leads to a more objective and evidence-driven DM process. Organizations may benefit from neat rankings generated by this methodology and select storage options that best meet their business needs. Effective method selection for cloud storage implies that advanced DM frameworks should be used since the task is very complex. Although the traditional MCGDM approaches provide obvious procedures, they tend not to possess the ability to handle uncertainty effectively. The utilization of the PULq-ROFSs method enables a dynamic and clear ranking of the cloud storage facilities by laying out the essential attributes that determine their performance. This approach combines probabilistic linguistic judgments with CRITIC weight allocations to increase accuracy in DM and provide meaningful recommendations for cloud storage selection.

1.1. Contributions

This study attempts to develop a new method by combining the MABAC approach with the CRITIC strategy, namely, the CRITIC-MABAC to prioritize alternative options in PULq-ROFSs situation of MCGDM. There are various underlying fundamental reasons for conducting this study. Using an extended framework for PULq-ROFSs, uncertainty can be modeled more adaptively. The capacity to combine a variety of fuzzy models by changing the parameter q increases its adaptability in order to manage uncertain information. When valuing alternatives in comparison with their location around the border approximation region, the MABAC procedure supports a straightforward and reliable ranking procedure. The accurate ranking system of the MABAC method allows it to be complementarily matched to the flexible PULq-ROFS structure, such as in making decisions under uncertainty. Justifying the weights of criteria and CRITIC analysis of correlations among various criteria. This makes DM process more objective, resulting in an unbiased thorough analysis of alternatives. This investigation introduces a unified approach grounded in these aspects, which exploits the advantages of CRITIC and MABAC in the PULq-ROFS setting. The key contributions of this study are highlighted as follows:

1. This study introduces a novel DM framework tailored for the flexible structure of PULq-ROFSs, referred to as the PULq-ROFSs-CRITIC-MABAC method. Within this approach, the criteria weights are determined using the CRITIC approach, ensuring an objective and data driven evaluation process.
2. A detailed step-by-step procedure is provided for the CRITIC-MABAC method, offering a clear and systematic guide for their implementation in DM scenarios.
3. In order to confirm the successful outcome of the suggested approach, it is applied to a real world problem, demonstrating it is practical utility in handling uncertainty and ranking alternatives.

4. Furthermore, a comparative analysis is conducted with existing approaches, highlighting the advantages and improvements achieved through our extended method.

1.2. Novelties and Motivations

In this study, we propose an advanced DM approach built upon the principles of PULq-ROFSs, termed the PULq-ROFSs-CRITIC-MABAC method. This framework integrates the CRITIC technique to determine the criterion weights, ensuring an objective and data driven weighting mechanism. A key aspect of our research is the novel combination of the MABAC and CRITIC methods, which, has not, as far as we are aware, been examined in the body of existing literature. This study is the first to introduce a dual-framework approach for MCGDM by utilizing both CRITIC and MABAC within the context of PULq-ROFSs. To enhance clarity, we illustrate the core steps of the proposed methodologies through diagrams, providing a visual representation of the developed strategies. Furthermore, we apply our method to a real-world problem centered on choosing the best cloud storage service in order to show it is practical usefulness. The PULq-ROFSs-CRITIC-MABAC method is employed to evaluate and rank alternatives, showcasing it is effectiveness in handling complex DM scenarios. We have emphasized the impact of our proposed approach to demonstrate it is significance, superiority, and reliability compared to existing DM techniques. Over the past years, the only known integration within this domain has been the CRITIC-VIKOR method combined with PULq-ROFSs, as reported by Naz, et al. [36]. For further study, the readers are referred to [37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48]. The distinctive aspects of our study are as follows: The PULq-ROFSs framework offers a high degree of flexibility, making it capable of managing diverse types of data, including probabilistic and linguistic information. Due to it is adaptable nature, we have incorporated this structure into our methodology to ensure more comprehensive DM under uncertainty. While numerous researchers have explored and expanded both the CRITIC and MABAC methods, their combined implementation within the PULq-ROFSs environment remains largely unexplored. Our study bridges this gap by introducing a novel integration that enhances the accuracy and applicability of these techniques. We introduce the PULq-ROFSs-CRITIC-MABAC approach as a novel solution to tackle real world DM challenges. To effectively address complex problems, we employ this extended methodology, which systematically converts expert evaluations into PULq-ROFSs numerical representations. Additionally, we provide a comprehensive breakdown of the step-by-step procedure and the necessary computations involved in the MCGDM process, ensuring clarity and precision in implementation.

The following describes the study's structure: Section 2 provides an overview of fundamental concepts related to PULq-ROFSs. In Section 3, we introduce the methodology behind the proposed approach, detailing the application of the MABAC and CRITIC methods for solving MCGDM problems within the PULq-ROFSs framework. Section 4 demonstrates the practical implementation of our approach through a real world case study, where we evaluate and select the most suitable cloud storage service on a global scale using the CRITIC-MABAC technique. In Section 5, we conduct a comparative analysis as well as sensitivity analysis to assess it is effectiveness against previously established methods. In conclusion, the findings of this research are summarized in Section 6.

For clarity, the notations employed in Section 2 are summarized in Table 1.

2. Preliminaries

Definition 2.1. [38] Let $\mathbb{S} = \{\bar{a}_{\varpi} | \varpi = -\lambda, \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, \lambda\}$ be a linguistic term set (LTS), a probabilistic LTS (PLTS) defines itself by:

$$\mathcal{N}_{\bar{a}}(\varrho) = \{\bar{a}^{o}(\varrho^{o}) | \bar{a}^{o} \in \mathbb{S}, \varrho^{o} \geq 0, o = 1, 2, \ldots, \#\mathcal{N}_{\bar{a}}(\varrho), \sum_{o=1}^{\#\mathcal{N}_{\bar{a}}(\varrho)} \varrho^{(o)} \leq 1\}, \tag{2.1}$$

Symbol	**Description**
$\mathbb{S}$	Linguistic term set (LTS).
λ	Linguistic scale parameter.
$o,\ f,\ h,\ l,\ \epsilon$	Generic index variables.
$\bar{a}_\omega$	Linguistic term at index ω.
$N_{\bar{a}}(\varrho)$	Probabilistic linguistic term set (PLTS).
$\bar{a}^{(o)},\ \mathfrak{A}^{(o)},\ \mathfrak{B}^{(o)}$	Linguistic terms.
$\varrho^{(o)}$	Probability of $\bar{a}_o$.
$\mathfrak{Z} = [\bar{a}_a, \bar{a}_b]$	Uncertain linguistic variable.
$\mathfrak{Z}(\varrho)$	Probabilistic uncertain linguistic term set (PULTS).
$\#$	Cardinality function.
$\dot{\mathfrak{Z}}(\varrho)$	Normalized PULTS.
$\dot{\varrho}^{(o)}$	Normalized probability.
Y	Universal set.
Q	q-ROFS.
$\sigma_Q(y)$	Membership degree of element y.
$\zeta_Q(y)$	Non-membership degree of element y.
$\pi_Q(y)$	Indeterminacy degree of element y.
γ	q-rung orthopair fuzzy number (q-ROFN).
$t : [\bar{a}_{-\lambda}, \bar{a}_\lambda] \to [0,1]$	Transformation function.
t^{-1}	Inverse transformation function.
$N(\varrho)$	PULq-ROFS.
$\sigma(\hat{\varrho})(y_\varepsilon)$	Membership function with probability distribution $\hat{\varrho}$.
$\zeta(\tilde{\varrho})(y_\varepsilon)$	Non-membership function with probability distribution $\tilde{\varrho}$.
$\phi^\varepsilon(f), \varphi^\varepsilon(h)$	Indexing functions mapping f, h to linguistic terms.
$[\mathfrak{L}_{\phi^\varepsilon(f)}, \mathfrak{U}_{\phi^\varepsilon(f)}]$	Membership linguistic interval.
$[\mathfrak{M}_{\varphi^\varepsilon(h)}, \mathfrak{N}_{\varphi^\varepsilon(h)}]$	Non-membership linguistic interval.
$\hat{\varrho}(f), \tilde{\varrho}(h)$	Probabilities for membership and non-membership terms.
F, H	Number of membership and non-membership terms.
q	Rung parameter.
$\digamma$	Score function.
$\daleth$	Deviation function.
$\oplus$	Addition operator.
$\otimes$	Multiplication operator.
HD	Hamming distance.
ξ	Scalar parameter.

Table 1: Notations used in Preliminaries

in which $\bar{a}^{(o)}(\varrho^{(o)})$ is the LT, $\bar{a}^{(o)}$ is associated with the probability $\varrho^{(o)}$ and $\#\mathcal{N}_{\bar{a}}(\varrho)$ represents the total amount of LTS in a $\mathcal{N}_{\bar{a}}(\varrho)$.

Definition 2.2. [39] Let $\mathfrak{Z} = [\bar{a}_a, \bar{a}_b]$, where $\bar{a}_a, \bar{a}_b \in \mathbb{S}_{[\lambda,-\lambda]}$, $\bar{a}_a$ and $\bar{a}_b$ are also the more severe and smaller restrictions; we then designate $\mathfrak{Z}$ as the uncertain linguistic variable.

2.1. The concept of PULTS

Probabilistic uncertain linguistic term set (PULTS), a novel concept put out by Lin et al. [40], relies on PLTSs and unclear linguistic variables [41] to accurately illustrate the DMs concern.

Definition 2.3. [40] A PULTS is characterized as follows:

$$\mathfrak{Z}(\varrho) = \{[\mathfrak{A}^{(o)}, \mathfrak{B}^{(o)}](\varrho^{(o)}) | \mathfrak{A}^{(o)}, \mathfrak{B}^{(o)} \in \mathbb{S}, \varrho^{(o)} \geq 0, o = 1, 2, \ldots, \#\mathfrak{Z}(\varrho), \sum_{o=1}^{\#\mathfrak{Z}(\varrho)} \varrho^{(o)} \leq 1\}, \tag{2.2}$$

in which $[\mathfrak{A}^{(o)}, \mathfrak{B}^{(o)}](\varrho^{(o)})$ signifies the uncertain linguistic term (ULT) $[\mathfrak{A}^{(o)}, \mathfrak{B}^{(o)}]$ related with the likelihood $\varrho^{(o)}$ and $\mathfrak{A}^{(o)}, \mathfrak{B}^{(o)}$ are LTs, $\mathfrak{A}^{(o)} \leq \mathfrak{B}^{(o)}$ and $\#\mathfrak{Z}(\varrho)$ is the cardinality of $\mathfrak{Z}(\varrho)$.
If $\sum_{o=1}^{\#\mathfrak{Z}(\varrho)} \varrho^{(o)} = 0$, there is not any evaluation data available yet. If $\sum_{o=1}^{\#\mathfrak{Z}(\varrho)} \varrho^{(o)} = 1$, it shows that complete information on their syntax evaluation is provided. If $0 < \sum_{q=1}^{\#\mathfrak{Z}(\varrho)} \varrho^{(o)} < 1$, that only a portion of the linguistic tests are provided because some DMs are capable of providing the full evaluation data or because some DMs are responsible for delivering the data. This is often the case in real-world MCGDM problems, and it is necessary to resolve uncertainty chance.

Definition 2.4. [40] Given a PULTS $\mathfrak{Z}(\varrho) = \{[\mathfrak{A}^{(o)}, \mathfrak{B}^{(o)}](\varrho^{(o)}) | o = 1, 2, \ldots, \#\mathfrak{Z}(\varrho)\}$, If each of its constituent parts are arranged in decreasing order, then $\mathfrak{Z}(\varrho)$ is referred to as an ordered PULTS. Two aspects $\langle[\mathfrak{A}^{(\iota)}, \mathfrak{B}^{(\iota)}](\varrho^{(\iota)})\rangle$ and $\langle[\mathfrak{A}^{(\epsilon)}, \mathfrak{B}^{(\epsilon)}](\varrho^{(\epsilon)})\rangle$ are contrasted with the chance ratio of $[\mathfrak{A}^{(\iota)} \times (\varrho^{(\iota)}), \mathfrak{B}^{(\iota)} \times (\varrho^{(\iota)})]$ over $[\mathfrak{A}^{(\epsilon)} \times (\varrho^{(\epsilon)}), \mathfrak{B}^{(\epsilon)} \times (\varrho^{(\epsilon)})]$.

2.2. The Normalization of PULTS

In order to address the remaining unexplained chance that makes the significance of PULTS comparable, PULTS are normalized. Normalization of PULTS involves two distinct functions: estimating a person's lack of statistical knowledge is the basic function, and normalizing a PULTS's circularity is the other part that involves computation.

Definition 2.5. [40] Given a PULTS $\mathfrak{Z}(\varrho)$ with $\sum_{o=1}^{\#\mathfrak{Z}(\varrho)} \varrho^{(o)} < 1$, hence the corresponding PULTS $\dot{\mathfrak{Z}}(\varrho)$ can be defined as:

$$\dot{\mathfrak{Z}}(\varrho) = \{[\mathfrak{A}^{(o)}, \mathfrak{B}^{(o)}](\dot{\varrho}^{(o)}) | o = 1, 2, \ldots, \#\mathfrak{Z}(\varrho)\}, \tag{2.3}$$

where $\dot{\varrho}^{(o)} = \frac{\varrho^{(o)}}{\sum_{o=1}^{\#\mathfrak{Z}(\varrho)} \varrho^{(o)}}$, for all $o = 1, 2, ..., \#\mathfrak{Z}(\varrho)$.
The cardinalities of PULTS are typically different in real DM applications, which causes significant operational difficulties. In this instance, we add uncertain linguistic concepts with probability 0 to PULTS with relatively few elements to raise their cardinalities.

Definition 2.6. [40] Let $\mathfrak{Z}_1(\varrho)$ and $\mathfrak{Z}_2(\varrho)$ be any two PULTS, where $\mathfrak{Z}_1(\varrho) = \{[\mathfrak{A}_1^{(o)}, \mathfrak{B}_1^{(o)}](\varrho_1^{(o)}) | o = 1, 2, \ldots, \#\mathfrak{Z}_1(\varrho)\}$ and $\mathfrak{Z}_2(\varrho) = \{[\mathfrak{A}_2^{(o)}, \mathfrak{B}_2^{(o)}](\varrho_2^{(o)}) | o = 1, 2, \ldots, \#\mathfrak{Z}_2(\varrho)\}$, and let $\#\mathfrak{Z}_1(\varrho)$ and $\#\mathfrak{Z}_2(\varrho)$ be the number of linguistic terms in $\mathfrak{Z}_1(\varrho)$ and $\mathfrak{Z}_2(\varrho)$, respectively. If $\#\mathfrak{Z}_1(\varrho) > \#\mathfrak{Z}_2(\varrho)$, then we will add $\#\mathfrak{Z}_1(\varrho) - \#\mathfrak{Z}_2(\varrho)$ linguistic terms to $\mathfrak{Z}_2(\varrho)$ so that number of terms in $\mathfrak{Z}_1(\varrho)$ and $\mathfrak{Z}_2(\varrho)$ are identical. In $\mathfrak{Z}_2(\varrho)$, the smallest linguistic terms are added, and all of the linguistic terms have zero probabilities.

Let $\mathfrak{Z}_1(\varrho) = \{[\mathfrak{A}_1^{(o)}, \mathfrak{B}_1^{(o)}](\varrho_1^{(o)})|o = 1, 2, \ldots, \#\mathfrak{Z}_1(\varrho)\}$ and $\mathfrak{Z}_2(\varrho) = \{[\mathfrak{A}_2^{(o)}, \mathfrak{B}_2^{(o)}](\varrho_2^{(o)})|o = 1, 2, \ldots, \#\mathfrak{Z}_2(\varrho)\}$, then the two stages listed below can be used to carry out the normalization process:

1. If $\sum_{o=1}^{\#\mathfrak{Z}(\varrho)} \varrho^{(o)} < 1$ then by 2.3, the value $\dot{\mathfrak{Z}}_\iota(\varrho), \iota = 1, 2$ can be calculated.
2. If $\#\mathfrak{Z}_1(\varrho) \neq \#\mathfrak{Z}_2(\varrho)$, then, in accordance with Definition 2.6, it is necessary to add some elements to the one that has fewer components.

We refer to the resulting PULTS as the normalized PULTS. The normalized PULTS are also represented by $\mathfrak{Z}_1(\varrho)$ and $\mathfrak{Z}_2(\varrho)$ for presentational convenience.

2.3. The concept of q-ROFS

Definition 2.7. [14] Let Y be an ordinary fixed set, then a q-ROFS represented by $\mathcal{Q}$ defined on Y is expressed as:

$$\mathcal{Q} = \{\langle y, \sigma_{\mathcal{Q}}(y), \zeta_{\mathcal{Q}}(y)\rangle | y \in Y\}, \tag{2.4}$$

where the membership and non-membership degrees of the element $y \in Y$ to the set $\mathcal{Q}$ are denoted by $\sigma_{\mathcal{Q}}(y)$ and $\zeta_{\mathcal{Q}}(y)$, respectively, satisfying $0 \leq \sigma_{\mathcal{Q}}(y), \zeta_{\mathcal{Q}}(y) \leq 1$ and $(\sigma_{\mathcal{Q}}(y))^q + (\zeta_{\mathcal{Q}}(y))^q \leq 1, (q \geq 1)$. The indeterminacy degree is defined as $\pi_{\mathcal{Q}}(y) = \sqrt[q]{(\sigma_{\mathcal{Q}}(y))^q + (\zeta_{\mathcal{Q}}(y))^q - (\sigma_{\mathcal{Q}}(y))^q(\zeta_{\mathcal{Q}}(y))^q}$. Liu and Wang [14] called the ordered pair $(\sigma_{\mathcal{Q}}(y), \zeta_{\mathcal{Q}}(y))$ a q-ROF number, which can be denoted as $\gamma = (\sigma, \zeta)$.

Definition 2.8. [42, 43] Let $\mathbb{S} = \{\bar{a}_\varpi | \varpi = -\lambda, \ldots, -2, -1, 0, 1, 2, \ldots, \lambda\}$ be a LTS [42]. The transformation function [43] is used to construct the linguistic phrases $\bar{a}_\varpi$, which can represent the equivalent information to ω:

$$t : [\bar{a}_{-\lambda}, \bar{a}_\lambda] \to [0, 1], t(\bar{a}_\omega) = \frac{\omega + \lambda}{2\lambda} = \omega. \tag{2.5}$$

In addition, ω can be obtained using the transformation function t^{-1}, it can be used to convey information that is comparable to the linguistic terms $\bar{a}_\omega$:

$$t^{-1} : [0, 1] \to [\bar{a}_{-\lambda}, \bar{a}_\lambda], t^{-1} = \bar{a}_{(2\omega-1)\lambda} = \bar{a}_\omega. \tag{2.6}$$

2.4. PULq-ROFS

Now we suggest a new probabilistic fuzzy set PULq-ROFS, it not only enables specialists to present assessment data using a variety of LTS, but also includes the potential for each LTS. The challenge is defining how to implement rules of PULq-ROFS appropriately in cases where the associated distributions of probability become distinct..

2.5. The basic definition of PULq-ROFS

Definition 2.9. Given a consistent set $Y = \{y_1, y_2, \ldots, y_\epsilon\}$, where
$\mathbb{S} = \{\bar{a}_\omega | \omega = -\lambda, \ldots, 0, 1, 2, 3, -3, -2, -1, \lambda\}$ be a LTS. Next, we obtain a PULq-ROFS $\mathcal{N}(\varrho)$ on Y by

$$\mathcal{N}(\varrho) = \{\langle y_\epsilon, \sigma(\hat{\varrho})(y_\epsilon), \zeta(\tilde{\varrho})(y_\epsilon)\rangle : y_\epsilon \in Y\}, \tag{2.7}$$

where $\sigma(\hat{\varrho})(y_\epsilon) = \{[\mathfrak{L}_{\phi^{\epsilon(f)}}, \mathfrak{U}_{\phi^{\epsilon(f)}}](\hat{\varrho}^{(f)}) : \mathfrak{L}_{\phi^{\epsilon(f)}}, \mathfrak{U}_{\phi^{\epsilon(f)}} \in \mathbb{S}_{[\lambda,-\lambda]}, \ \hat{\varrho}^{(f)} \geq 0, \ \sum_{f=1}^{f} \hat{\varrho}^{(f)} \leq 1\}$ and $\zeta(\tilde{\varrho})(y_\epsilon) = \{[\mathfrak{M}_{\varphi^{\epsilon(h)}}, \mathfrak{N}_{\varphi^{\epsilon(h)}}](\tilde{\varrho}^{(h)}) : \mathfrak{M}_{\varphi^{\epsilon(h)}}, \mathfrak{N}_{\varphi^{\epsilon(h)}} \in \mathbb{S}_{[\lambda,-\lambda]}, \ \tilde{\varrho}^{(h)} \geq 0, \ \sum_{h=1}^{H} \tilde{\varrho}^h \leq 1\}$ shows the membership and non-membership degrees, respectively, of $y_\epsilon \in Y$ and the associated probabilities are $\hat{\varrho}^{(f)}$ and $\tilde{\varrho}^{(h)}$, respectively; $\phi^{\epsilon(f)}$ and $\varphi^{\epsilon(h)}$ are the subscripts of the ULTs $[\mathfrak{L}_{\phi^{\epsilon(f)}}, \mathfrak{U}_{\phi^{\epsilon(f)}}]$ and $[\mathfrak{M}_{\varphi^{\epsilon(h)}}, \mathfrak{N}_{\varphi^{\epsilon(h)}}]$, respectively; satisfying the condition $0 \leq (\max_{f=1}^{f} \phi^{\epsilon(f)})^q + (\max_{h=1}^{h} \varphi^{\epsilon(h)})^q \leq \lambda^q \ (q \geq 1)$.

If the set $\mathcal{N}(\varrho)$ contain only unique element, then it minimize the PULq-ROFN and we write it as $\mathcal{N}(\varrho) = \langle\{[\mathfrak{L}_{\phi^{(f)}}, \mathfrak{U}_{\phi^{(f)}}](\hat{\varrho}^{(f)})\}, \{[\mathfrak{M}_{\varphi^{(h)}}, \mathcal{N}_{\varphi^{(h)}}](\tilde{\varrho}^{(h)})\}\rangle$, where $[\mathfrak{L}_{\phi(f)}, \mathfrak{U}_{\phi(f)}], [\mathfrak{M}_{\varphi^{(h)}}, \mathfrak{N}_{\varphi^{(h)}}] \in \mathbb{S}_{[\lambda,-\lambda]}$ and $\hat{\varrho}^{(f)}$, $\tilde{\varrho}^{(h)} \geq 0$, $\sum_{f=1}^{F} \hat{\varrho}^{(f)} \leq 1$, $\sum_{h=1}^{H} \tilde{\varrho}^{(h)} \leq 1$.

Example 2.1
Let the linguistic term set be $\mathbb{S} = \{s_{-3} : \text{VP}, s_{-2} : \text{P}, s_{-1} : \text{SP}, s_0 : \text{N}, s_1 : \text{SG}, s_2 : \text{G}, s_3 : \text{VG}\}$, where $\lambda = 3$. Let the rung parameter be $q = 3$.

A valid PULq-ROFN is:

$$\mathcal{N}(\varrho) = \langle\{[s_1, s_2](0.6), [s_0, s_1](0.4)\}, \{[s_{-2}, s_{-1}](0.8), [s_{-1}, s_0](0.2)\}\rangle$$

Verification of Conditions:

- **Probability Sums:** The sum of membership probabilities is $0.6 + 0.4 = 1.0$. The sum of non-membership probabilities is $0.8 + 0.2 = 1.0$.
- **q-rung Condition:** The condition is $(\max_f \phi(f))^q + (\max_h \varphi(h))^q \leq \lambda^q$.
  - **For Membership:** The subscripts in $\{[s_1, s_2], [s_0, s_1]\}$ are $\{1, 2\}$ and $\{0, 1\}$. The maximum value is $\max(2, 1) = 2$.
  - **For Non-Membership:** The subscripts in $\{[s_{-2}, s_{-1}], [s_{-1}, s_0]\}$ are $\{-2, -1\}$ and $\{-1, 0\}$. The maximum value is $\max(-1, 0) = 0$.
  - **Calculation:** We check: $(2)^3 + (0)^3 = 8 + 0 = 8 \leq 3^3 = 27$ ✓

Definition 2.10. Let $\mathbb{S}_{[-\lambda,\lambda]}$ be a LTS, for any adjusted PULq-ROFN $\mathcal{N}(\varrho)=\langle\{[\mathfrak{L}_{\phi^{(f)}}, \mathfrak{U}_{\phi^{(f)}}](\hat{\varrho}^{(f)}), [\mathfrak{M}_{\varphi^{(h)}}, \mathfrak{N}_{\varphi^{(h)}}](\tilde{\varrho}^{(h)})\}\rangle$, where $\mathfrak{L}_{\phi^{(f)}}, \mathfrak{U}_{\phi^{(f)}}, \mathfrak{M}_{\varphi^{(h)}}$, and $\mathfrak{N}_{\varphi^{(h)}} \in \mathbb{S}_{[-\lambda,\lambda]}$, $(f = 1, 2, 3 \ldots F; h = 1, 2, 3 \ldots, H)$, the score function of $\mathcal{N}(\varrho)$ is defined as:

$$\digamma(\mathcal{N}(\varrho)) = \frac{\sum_{f=1}^{\#F_\phi} \left(\frac{t(\mathfrak{L}_{\phi^{(f)}})\hat{\varrho}^{(f)} + t(\mathfrak{U}_{\phi^{(f)}})\hat{\varrho}^{(f)}}{2}\right)^q}{\sum_{f=1}^{\#F_\phi} \hat{\varrho}^{(f)}} - \frac{\sum_{h=1}^{\#H_\varphi} \left(\frac{t(\mathfrak{M}_{\varphi^{(h)}})\tilde{\varrho}^{(h)} + t(\mathfrak{N}_{\varphi^{(h)}})\tilde{\varrho}^{(h)}}{2}\right)^q}{\sum_{h=1}^{\#H_\varphi} \tilde{\varrho}^{(h)}} \tag{2.8}$$

where $t(\mathfrak{L}_{\phi^{(f)}}), t(\mathfrak{U}_{\phi^{(f)}}), t(\mathfrak{M}_{\varphi^{(h)}})$, and $t(\mathfrak{N}_{\varphi^{(h)}}) \in [0, 1]$, $\#F_\phi$ and $\#H_\varphi$ indicate, correspondingly, how many elements there are in the matching set. The standard deviation of $\mathcal{N}(\varrho)$ is defined as:

$$\daleth(\mathcal{N}(\varrho)) = \frac{\sqrt{\sum_{f=1}^{\#F_\phi} \left(\frac{t(\mathfrak{L}_{\phi^{(f)}})\hat{\varrho}^{(f)} + t(\mathfrak{U}_{\phi^{(f)}})\hat{\varrho}^{(f)}}{2} - \beth(\mathfrak{N}(\varrho))\right)^q}}{\sum_{f=1}^{\#F_\phi} \hat{\varrho}^{(f)}} + \frac{\sqrt{\sum_{h=1}^{\#H_\varphi} \left(\frac{t(\mathfrak{M}_{\varphi^h})\tilde{\varrho}^{h} + t(\mathfrak{N}_{\varphi^{(h)}})\tilde{\varrho}^{(h)}}{2} - \beth(\mathfrak{N}(\varrho))\right)^q}}{\sum_{h=1}^{\#H_\varphi} \tilde{\varrho}^{(h)}} \tag{2.9}$$

where $t(\mathfrak{L}_{\phi^{(f)}}), t(\mathfrak{U}_{\phi^{(f)}}), t(\mathfrak{M}_{\varphi^{(h)}})$, and $t(\mathfrak{N}_{\varphi^{(h)}}) \in [0, 1]$, $\#F_\phi$ and $\#H_\varphi$ are cardinalities of the corresponding sets, respectively.

Example 2.2
Let the linguistic term set be $\mathbb{S} = \{s_{-3} : \text{VP}, s_{-2} : \text{P}, s_{-1} : \text{SP}, s_0 : \text{N}, s_1 : \text{SG}, s_2 : \text{G}, s_3 : \text{VG}\}$, where $\lambda = 3$. The transformation function is $t(s_\omega) = \frac{\omega+3}{6}$. Let the rung parameter be $q = 3$.
Consider a PULq-ROFN:

$$\mathcal{N}(\varrho) = \langle \{[s_1, s_2](0.6), [s_2, s_3](0.4)\}, \{[s_{-1}, s_0](0.7), [s_0, s_1](0.3)\}\rangle$$

Step 1: Calculate Transformed Values

$$t(s_{-1}) = \frac{-1+3}{6} = \frac{2}{6} \approx 0.3333 \qquad t(s_0) = \frac{0+3}{6} = \frac{3}{6} = 0.5000$$
$$t(s_1) = \frac{1+3}{6} = \frac{4}{6} \approx 0.6667 \qquad t(s_2) = \frac{2+3}{6} = \frac{5}{6} \approx 0.8333$$
$$t(s_3) = \frac{3+3}{6} = \frac{6}{6} = 1.0000$$

Step 2: Compute the Score Function $\mathbb{S}(\gamma)$ Using the formula from Def. 2.10:

$$\digamma(\mathcal{N}(\varrho)) = \frac{\sum_{f=1}^{\#F_\phi} \left(\frac{t(\mathfrak{L}_{\phi(f)})+t(\mathfrak{U}_{\phi(f)})}{2} \cdot \hat{\varrho}(f)\right)^q}{\sum_{f=1}^{\#F_\phi} \hat{\varrho}(f)} - \frac{\sum_{h=1}^{\#H_\varphi} \left(\frac{t(\mathfrak{M}_{\varphi(h)})+t(\mathfrak{N}_{\varphi(h)})}{2} \cdot \tilde{\varrho}(h)\right)^q}{\sum_{h=1}^{\#H_\varphi} \tilde{\varrho}(h)}$$

$$\begin{aligned}
\digamma(\mathcal{N}(\varrho)) &= \frac{\left(\frac{0.6667+0.8333}{2} \cdot 0.6\right)^3 + \left(\frac{0.8333+1.0000}{2} \cdot 0.4\right)^3}{0.6+0.4} \\
&\quad - \frac{\left(\frac{0.3333+0.5000}{2} \cdot 0.7\right)^3 + \left(\frac{0.5000+0.6667}{2} \cdot 0.3\right)^3}{0.7+0.3} \\
&= \frac{(0.75 \cdot 0.6)^3 + (0.91665 \cdot 0.4)^3}{1.0} - \frac{(0.41665 \cdot 0.7)^3 + (0.58335 \cdot 0.3)^3}{1.0} \\
&= (0.45)^3 + (0.36666)^3 - (0.29166)^3 - (0.17500)^3 \\
&= 0.091125 + 0.04930 - 0.02480 - 0.00536 \\
\digamma(\mathcal{N}(\varrho)) &\approx 0.1103
\end{aligned}$$

A positive score (≈ 0.1103) indicates a membership-dominant evaluation.
Step 3: Compute the Deviation Function $\daleth(\mathcal{N}(\varrho))$ Using the formula from Def. 2.10:

$$\daleth(\mathcal{N}(\varrho)) = \sqrt{\frac{\sum_{f=1}^{\#F_\phi} \left(\frac{t(\mathfrak{L}_{\phi(f)})+t(\mathfrak{U}_{\phi(f)})}{2} - \mathbb{S}(\mathcal{N}(\varrho))\right)^q \cdot \hat{\varrho}(f)}{\sum_{f=1}^{\#F_\phi} \hat{\varrho}(f)} + \frac{\sum_{h=1}^{\#H_\varphi} \left(\frac{t(\mathfrak{M}_{\varphi(h)})+t(\mathfrak{N}_{\varphi(h)})}{2} - \mathbb{S}(\mathcal{N}(\varrho))\right)^q \cdot \tilde{\varrho}(h)}{\sum_{h=1}^{\#H_\varphi} \tilde{\varrho}(h)}}$$

For brevity, we compute the result:

$$\begin{aligned}
\daleth(\mathcal{N}(\varrho)) &\approx \sqrt{\frac{(0.75-0.1103)^3 \cdot 0.6 + (0.91665-0.1103)^3 \cdot 0.4}{1.0} + \frac{(0.41665-0.1103)^3 \cdot 0.7 + (0.58335-0.1103)^3 \cdot 0.3}{1.0}} \\
&\approx \sqrt{\frac{(0.6397)^3 \cdot 0.6 + (0.80635)^3 \cdot 0.4}{1} + \frac{(0.30635)^3 \cdot 0.7 + (0.47305)^3 \cdot 0.3}{1}} \\
&\approx \sqrt{\frac{(0.2618) \cdot 0.6 + (0.5243) \cdot 0.4}{1} + \frac{(0.02875) \cdot 0.7 + (0.1058) \cdot 0.3}{1}} \\
&\approx \sqrt{(0.1571 + 0.2097) + (0.0201 + 0.0317)} \\
&\approx \sqrt{0.3668 + 0.0518} \\
&\approx \sqrt{0.4186} \\
\daleth(\mathcal{N}(\varrho)) &\approx 0.6470
\end{aligned}$$

Definition 2.11. Let $\mathcal{N}^{(1)}(\varrho)$ and $\mathcal{N}^{(2)}(\varrho)$ be two PULq-ROFNs. Consequently, PULq-ROFNs comparison structure is shown as:

- If $\digamma(\mathcal{N}^{(1)}(\varrho)) \succ \digamma(\mathcal{N}^{(2)}(\varrho))$, then $\mathcal{N}^{(1)}(\varrho) \succ \mathcal{N}^{(2)}(\varrho)$.
- If $\digamma(\mathcal{N}^{(1)}(\varrho)) \prec \digamma(\mathcal{N}^{(2)}(\varrho))$, then $\mathcal{N}^{(1)}(\varrho) \prec \mathcal{N}^{(2)}(\varrho)$.
- If $\digamma(\mathcal{N}^{(1)}(\varrho)) = \digamma(\mathcal{N}^{(2)}(\varrho))$, then
  - If $\urcorner(\mathcal{N}^{(1)}(\varrho)) \succ \urcorner(\mathcal{N}^{(2)}(\varrho))$, then $\mathcal{N}^{(1)}(\varrho) \prec \mathcal{N}^{(2)}(\varrho)$.
  - If $\urcorner(\mathcal{N}^{(1)}(\varrho)) \prec \urcorner(\mathcal{N}^{(2)}(\varrho))$, then $\mathcal{N}^{(1)}(\varrho) \succ \mathcal{N}^{(2)}(\varrho)$.
  - If $\urcorner(\mathcal{N}^{(1)}(\varrho)) = \urcorner(\mathcal{N}^{(2)}(\varrho))$, then $\mathcal{N}^{(1)}(\varrho) \approx \mathcal{N}^{(2)}(\varrho)$.

Example 2.3
Let the linguistic term set be $\mathbb{S} = \{s_{-3} : \mathrm{VP}, s_{-2} : \mathrm{P}, s_{-1} : \mathrm{SP}, s_0 : \mathrm{N}, s_1 : \mathrm{SG}, s_2 : \mathrm{G}, s_3 : \mathrm{VG}\}$, where $\lambda = 3$. The transformation function is $t(s_\omega) = \frac{\omega+3}{6}$. Let the rung parameter be $q = 3$.
Consider two PULq-ROFNs to be compared:

$$\mathcal{N}_1(\varrho) = \langle \{[s_1, s_2](0.8)\}, \{[s_{-1}, s_0](0.6), [s_0, s_1](0.4)\} \rangle$$

$$\mathcal{N}_2(\varrho) = \langle \{[s_0, s_1](0.9)\}, \{[s_{-2}, s_{-1}](1.0)\} \rangle$$

Step 1: Calculate Transformed Values

$$t(s_{-2}) = \frac{-2+3}{6} = \frac{1}{6} \approx 0.1667 \qquad t(s_{-1}) = \frac{-1+3}{6} = \frac{2}{6} \approx 0.3333$$
$$t(s_0) = \frac{0+3}{6} = \frac{3}{6} = 0.5000 \qquad t(s_1) = \frac{1+3}{6} = \frac{4}{6} \approx 0.6667$$
$$t(s_2) = \frac{2+3}{6} = \frac{5}{6} \approx 0.8333$$

Step 2: Compute Score for $\mathcal{N}_1(\varrho)$

$$\begin{aligned}
\digamma(\mathcal{N}_1(\varrho)) &= \frac{\left(\frac{0.6667+0.8333}{2} \cdot 0.8\right)^3}{0.8} - \frac{\left(\frac{0.3333+0.5000}{2} \cdot 0.6\right)^3 + \left(\frac{0.5000+0.6667}{2} \cdot 0.4\right)^3}{1.0} \\
&= (0.75 \cdot 0.8)^3 - \left[(0.41665 \cdot 0.6)^3 + (0.58335 \cdot 0.4)^3\right] \\
&= (0.60)^3 - \left[(0.2500)^3 + (0.2333)^3\right] \\
&= 0.2160 - [0.0156 + 0.0127] = 0.2160 - 0.0283 = 0.1877
\end{aligned}$$

Step 3: Compute Score for $\mathcal{N}_2(\varrho)$)

$$\begin{aligned}
\digamma(\mathcal{N}_2(\varrho)) &= \frac{\left(\frac{0.5000+0.6667}{2} \cdot 0.9\right)^3}{0.9} - \frac{\left(\frac{0.1667+0.3333}{2} \cdot 1.0\right)^3}{1.0} \\
&= (0.58335 \cdot 0.9)^3 - (0.2500)^3 \\
&= (0.5250)^3 - 0.0156 = 0.1447 - 0.0156 = 0.1291
\end{aligned}$$

Step 4: Apply Comparison Law

$$\digamma(\mathcal{N}_1(\varrho)) \approx 0.1877 > \digamma(\mathcal{N}_2(\varrho)) \approx 0.1291$$

According to Definition 2.11, since $\digamma(\mathcal{N}_1(\varrho)) > \digamma(\mathcal{N}_2(\varrho))$, we conclude that $\mathcal{N}_1(\varrho)$ is superior to $\mathcal{N}_2(\varrho)$, denoted as $\mathcal{N}_1(\varrho) \succ \mathcal{N}_2(\varrho)$.

Definition 2.12. Let $\mathcal{N}^1(\varrho) = \langle\{[\mathfrak{L}_{\phi^{1(f)}}, \mathfrak{U}_{\phi^{1(f)}}](\hat{\varrho}^{(f)}), [\mathfrak{M}_{\varphi^{1(h)}}, \mathfrak{N}_{\varphi^{1(h)}}](\tilde{\varrho}^{(h)})\}\rangle$ and $\mathcal{N}^2(\varrho) = \langle\{[\mathfrak{L}_{\phi^{2(f)}}, \mathfrak{U}_{\phi^{2(f)}}](\hat{\varrho}^{(f)}),$ $[\mathfrak{M}_{\varphi^{2(h)}}, \mathfrak{N}_{\varphi^{2(h)}}](\tilde{\varrho}^{(h)})\}\rangle$ $(f = 1, 2, \ldots, F; h = 1, 2, \ldots, H)$ include two altered PULq-ROFNs where $\phi^{\epsilon(f)}$ and $\varphi^{\epsilon(h)}(\epsilon = 1, 2)$ contains the related subtitle of $[\mathfrak{L}_{\phi^{\epsilon(f)}}, \mathfrak{U}_{\phi^{\epsilon(f)}}]$ and $[\mathfrak{M}_{\varphi^{\epsilon(h)}}, \mathfrak{N}_{\varphi^{\epsilon(h)}}](\epsilon = 1, 2)\eta > 0$, afterward, the essential characteristics of PULq-ROFNs are described as below:

(1) $neg(\mathcal{N}^1(\varrho)) = \left\langle\left\{\left[\mathfrak{M}_{\varphi^{1(h)}}, \mathfrak{N}_{\varphi^{1(h)}}\right](\tilde{\varrho}^{(h)}), \left[\mathfrak{L}_{\phi^{1(f)}}, \mathfrak{U}_{\phi^{1(f)}}\right](\hat{\varrho}^{(f)})\right\}\right\rangle;$

(2) $\mathcal{N}^1(\varrho) \oplus \mathcal{N}^2(\varrho) = \left\langle\left\{\left[\mathfrak{L}_{\sqrt[q]{(\phi^{1(f)})^q + (\phi^{2(f)})^q - \left(\frac{(\phi^{1(f)})(\phi^{2(f)})}{\lambda}\right)^q}}, \mathfrak{U}_{\sqrt[q]{(\phi^{1(f)})^q + (\phi^{2(f)})^q - \left(\frac{(\phi^{1(f)})(\phi^{2(f)})}{\lambda}\right)^q}}\right](\hat{\varrho}^{(f)}), \left[\mathfrak{M}_{\frac{\varphi^{1(h)}\varphi^{2(h)}}{\lambda}}, \mathfrak{N}_{\frac{\varphi^{1(h)}\varphi^{2(h)}}{\lambda}}\right](\tilde{\varrho}^{(h)})\right\}\right\rangle;$

(3) $\mathcal{N}^1(\varrho) \otimes \mathcal{N}^2(\varrho) = \left\langle\left\{\left[\mathfrak{L}_{\frac{\phi^{1(f)}\phi^{2(f)}}{\lambda}}, \mathfrak{U}_{\frac{\phi^{1(f)}\phi^{2(f)}}{\lambda}}\right](\hat{\varrho}^{(f)}), \left[\mathfrak{M}_{\sqrt[q]{(\varphi^{1(h)})^q + (\varphi^{2(h)})^q - \left(\frac{(\varphi^{1(h)})(\varphi^{2(h)})}{\lambda}\right)^q}}, \mathfrak{N}_{\sqrt[q]{(\varphi^{1(h)})^q + (\varphi^{2(h)})^q - \left(\frac{(\varphi^{1(h)})(\varphi^{2(h)})}{\lambda}\right)^q}}\right](\tilde{\varrho}^{(h)})\right\}\right\rangle;$

(4) $\eta\mathcal{N}^1(\varrho) = \left\langle\left\{\left[\mathfrak{L}_{\sqrt[q]{\delta^q - \lambda^q\left(1 - \frac{(\phi^{1(f)})^q}{\lambda^q}\right)^\eta}}, \mathfrak{U}_{\sqrt[q]{\lambda^q - \lambda^q\left(1 - \frac{(\phi^{1(f)})^q}{\lambda^q}\right)^\eta}}\right](\hat{\varrho}^{(f)}), \left[\mathfrak{M}_{\lambda\left(\frac{\varphi^{1(h)}}{\lambda}\right)^\eta}, \mathfrak{N}_{\lambda\left(\frac{\varphi^{1(h)}}{\lambda}\right)^\eta}\right](\tilde{\varrho}^{(h)})\right\}\right\rangle;$

(5) $(\mathcal{N}^1(\varrho))^\eta = \left\langle\left\{\left[\mathfrak{L}_{\lambda\left(\frac{\phi^{1(f)}}{\lambda}\right)}, \mathfrak{U}_{\lambda\left(\frac{\phi^{1(f)}}{\lambda}\right)}\right]^\eta(\hat{\varrho}^{(f)}), \left[\mathfrak{M}_{\sqrt[q]{\lambda^q - \lambda^q\left(1 - \frac{(\varphi^{1(h)})^q}{\lambda^q}\right)}}, \mathfrak{N}_{\sqrt[q]{\lambda^q - \lambda^q\left(1 - \frac{(\varphi^{1(h)})^q}{\lambda^q}\right)}}\right]^\eta(\tilde{\varrho}^{(h)})\right\}\right\rangle.$

Example 2.4
Let the linguistic term set be $\mathbb{S} = \{s_{-3} : \text{VP}, s_{-2} : \text{P}, s_{-1} : \text{SP}, s_0 : \text{N}, s_1 : \text{SG}, s_2 : \text{G}, s_3 : \text{VG}\}$, where $\lambda = 3$. Let the rung parameter be $q = 3$.

Consider two PULq-ROFNs and a scalar value:

$$\mathcal{N}_1(\varrho) = \langle\{[s_1, s_2](1.0)\}, \{[s_{-1}, s_0](1.0)\}\rangle$$

$$\mathcal{N}_2(\varrho) = \langle\{[s_0, s_1](1.0)\}, \{[s_{-2}, s_{-1}](1.0)\}\rangle$$

$$\eta = 2$$

1. Negation Operation:

$$\text{neg}(\mathcal{N}_1(\varrho)) = \langle\{[s_{-1}, s_0](1.0)\}, \{[s_1, s_2](1.0)\}\rangle$$

2. Addition Operation:

$$\mathcal{N}_1(\varrho) \oplus \mathcal{N}_2(\varrho) = \left\langle\left\{\left[s_{\sqrt[3]{1^3 + 0^3 - (1\cdot 0/3)^3}},\ s_{\sqrt[3]{2^3 + 1^3 - (2\cdot 1/3)^3}}\right](1.0)\right\}, \left\{\left[s_{(-1)(-2)/3},\ s_{0\cdot(-1)/3}\right](1.0)\right\}\right\rangle$$

3. Multiplication Operation:

$$\mathcal{N}_1(\varrho) \otimes \mathcal{N}_2(\varrho) = \left\langle\left\{\left[s_{1\cdot 0/3},\ s_{2\cdot 1/3}\right](1.0)\right\}, \left\{\left[s_{\sqrt[3]{(-1)^3 + (-2)^3 - ((-1)(-2)/3)^3}},\ s_{\sqrt[3]{0^3 + (-1)^3 - (0\cdot(-1)/3)^3}}\right](1.0)\right\}\right\rangle$$

4. Scalar Multiplication:

$$\eta\mathcal{N}_1(\varrho) = \left\langle\left\{\left[s_{3\sqrt[3]{1 - (1 - (1/3)^3)^2}},\ s_{3\sqrt[3]{1 - (1 - (2/3)^3)^2}}\right](1.0)\right\}, \left\{\left[s_{3\cdot(-1/3)^2},\ s_{3\cdot(0/3)^2}\right](1.0)\right\}\right\rangle$$

5. Power Operation:

$$(\mathcal{N}_1(\varrho))^\eta = \left\langle\left\{\left[s_{3\cdot(1/3)^2},\ s_{3\cdot(2/3)^2}\right](1.0)\right\}, \left\{\left[s_{3\sqrt[3]{1 - (1 - (-1/3)^3)^2}},\ s_{3\sqrt[3]{1 - (1 - (0/3)^3)^2}}\right](1.0)\right\}\right\rangle$$

Calculation Results (Approximated):

$$\begin{aligned} \text{2. Addition:} &\approx \langle \{[s_{1.0},\ s_{2.08}](1.0)\}, \{[s_{0.67},\ s_{0.0}](1.0)\} \rangle \\ \text{3. Multiplication:} &\approx \langle \{[s_{0.0},\ s_{0.67}](1.0)\}, \{[s_{-2.08},\ s_{-1.0}](1.0)\} \rangle \\ \text{4. Scalar Mult:} &\approx \langle \{[s_{1.25},\ s_{2.39}](1.0)\}, \{[s_{0.33},\ s_{0.0}](1.0)\} \rangle \\ \text{5. Power:} &\approx \langle \{[s_{0.33},\ s_{1.33}](1.0)\}, \{[s_{-2.39},\ s_{-1.25}](1.0)\} \rangle \end{aligned}$$

Theorem 2.13. *Let* $\mathcal{N}^1(\varrho) = \langle \{[\mathfrak{L}_{\phi^{1(f)}}, \mathfrak{U}_{\phi^{1(f)}}](\hat{\varrho}^{(f)}), [\mathfrak{M}_{\varphi^{1(h)}}, \mathfrak{N}_{\varphi^{1(h)}}](\tilde{\varrho}^{(h)})\} \rangle$ *and* $\mathcal{N}^2(\varrho) = \langle \{[\mathfrak{L}_{\phi^{2(f)}}, \mathfrak{U}_{\phi^{2(f)}}](\hat{\varrho}^{(f)}), [\mathfrak{M}_{\varphi^{2(h)}}, \mathfrak{N}_{\varphi^{2(h)}}](\tilde{\varrho}^{(h)})\} \rangle$ *($f = 1, 2, \ldots, F; h = 1, 2, \ldots, H$) be any two adjusted PULq-ROFNs,* $\xi,\ \xi_1,\ \xi_2, > 0$*, then*

- $\mathcal{N}^1(\varrho) \oplus \mathcal{N}^2(\varrho) = \mathcal{N}^2(\varrho) \oplus \mathcal{N}^1(\varrho)$;
- $\mathcal{N}^1(\varrho) \otimes \mathcal{N}^2(\varrho) = \mathcal{N}^2(\varrho) \otimes \mathcal{N}^1(\varrho)$;
- $\xi(\mathcal{N}^1(\varrho) \oplus \mathcal{N}^2(\varrho)) = \xi\mathcal{N}^1(\varrho) \oplus \xi\mathcal{N}^2(\varrho)$;
- $\xi_1\mathcal{N}^1(\varrho) \oplus \xi_2\mathcal{N}^1(\varrho) = (\xi_1 + \xi_2)\mathcal{N}^1(\varrho)$;
- $(\mathcal{N}^1(\varrho))^{\xi_1} \otimes (\mathcal{N}^1(\varrho))^{\xi_2} = (\mathcal{N}^1(\varrho))^{\xi_1+\xi_2}$;
- $(\mathcal{N}^1(\varrho))^{\xi} \otimes (\mathcal{N}^2(\varrho))^{\xi} = (\mathcal{N}^1(\varrho) \otimes \mathcal{N}^2(\varrho))^{\xi}$.

Definition 2.14. Let $\mathcal{N}^1(\varrho) = \langle \{[\mathfrak{L}_{\phi^{1(f)}}, \mathfrak{U}_{\phi^{1(f)}}](\hat{\varrho}^{(f)}), [\mathfrak{M}_{\varphi^{1(h)}}, \mathfrak{N}_{\varphi^{1(h)}}](\tilde{\varrho}^{(h)})\} \rangle$ and $\mathcal{N}^2(\varrho) = \langle \{[\mathfrak{L}_{\phi^{2(f)}}, \mathfrak{U}_{\phi^{2(f)}}](\hat{\varrho}^{(f)}), [\mathfrak{M}_{\varphi^{2(h)}}, \mathfrak{N}_{\varphi^{2(h)}}](\tilde{\varrho}^{(h)})\} \rangle$ ($f = 1, 2, \ldots, F; h = 1, 2, \ldots, H$) comprise two modified PULq-ROFNs, followed by Hamming distance (HD) $(\mathcal{N}^1(\varrho), \mathcal{N}^2(\varrho))$. The distance between $\mathcal{N}^1(\varrho)$ and $\mathcal{N}^2(\varrho)$ has the following definition:

$$HD(\mathcal{N}^1(\varrho), \mathcal{N}^2(\varrho)) = \sqrt{\frac{\sum_{f=1}^{\#F_\phi} \left(|t(\mathfrak{L}_{\phi^{1(f)}})\hat{\varrho}^{(f)} - t(\mathfrak{L}_{\phi^{2(f)}})\hat{\varrho}^{(f)}|^q + |t(\mathfrak{U}_{\phi^{1(f)}})\hat{\varrho}^{(f)} - t(\mathfrak{U}_{\phi^{2(f)}})\hat{\varrho}^{(f)}|^q \right)}{2\#F_\phi}} + \sqrt{\frac{\sum_{h=1}^{\#H_\varphi} \left(|t(\mathfrak{M}_{\varphi^{1(h)}})\tilde{\varrho}^{(h)} - t(\mathfrak{M}_{\varphi^{2(h)}})\tilde{\varrho}^{(h)}|^q + |t(\mathfrak{N}_{\varphi^{1(h)}})\tilde{\varrho}^{(h)} - t(\mathfrak{N}_{\varphi^{2(h)}})\tilde{\varrho}^{(h)}|^q \right)}{2\#H_\varphi}}. \quad (2.10)$$

3. PULq-ROF-CRITIC-MABAC method

The MABAC method is known for it is precision in addressing DM challenges, thanks to it is simplicity and adaptability. This part contains introduction of an MABAC method tailored for group DM problems within the PULq-ROFSs enviornment. This approach is designed to handle assessment information expressed in PULq-ROFSs effectively. Additionally, we outline a methodology for calculating criteria weights. First, we describe the CRITIC method adapted for PULq-ROFSs. Let $A = \{A_1, A_2, \ldots, A_m\}$, where $(f = 1, 2, \ldots, m)$ represent a set of m alternatives measured against n criteria $C = \{C_1, C_2, \ldots, C_n\}$. Let $E = (E_1, E_2, \ldots E_p)$ denote the p DMs involved in the evaluation process. Assume that $W = (W_1, W_2, \ldots, W_n)$ and $W = (W_1, W_2, \ldots, W_p)^T$ represent the weights of the criteria and the weights of the DMs, respectively, where $W_j \in [0, 1]$, $\sum_j^n W_j = 1$, $W_l \in [0, 1]$, $\sum_l^p W_l = 1$. In this approach, PULq-ROFNs are used to assess alternatives against all criteria. The procedural steps of the CRITIC method are illustrated graphically in Figure 1, while the MABAC method are illustrated as a flowchart in Figure 2.

3.1. MABAC method for PULq-ROFSs

The following is an explanation of the PULq-ROF-MABAC technique application methodology:

Step 1. *Organize the uncertain probabilistic linguistic term.*

The DMs choose a set of appropriate linguistic terms to evaluate the possibilities in most DM situations. The DMs first defined the PULTS and then gave them numerical values. The PUL phrases are constructed by the DMs in this step, and they are given their corresponding PULq-ROFNs.

Step 2. *Experts from the decision matrices.*

$$R^l = [A^l_{fj}]_{m\times n} \tag{3.1}$$
$$= \begin{bmatrix} ([\mathfrak{L}^l_{\phi_{11}}, \mathfrak{U}^l_{\phi_{11}}](\hat{\varrho}^l_{11}), [\mathfrak{M}^l_{\varphi_{11}}, \mathfrak{N}^l_{\varphi_{11}}](\tilde{\varrho}^l_{11})) & ([\mathfrak{L}^l_{\phi_{12}}, \mathfrak{U}^l_{\phi_{12}}](\hat{\varrho}^l_{12}), [\mathfrak{M}^l_{\varphi_{12}}, \mathfrak{N}^l_{\varphi_{12}}](\tilde{\varrho}^l_{12})) & \cdots & ([\mathfrak{L}^l_{\phi_{1n}}, \mathfrak{U}^l_{\phi_{1n}}](\hat{\varrho}^l_{1n}), [\mathfrak{M}^l_{\varphi_{1n}}, \mathfrak{N}^l_{\varphi_{1n}}](\tilde{\varrho}^l_{1n})) \\ ([\mathfrak{L}^l_{\phi_{21}}, \mathfrak{U}^l_{\phi_{21}}](\hat{\varrho}^l_{21}), [\mathfrak{M}^l_{\varphi_{21}}, \mathfrak{N}^l_{\varphi_{21}}](\tilde{\varrho}^l_{21})) & ([\mathfrak{L}^l_{\phi_{22}}, \mathfrak{U}^l_{\phi_{22}}](\hat{\varrho}^l_{22}), [\mathfrak{M}^l_{\varphi_{22}}, \mathfrak{N}^l_{\varphi_{22}}](\tilde{\varrho}^l_{22})) & \cdots & ([\mathfrak{L}^l_{\phi_{2n}}, \mathfrak{U}^l_{\phi_{2n}}](\hat{\varrho}^l_{2n}), [\mathfrak{M}^l_{\varphi_{2n}}, \mathfrak{N}^l_{\varphi_{2n}}](\tilde{\varrho}^l_{2n})) \\ \vdots & \vdots & \vdots & \vdots \\ ([\mathfrak{L}^l_{\phi_{m1}}, \mathfrak{U}^l_{\phi_{m1}}](\hat{\varrho}^l_{m1}), [\mathfrak{M}^l_{\varphi_{m1}}, \mathfrak{N}^l_{\varphi_{m1}}](\tilde{\varrho}^l_{m1})) & ([\mathfrak{L}^l_{\phi_{m2}}, \mathfrak{U}^l_{\phi_{m2}}](\hat{\varrho}^l_{m2}), [\mathfrak{M}^l_{\varphi_{m2}}, \mathfrak{N}^l_{\varphi_{m2}}](\tilde{\varrho}^l_{m2})) & \cdots & ([\mathfrak{L}^l_{\phi_{mn}}, \mathfrak{U}^l_{\phi_{mn}}](\hat{\varrho}^l_{mn}), [\mathfrak{M}^l_{\varphi_{mn}}, \mathfrak{N}^l_{\varphi_{mn}}](\tilde{\varrho}^l_{mn})) \end{bmatrix};$$

where $A^l_{fj} = \langle[\mathfrak{L}^l_{\phi_{fj}}, \mathfrak{U}^l_{\phi_{fj}}](\hat{\varrho}^l_{fj}), [\mathfrak{M}^l_{\varphi_{fj}}, \mathfrak{N}^l_{\varphi fj}](\hat{\varrho}^l_{fj})\rangle$ $f = (1, 2, \ldots, m), j = (1, 2, \ldots, n)$ represents the PULq-ROFSs evaluation data of the alternative A_f provided by the expert E_l in relation to the criterion C_j.

Step 3. *Create the combined matrix.*

To create a combined matrix, the DMs decision matrices from step 2 must be used. As a result, the isolated decision matrices are subjected to the probabilistic uncertain linguistic q-rung orthopair fuzzy weighted average (PULq-ROFWA) operator (Eq. (3.3)), and an overall PULq-ROFSs matrix $r = [A_{fj}]_{m\times n}$ is built as:

$$R = [A_{fj}]_{m\times n} \tag{3.2}$$
$$= \begin{bmatrix} ([\mathfrak{L}_{\phi_{11}}, \mathfrak{U}_{\phi_{11}}](\hat{\varrho}_{11}), [\mathfrak{M}_{\varphi_{11}}, \mathfrak{N}_{\varphi_{11}}](\tilde{\varrho}_{11})) & ([\mathfrak{L}_{\phi_{12}}, \mathfrak{U}_{\phi_{12}}](\hat{\varrho}_{12}), [\mathfrak{M}_{\varphi_{12}}, \mathfrak{N}_{\varphi_{12}}](\tilde{\varrho}^l_{12})) & \cdots & ([\mathfrak{L}_{\phi_{1n}}, \mathfrak{U}_{\phi_{1n}}](\hat{\varrho}_{1n}), [\mathfrak{M}_{\varphi_{1n}}, \mathfrak{N}_{\varphi_{1n}}](\tilde{\varrho}_{1n})) \\ ([\mathfrak{L}_{\phi_{21}}, \mathfrak{U}_{\phi_{21}}](\hat{\varrho}_{21}), [\mathfrak{M}_{\varphi_{21}}, \mathfrak{N}_{\varphi_{21}}](\tilde{\varrho}_{21})) & ([\mathfrak{L}_{\phi_{22}}, \mathfrak{U}_{\phi_{22}}](\hat{\varrho}_{22}), [\mathfrak{M}_{\varphi_{22}}, \mathfrak{N}_{\varphi_{22}}](\tilde{\varrho}_{22})) & \cdots & ([\mathfrak{L}^l_{\phi_{2n}}, \mathfrak{U}^l_{\phi_{2n}}](\hat{\varrho}^l_{2n}), [\mathfrak{M}^l_{\varphi_{2n}}, \mathfrak{N}^l_{\varphi_{2n}}](\tilde{\varrho}^l_{2n})) \\ \vdots & \vdots & \vdots & \vdots \\ ([\mathfrak{L}_{\phi_{m1}}, \mathfrak{U}_{\phi_{m1}}](\hat{\varrho}_{m1}), [\mathfrak{M}_{\varphi_{m1}}, \mathfrak{N}_{\varphi_{m1}}](\tilde{\varrho}_{m1})) & ([\mathfrak{M}_{\phi_{m2}}, \mathfrak{N}_{\phi_{m2}}](\hat{\varrho}_{m2}), [\mathfrak{M}_{\varphi_{m2}}, \mathfrak{N}_{\varphi_{m2}}](\tilde{\varrho}_{m2})) & \cdots & ([\mathfrak{L}_{\phi_{mn}}, \mathfrak{U}_{\phi_{mn}}](\hat{\varrho}_{mn}), [\mathfrak{M}_{\varphi_{mn}}, \mathfrak{N}_{\varphi_{mn}}](\tilde{\varrho}_{mn})) \end{bmatrix};$$

where $A_{fj} = \langle[\mathfrak{L}_{\phi_{fj}}, \mathfrak{U}_{\phi_{fj}}](\hat{\varrho}_{fj}), [\mathfrak{M}_{\varphi_{fj}}, \mathfrak{N}_{\varphi fj}](\hat{\varrho}_{fj})\rangle$f = (1, 2, ..., m), j = (1, 2, ..., n) represents the combined PULq-ROF evaluation information of the alternatives A_f(f = 1, 2, ..., m) with respect to the criteria C_j(j= 1, 2, ..., n).

$$PULq-ROFWA(A^1_{fj}, A^2_{fj}, \ldots, A^p_{fj}) =$$
$$\left(\left[((1-\prod_{l=1}^{p}((1-t(\mathfrak{L}^l_{\phi_{fj}})^q)^{W_l})^{\frac{1}{q}}, (1-\prod_{l=1}^{p}((1-t(\mathfrak{U}^l_{\phi_{fj}})^q)^{W_l})^{\frac{1}{q}}\right](\hat{\varrho}_{fj}), \quad \left[\prod_{l=1}^{p} t(\mathfrak{M}_{\varphi fj})^{W_l}, \prod_{l=1}^{p} t(\mathfrak{N}_{\varphi fj})^{W_l}\right](\hat{\varrho}_{fj})\right) \tag{3.3}$$

Step 4. *Determining the weights of the criteria.*

Calculate the criteria weights using any suitable technique.

Step 5. *Create the weighted combined decision matrix.*

Calculate the PULq-ROFS weighted matrix $W\mathcal{N}_{fj} = \langle[\mathfrak{L}_{\phi_{fj}}, \mathfrak{U}_{\phi_{fj}}](\hat{\varrho}_{fj}), [\mathfrak{M}_{\varphi fj}, \mathfrak{N}_{\varphi fj}](\hat{\varrho}_{fj})\rangle$ (f = 1, 2, ..., m, j = 1, 2, ..., n) by using the matrix r=$[A^l_{fj}]_{m\times n} = \langle[\mathfrak{L}_{\phi_{fj}}, \mathfrak{U}_{\phi_{fj}}](\hat{\varrho}_{fj}), [\mathfrak{M}_{\varphi fj}, \mathfrak{N}_{\varphi fj}](\hat{\varrho}_{fj})\rangle$(f = 1, 2, ..., m, j = 1, 2, ..., n) and criteria weights W_j(j=1, 2, ..., n), the formula is given as follows:

$$W\mathcal{N}_{fj} = W_j \bigotimes \mathcal{N}_{fj} = \left(\left[(1-\left(1-t(\mathfrak{L}_{\phi_{fj}})^q\right)^{W_j}\right)^{\frac{1}{q}}, (1-\left(1-t(\mathfrak{U}_{\phi_{fj}})^q\right)^{W_j}\right)^{\frac{1}{q}}\right](\hat{\varrho}_{fj}), \quad \left[t(\mathfrak{M}^{W_j}_{\varphi fj}), t(\mathfrak{N}^{W_j}_{\varphi fj})\right](\hat{\varrho}_{fj})\right). \tag{3.4}$$

Step 6. *Determine boundary approximation area.*

$G = (\tilde{g}_j)_{1\times n}$ is the boundary approximation area (BAA) matrix, whose elements may be found using the Eq. (**??**):

$$\tilde{g}_j = \left(\prod_{f=1}^{m} W\mathcal{N}_{fj}\right)^{\frac{1}{m}}$$

$$=\Bigg(\left[(\prod_{f=1}^{m} t(\mathfrak{L}_{\phi_f}))^{\frac{1}{m}}, (\prod_{f=1}^{m} t(\mathfrak{U}_{\phi_f}))^{\frac{1}{m}}\right](\prod_{f=1}^{m}(\hat{\varrho}_f))^{\frac{1}{m}},$$
$$\left[(1-\prod_{f=1}^{m}(1-t(\mathfrak{M}_{\varphi f})^q)^{\frac{1}{m}})^{\frac{1}{q}}, (1-\prod_{f=1}^{m}(1-t(\mathfrak{N}_{\varphi f})^q)^{\frac{1}{m}})^{\frac{1}{q}}\right](\prod_{f=1}^{m}(\hat{\varrho}_f))^{\frac{1}{m}}\Bigg). \tag{3.5}$$

Step 7. *Create a distance matrix.*
Determine the distance of each choice from BAA Table 8 to create the distance matrix $D = (d_{fj})_{m\times n}$ as follows:

$$d_{fj} = \begin{cases} d(W\mathcal{N}_{fj}, \tilde{g}_j), & W\mathcal{N}_{fj} > \tilde{g}_j, \\ 0, & W\mathcal{N}_{fj} = \tilde{g}_j, \\ -d(W\mathcal{N}_{fj}, \tilde{g}_j), & W\mathcal{N}_{fj} < \tilde{g}_j, \end{cases} \tag{3.6}$$

where $W\mathcal{N}_{fj}$ shows the distance between $W\mathcal{N}_{fj}$ and $\tilde{g}_j$ as given below:

$$d_{fj} = d(W\mathcal{N}_{fj}, \tilde{g}_j) = \frac{1}{3}\left(\left|t(\mathfrak{L}_{\phi_f^1})(\hat{\varrho}_f^1) - t(\mathfrak{L}_{\phi_f^2})(\hat{\varrho}_f^2)\right|^q + \left|t(\mathfrak{U}_{\phi_f^1})(\hat{\varrho}_f^1) - t(\mathfrak{U}_{\phi_f^2})(\hat{\varrho}_f^2)\right|^q + \left|t(\mathfrak{M}_{\varphi_f^1})(\hat{\varrho}_f^1) - t(\mathfrak{M}_{\varphi_f^2})(\hat{\varrho}_f^2)\right|^q + \left|t(\mathfrak{N}_{\varphi_f^1})(\hat{\varrho}_f^1) - t(\mathfrak{N}_{\varphi_f^2})(\hat{\varrho}_f^2)\right|^q\right). \tag{3.7}$$

Step 8. *Add the values.*
Each alternative value should be added as:

$$S_f = \sum_{j=1}^{n} d_{fj}. \tag{3.8}$$

Step 9. *Rank the alternatives.*
Using the values obtained in Step 8, arrange the options in decreasing order. The best option will be determined by the alternative with the highest value.

3.2. CRITIC method

In the MCGDM process, the importance of different criteria plays a key role in determining the ranking of alternatives. There are various methods available to calculate these criteria weights. Since MCGDM problems involve multiple conflicting criteria with varying levels of significance, assigning accurate weights can sometimes be challenging for DMs, especially when they have limited time or knowledge. To address this issue, Diakoulaki et al. [34] introduced the CRITIC method, which helps in determining the weights of criteria. The main steps of the CRITIC method are as follows:
Step 1^*. *Correlation coefficient calculation.*
Utilizing the score function formula by Akram and Shumaiza [37] given in Eq. (3.9), determine the combined decision matrix's scores, which were determined in step 3. Once the score matrix has been calculated, use Eq. (3.10) to determine the correlation coefficient (ϱ_{jk}) between the criteria of the combined decision matrix $(A_{fj})_{m\times n}$.

$$S(A_{fj}) = \frac{1}{3}\left(1 + t(\mathfrak{L}_{\phi_{fj}})^q(\hat{\varrho}_{fj}) + t(\mathfrak{U}_{\phi_{fj}})^q(\hat{\varrho}_{fj}) - (t(\mathfrak{M}_{\varphi_{fj}})^q(\hat{\varrho}_{fj}) + t(\mathfrak{N}_{\varphi fj})^q(\hat{\varrho}_{fj})))\right), \tag{3.9}$$

$$\varrho_{jk} = \frac{\sum_{f=1}^{m}\left(S(A_{fj}) - S(A_j)\right)\left(S(A_{fk}) - S(A_k)\right)}{\sqrt{\sum_{f=1}^{m}\left(S(A_{fj}) - S(A_j)\right)^2}\sqrt{\sum_{f=1}^{m}\left(S(A_{fk}) - S(A_k)\right)^2}}, \quad j,k = 1,2,\ldots,n, \tag{3.10}$$

where

$$S(A_j) = \frac{1}{m}\sum_{f=1}^{m} S(A_{fj}),\ and \quad S(A_k) = \frac{1}{m}\sum_{f=1}^{m} S(A_{fk})$$

. **Step 2*.** *Calculating the standard deviation.*
In this stage, use Eq. (3.11) to get the standard deviation $(\sigma)_j$ of each criterion

$$\sigma_j = \sqrt{\frac{\sum_{f=1}^{m}\left(S(A_{fj}) - S(A_j)\right)^2}{m}}, \quad j = 1, 2, \ldots, n. \tag{3.11}$$

Step 3* *Computation of criteria weights.*
Determine the weights of the criteria using Eq. (3.12).

$$W_j = \frac{\sigma_j \sum_{k=1}^{n}(1 - \varrho_{jk})}{\sum_{j=1}^{n}\left(\sigma_j \sum_{k=1}^{n}(1 - \varrho_{jk})\right)}, \quad j = 1, 2, \ldots, n, \tag{3.12}$$

with $W_j \epsilon [0, 1]$ and $\sum_{j=1}^{n} W_j = 1$.

We provide the following algorithm, which outlines the step-by-step procedure of the CRITIC-MABAC method, including normalization, weight calculation, and ranking, under the framework of PULq-ROFSs.

Algorithm PULq-ROF-CRITIC-MABAC Method

Require: Alternatives: $A = \{A_1, A_2, \ldots, A_m\}$
Criteria: $C = \{C_1, C_2, \ldots, C_n\}$
DMs: $E = \{E_1, E_2, \ldots, E_p\}$ with weights W_l
Linguistic term set $\mathbb{S}$, parameter $q \geq 1$

Ensure: Ranking of alternatives

Step 1: Construct PULq-ROF decision matrices R^l for each DM E_l
Step 2: Aggregate all R^l into combined matrix R using PULq-ROFWA operator
Step 3: Compute criteria weights via CRITIC:
3.1. Compute score matrix $S(A_{fj})$
3.2. Compute correlation matrix ϱ_{jk}
3.3. Compute standard deviation σ_j
3.4. Compute weights W_j
Step 4: Construct weighted combined matrix $WN_{fj} = W_j \otimes A_{fj}$
Step 5: Compute BAA matrix $\tilde{g}_j = \left(\prod_{f=1}^{m} WN_{fj}\right)^{1/m}$
Step 6: Compute distance matrix $D = (d_{fj})_{m\times n}$ by equation 17 and:

$$d_{fj} = \begin{cases} d(WN_{fj}, \tilde{g}_j), & \text{if } WN_{fj} > \tilde{g}_j \\ 0, & \text{if } WN_{fj} = \tilde{g}_j \\ -d(WN_{fj}, \tilde{g}_j), & \text{if } WN_{fj} < \tilde{g}_j \end{cases}$$

Step 7: Compute total score $S_f = \sum_{j=1}^{n} d_{fj}$ for each alternative
Step 8: Rank alternatives in descending order of S_f
return Ranked alternatives

4. Application in cloud storage service

A numerical example of the suggested method is given in this section. We begin by providing a brief introduction of the problem under consideration. Next, we apply the method that have been presented to

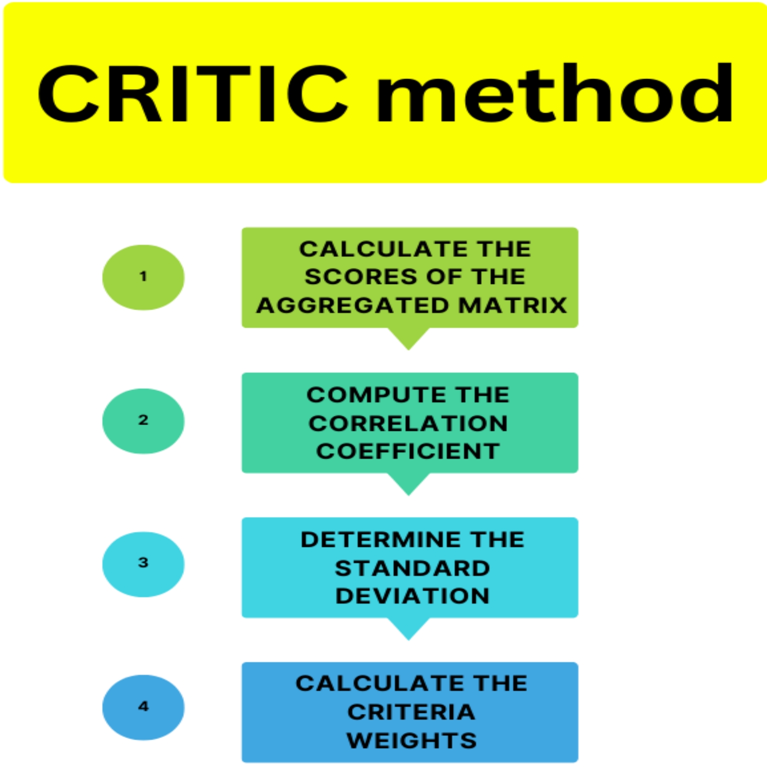

Figure 1: Steps of CRITIC method

the paper.

4.1. Description of the problem

In this digital era, cloud storage services have become a cornerstone for both personal and professional use. They offer a flexible and scalable way to store, access, and manage data online. Whether it is backing up critical files, hosting applications, or sharing large datasets with a team, cloud storage provides unmatched convenience and accessibility compared to traditional methods. Popular platforms like Google Drive, Dropbox, and Microsoft dominate the market, each catering to specific needs with their unique features. However, choosing the right cloud storage service is not straightforward. With so many options available, evaluating key factors like cost, reliability, security, and performance can be overwhelming. Adding to the complexity is the absence of predefined criteria to weigh these criteria. This is where the our proposed technique for PULq-ROFSs comes into play. This innovative approach leverages PULq-ROFSs to simplify DM. The q-rung structure of these fuzzy sets adds depth to the evaluation, enabling a more nuanced representation of uncertainty. Unlike traditional methods, this dual membership framework captures varying degrees of satisfaction and hesitation, offering a more accurate reflection of real world DM. By aggregating these evaluations, our proposed technique generates a clear ranking of cloud storage services, even without predefined weights. This makes the DM process more dynamic, transparent, and logical. it is a practical solution for navigating the complexities and uncertainties of choosing the right service, ensuring that the final choice aligns with both individual and organizational needs. This example demonstrates the way to take on a complex MCGDM problem using PULq-ROFSs approach. A business looks for the best cloud storage option to satisfy it is operational needs. The business evaluates five options: A_1: Google Drive, A_2: Drop box , A_3: Microsoft One Drive , A_4: Amazon Drive, and A_5: iCloud with the help of a set of four DMs $D = \{D_1, D_2, D_3, D_4\}$. To reflect their differential expertise, a weighting vector of $(0.1, 0.2, 0.3, 0.4)^T$ was assigned, ensuring that the judgments of more

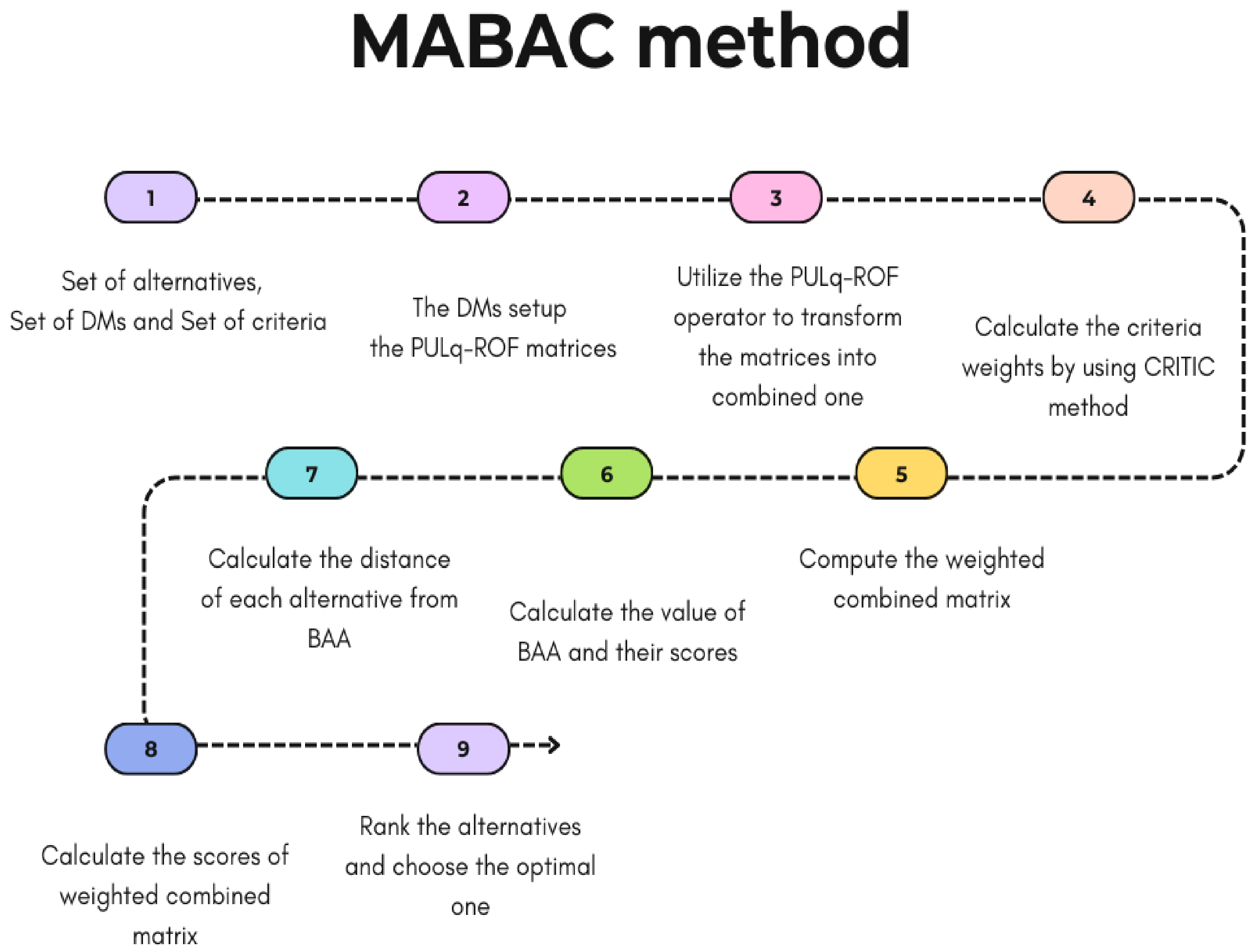

Figure 2: Flowchart containing steps of MABAC method

senior experts proportionally influenced the aggregate evaluation. The weights were determined based on the experts' years of experience and specialized knowledge in cloud storage technologies, with the most senior expert receiving the highest weight. The evaluation is based on four important criteria: C_1: cost, C_2: reliability, C_3: security, and C_4: performance. The PULq-ROFSs framework allows DMs to use linguistic terms set $S = \{s_{-6} : \text{Extremely Poor}(EP), s_{-5} : \text{Very Poor}(VP), s_{-4} : \text{Poor}(P), s_{-3} : \text{Below Average}(BA), s_{-2} :$ $\text{Somewhat Poor}(SP), s_{-1} : \text{Slightly Poor}(SP), s_0 : \text{Neutral}(N), s_1 : \text{Slightly Good}(SG), s_2 :$ $\text{Somewhat Good}(SG), s_3 : \text{Above Average}(AA), s_4 : \text{Good}(G), s_5 : \text{Very Good}(VG), s_6 :$ $\text{Extremely Good}(EG)\}$ to express their evaluations while assigning probabilities to these terms to capture both evaluations and their associated uncertainties. Data collection begins when DMs submit their evaluations for each service based on four specific criteria. The evaluations convert into PULq-ROFSs that represent the various perspectives and preferences of DMs. The framework enables evaluations to be aggregated and analyzed to create a comprehensive ranking system for cloud storage solutions. The main challenge throughout the evaluation process is the lack of established weights for these criteria, which makes DMs choice more complex. The evaluation procedure treats all criteria equally and operates without specified weight distri-

butions. The final rankings give actionable insights which help an organization to select the best solution and develop a clear framework for future DM records. The PULq-ROFSs technique produces reliable and equitable findings while demonstrating it is flexibility and practical application for real world DM situations when ambiguity and inadequate information exist.
Here are the brief description of each alternative:

- A_1**:** A cloud service that integrates with Google Workspace, offering seamless collaboration and generous free storage.

- A_2**:** A widely used platform for file sharing, synchronization, and cloud storage with easy to use file management.

- A_3**:** Cloud storage that integrates tightly with Microsoft 365, providing easy access and collaboration on files.

- A_4**:** Amazon cloud storage service offering secure and scalable storage with integration into Amazon ecosystem.

- A_5**:** Apple cloud storage solution that syncs photos, documents, and app data across Apple devices seamlessly.

The criteria are defined as follows:

- C_1**:** Represents the financial expenditure involved in using the cloud storage service, including subscription fees, extra charges, and hidden costs.

- C_2**:** Measures the services ability to operate consistently without interruptions, failures, or data loss, ensuring minimal downtime.

- C_3**:** Evaluates the robustness of data protection measures, such as encryption, fire wall configurations, and compliance with privacy regulations.

- C_4**:** Reflects the overall efficiency, speed, and responsiveness of the service, including upload/download speeds and system scalability.

4.2. Use of the CRITIC-MABAC technique on PULq-ROFSs

The key procedural steps for applying the CRITIC-MABAC method in the given context are outlined as follows:
Step 1. Formulate the decision matrices within the PULq-ROFS framework.
$R^l = [A^l_{fj}]_{5\times 4} = \langle [\mathfrak{L}^l_{\phi_{fj}}, \mathfrak{U}^l_{\phi_{fj}}](\hat{\varrho}^l_{fj}), [\mathfrak{M}^l_{\varphi_{fj}}, \mathfrak{N}^l_{\varphi fj}](\hat{\varrho}^l_{fj})\rangle_{5\times 4}$ where f = (1, 2, ..., 5), j = (1, 2, ..., 4) as shown in Table 2, Table **??**, Table **??** and Table **??**.
Step 2. Since all the given criteria are of the benefit type, normalization of the matrix is not required.
Step 3. Based on Eq. (3.3), the PULq-ROFWA operator is applied to merge the decision matrices R^l where $l = 1, 2, 3, 4$ into a combined matrix R, as presented in Table **??**.
Step 4. The CRITIC method is employed to determine the criteria weights.

- Calculate the score of combined matrix as shown in Table 10.
- The correlation coefficient matrix ϱ_{jk} (where $j, k = 1, 2, 3, 4$) is derived by computing the correlation coefficients between criteria pairs using Eq. (3.10). The resulting matrix is displayed in Table 3.
- By applying Eq. (3.11), the standard deviation of each criteria are computed, and the results are displayed in Table 4.
- By utilizing Eq. (3.12), the objective weight for each criteria are derived, and the results are displayed in Table 5.

Step 5. The weighted combined decision matrix values are presented in Table 6.
Step 6. Table 8 has the BAA numerical values.
Step 7. The weighted combined matrix and BAA matrix score values as shown in Table 7 as well as Table 9 used to calculate distance matrix, while the calculated distance matrix is contained in Table 11.
Step 8. Arrange the alternatives according to the MABAC index, where the option with the highest A_f value is likely to be the optimal choice. The rankings of cloud storage services are displayed in Table 12 as like $A_5 \succ A_1 \succ A_4 \succ A_3 \succ A_2$.

Table 2: **PULq-ROF decision matrix provided by $\mathcal{D}_1$.**

Alternatives	C_1
A_1	$\langle\{[\mathfrak{L}_{-6},\mathfrak{U}_{-5}](0.1),[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.4),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.5)\},\{[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.4),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.2),[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.4)\}\rangle$
A_2	$\langle\{[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.3),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.3),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.4)\},\{[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.1),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.1),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.8)\}\rangle$
A_3	$\langle\{[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.3),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.6),[\mathfrak{L}_{2},\mathfrak{U}_{3}](0.1)\},\{[\mathfrak{M}_{-6},\mathfrak{N}_{-5}](0.2),[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.4),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4)\}\rangle$
A_4	$\langle\{[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.3),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.2),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.5)\},\{[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.4),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.4),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.2)\}\rangle$
A_5	$\langle\{[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.6),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.1),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.3)\},\{[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.5),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.4),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.1)\}\rangle$
Alternatives	C_2
A_1	$\langle\{[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.1),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.4),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.5)\},\{[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.4),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.2),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.4)\}\rangle$
A_2	$\langle\{[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.3),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.3),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.4)\},\{[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.1),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.1),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.8)\}\rangle$
A_3	$\langle\{[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.3),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.6),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.1)\},\{[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.2),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.4),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.4)\}\rangle$
A_4	$\langle\{[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.3),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.2),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.5)\},\{[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.4),[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.4),[\mathfrak{M}_{2},\mathfrak{N}_{3}](0.2)\}\rangle$
A_5	$\langle\{[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.6),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.1),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.3)\},\{[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.5),[\mathfrak{M}_{2},\mathfrak{N}_{3}](0.4),[\mathfrak{M}_{3},\mathfrak{N}_{4}](0.1)\}\rangle$
Alternatives	C_3
A_1	$\langle\{[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.1),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.4),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.5)\},\{[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.2),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.4)\}\rangle$
A_2	$\langle\{[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.3),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.3),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.4)\},\{[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.1),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.1),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.8)\}\rangle$
A_3	$\langle\{[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.3),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.6),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.1)\},\{[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.2),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.4),[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.4)\}\rangle$
A_4	$\langle\{[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.3),[\mathfrak{L}_{2},\mathfrak{U}_{3}](0.2),[\mathfrak{L}_{3},\mathfrak{U}_{4}](0.5)\},\{[\mathfrak{M}_{-6},\mathfrak{N}_{-5}](0.4),[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.4),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.2)\}\rangle$
A_5	$\langle\{[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.6),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.1),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.3)\},\{[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.5),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.1)\}\rangle$
Alternatives	C_4
A_1	$\langle\{[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.1),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.4),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.5)\},\{[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.4),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.2),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.4)\}\rangle$
A_2	$\langle\{[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.3),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.3),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.4)\},\{[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.1),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.1),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.8)\}\rangle$
A_3	$\langle\{[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.3),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.6),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.1)\},\{[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.2),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.4),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.4)\}\rangle$
A_4	$\langle\{[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.3),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.2),[\mathfrak{L}_{2},\mathfrak{U}_{3}](0.5)\},\{[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.4),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.2)\}\rangle$
A_5	$\langle\{[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.6),[\mathfrak{L}_{2},\mathfrak{U}_{3}](0.1),[\mathfrak{L}_{3},\mathfrak{U}_{4}](0.3)\},\{[\mathfrak{M}_{-6},\mathfrak{N}_{-5}](0.5),[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.4),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.1)\}\rangle$

Table 3: **PULq-ROF decision matrix provided by $\mathcal{D}_2$.**

Alternatives	C_1
A_1	$\langle\{[\mathfrak{L}_{-6},\mathfrak{U}_{-5}](0.1),[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.4),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.5)\},\{[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.2),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.4)\}\rangle$
A_2	$\langle\{[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.3),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.3),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.4)\},\{[\mathfrak{M}_{-6},\mathfrak{N}_{-5}](0.1),[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.1),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.8)\}\rangle$
A_3	$\langle\{[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.3),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.6),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.1)\},\{[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.2),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.4),[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.4)\}\rangle$
A_4	$\langle\{[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.3),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.2),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.5)\},\{[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.4),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.2)\}\rangle$
A_5	$\langle\{[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.6),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.1),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.3)\},\{[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.5),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.4),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.1)\}\rangle$
Alternatives	C_2
A_1	$\langle\{[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.1),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.4),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.5)\},\{[\mathfrak{M}_{-6},\mathfrak{N}_{-5}](0.4),[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.2),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4)\}\rangle$
A_2	$\langle\{[\mathfrak{L}_{-6},\mathfrak{U}_{-5}](0.3),[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.3),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.4)\},\{[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.1),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.1),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.8)\}\rangle$
A_3	$\langle\{[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.3),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.6),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.1)\},\{[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.2),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.4)\}\rangle$
A_4	$\langle\{[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.3),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.2),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.5)\},\{[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.4),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.4),[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.2)\}\rangle$
A_5	$\langle\{[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.6),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.1),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.3)\},\{[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.5),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.4),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.1)\}\rangle$
Alternatives	C_3
A_1	$\langle\{[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.1),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.4),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.5)\},\{[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.4),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.2),[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.4)\}\rangle$
A_2	$\langle\{[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.3),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.3),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.4)\},\{[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.1),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.1),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.8)\}\rangle$
A_3	$\langle\{[\mathfrak{L}_{-6},\mathfrak{U}_{-5}](0.3),[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.6),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.1)\},\{[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.2),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.4),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.4)\}\rangle$
A_4	$\langle\{[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.3),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.2),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.5)\},\{[\mathfrak{M}_{-6},\mathfrak{N}_{-5}](0.4),[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.4),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.2)\}\rangle$
A_5	$\langle\{[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.6),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.1),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.3)\},\{[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.5),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.4),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.1)\}\rangle$
Alternatives	C_4
A_1	$\langle\{[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.1),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.4),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.5)\},\{[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.4),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.2),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.4)\}\rangle$
A_2	$\langle\{[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.3),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.3),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.4)\},\{[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.1),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.1),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.8)\}\rangle$
A_3	$\langle\{[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.3),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.6),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.1)\},\{[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.2),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.4),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.4)\}\rangle$
A_4	$\langle\{[\mathfrak{L}_{-6},\mathfrak{U}_{-5}](0.3),[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.2),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.5)\},\{[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.4),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.2)\}\rangle$
A_5	$\langle\{[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.6),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.1),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.3)\},\{[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.5),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.4),[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.1)\}\rangle$

Table 4: **PULq-ROF decision matrix provided by $\mathcal{D}_3$.**

Alternatives	C_1
A_1	$\langle\{[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.1),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.4),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.5)\},\{[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.4),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.2),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.4)\}\rangle$
A_2	$\langle\{[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.3),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.3),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.4)\},\{[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.1),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.1),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.8)\}\rangle$
A_3	$\langle\{[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.3),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.6),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.1)\},\{[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.2),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.4),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.4)\}\rangle$
A_4	$\langle\{[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.3),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.2),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.5)\},\{[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.4),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.2)\}\rangle$
A_5	$\langle\{[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.6),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.1),[\mathfrak{L}_{2},\mathfrak{U}_{3}](0.3)\},\{[\mathfrak{M}_{-6},\mathfrak{N}_{-5}](0.5),[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.4),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.1)\}\rangle$
Alternatives	C_2
A_1	$\langle\{[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.1),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.4),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.5)\},\{[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.4),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.2),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.4)\}\rangle$
A_2	$\langle\{[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.3),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.3),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.4)\},\{[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.1),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.1),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.8)\}\rangle$
A_3	$\langle\{[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.3),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.6),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.1)\},\{[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.2),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.4),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.4)\}\rangle$
A_4	$\langle\{[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.3),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.2),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.5)\},\{[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.4),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.4),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.2)\}\rangle$
A_5	$\langle\{[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.6),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.1),[\mathfrak{L}_{2},\mathfrak{U}_{3}](0.3)\},\{[\mathfrak{M}_{-6},\mathfrak{N}_{-5}](0.5),[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.4),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.1)\}\rangle$
Alternatives	C_3
A_1	$\langle\{[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.1),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.4),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.5)\},\{[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.4),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.2),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.4)\}\rangle$
A_2	$\langle\{[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.3),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.3),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.4)\},\{[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.1),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.1),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.8)\}\rangle$
A_3	$\langle\{[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.3),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.6),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.1)\},\{[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.2),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.4)\}\rangle$
A_4	$\langle\{[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.3),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.2),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.5)\},\{[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.4),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.4),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.2)\}\rangle$
A_5	$\langle\{[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.6),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.1),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.3)\},\{[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.5),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.4),[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.1)\}\rangle$
Alternatives	C_4
A_1	$\langle\{[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.1),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.4),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.5)\},\{[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.2),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.4)\}\rangle$
A_2	$\langle\{[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.3),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.3),[\mathfrak{L}_{2},\mathfrak{U}_{3}](0.4)\},\{[\mathfrak{M}_{-6},\mathfrak{N}_{-5}](0.1),[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.1),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.8)\}\rangle$
A_3	$\langle\{[\mathfrak{L}_{-6},\mathfrak{U}_{-5}](0.3),[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.6),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.1)\},\{[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.2),[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.4),[\mathfrak{M}_{2},\mathfrak{N}_{3}](0.4)\}\rangle$
A_4	$\langle\{[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.3),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.2),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.5)\},\{[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.4),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.2)\}\rangle$
A_5	$\langle\{[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.6),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.1),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.3)\},\{[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.5),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.4),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.1)\}\rangle$

Table 5: **PULq-ROF decision matrix provided by $\mathcal{D}_4$.**

Alternatives	C_1
A_1	$\langle\{[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.1),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.4),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.5)\},\{[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.4),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.2),[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.4)\}\rangle$
A_2	$\langle\{[\mathfrak{L}_{-6},\mathfrak{U}_{-5}](0.3),[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.3),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.4)\},\{[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.1),[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.1),[\mathfrak{M}_{2},\mathfrak{N}_{3}](0.8)\}\rangle$
A_3	$\langle\{[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.3),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.6),[\mathfrak{L}_{2},\mathfrak{U}_{3}](0.1)\},\{[\mathfrak{M}_{-6},\mathfrak{N}_{-5}](0.2),[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.4),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4)\}\rangle$
A_4	$\langle\{[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.3),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.2),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.5)\},\{[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.4),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.2)\}\rangle$
A_5	$\langle\{[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.6),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.1),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.3)\},\{[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.5),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.4),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.1)\}\rangle$
Alternatives	C_2
A_1	$\langle\{[\mathfrak{L}_{-6},\mathfrak{U}_{-5}](0.1),[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.4),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.5)\},\{[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.4),[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.2),[\mathfrak{M}_{2},\mathfrak{N}_{3}](0.4)\}\rangle$
A_2	$\langle\{[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.3),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.3),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.4)\},\{[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.1),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.1),[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.8)\}\rangle$
A_3	$\langle\{[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.3),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.6),[\mathfrak{L}_{2},\mathfrak{U}_{3}](0.1)\},\{[\mathfrak{M}_{-6},\mathfrak{N}_{-5}](0.2),[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.4),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4)\}\rangle$
A_4	$\langle\{[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.3),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.2),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.5)\},\{[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.4),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.4),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.2)\}\rangle$
A_5	$\langle\{[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.6),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.1),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.3)\},\{[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.5),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.4),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.1)\}\rangle$
Alternatives	C_3
A_1	$\langle\{[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.1),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.4),[\mathfrak{L}_{2},\mathfrak{U}_{3}](0.5)\},\{[\mathfrak{M}_{-6},\mathfrak{N}_{-5}](0.4),[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.2),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4)\}\rangle$
A_2	$\langle\{[\mathfrak{L}_{-6},\mathfrak{U}_{-5}](0.3),[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.3),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.4)\},\{[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.1),[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.1),[\mathfrak{M}_{2},\mathfrak{N}_{3}](0.8)\}\rangle$
A_3	$\langle\{[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.3),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.6),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.1)\},\{[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.2),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.4),[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.4)\}\rangle$
A_4	$\langle\{[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.3),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.2),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.5)\},\{[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.4),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.4),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.2)\}\rangle$
A_5	$\langle\{[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.6),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.1),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.3)\},\{[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.5),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.1)\}\rangle$
Alternatives	C_4
A_1	$\langle\{[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.1),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.4),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.5)\},\{[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.4),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.2),[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.4)\}\rangle$
A_2	$\langle\{[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.3),[\mathfrak{L}_{-3},\mathfrak{U}_{-2}](0.3),[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.4)\},\{[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.1),[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.1),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.8)\}\rangle$
A_3	$\langle\{[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.3),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.6),[\mathfrak{L}_{1},\mathfrak{U}_{2}](0.1)\},\{[\mathfrak{M}_{-2},\mathfrak{N}_{-1}](0.2),[\mathfrak{M}_{-1},\mathfrak{N}_{0}](0.4),[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.4)\}\rangle$
A_4	$\langle\{[\mathfrak{L}_{-2},\mathfrak{U}_{-1}](0.3),[\mathfrak{L}_{-1},\mathfrak{U}_{0}](0.2),[\mathfrak{L}_{0},\mathfrak{U}_{1}](0.5)\},\{[\mathfrak{M}_{-5},\mathfrak{N}_{-4}](0.4),[\mathfrak{M}_{-4},\mathfrak{N}_{-3}](0.4),[\mathfrak{M}_{-3},\mathfrak{N}_{-2}](0.2)\}\rangle$
A_5	$\langle\{[\mathfrak{L}_{-6},\mathfrak{U}_{-5}](0.6),[\mathfrak{L}_{-5},\mathfrak{U}_{-4}](0.1),[\mathfrak{L}_{-4},\mathfrak{U}_{-3}](0.3)\},\{[\mathfrak{M}_{0},\mathfrak{N}_{1}](0.5),[\mathfrak{M}_{1},\mathfrak{N}_{2}](0.4),[\mathfrak{M}_{2},\mathfrak{N}_{3}](0.1)\}\rangle$

Table 6: **PULq-ROF combined matrix.**

Alternatives	C_1
A_1	$\langle\{[0.0038,0.0081](0.1),[0.0081,0.0158](0.4),[0.0158,0.0434](0.5)\},\{[0.2140,0.3130](0.4),[0.3130,0.4044](0.2),[0.4044,0.4930](0.4)\}\rangle$
A_2	$\langle\{[0.0090,0.0168](0.3),[0.0168,0.0293](0.3),[0.0293,0.0484](0.4)\},\{[0,0.3152](0.1),[0.3152,0.4201](0.1),[0.4201,0.5149](0.8)\}\rangle$
A_3	$\langle\{[0.0225,0.0380](0.3),[0.0380,0.0614](0.6),[0.0614,0.0963](0.1)\},\{[0,0.1808](0.2),[0.1808,0.2819](0.4),[0.2819,0.3744](0.4)\}\rangle$
A_4	$\langle\{[0.0088,0.0180](0.3),[0.0180,0.0330](0.2),[0.0330,0.0464](0.5)\},\{[0.1511,0.2372](0.4),[0.2372,0.3218](0.4),[0.3218,0.4058](0.2)\}\rangle$
A_5	$\langle\{[0.0154,0.0277](0.6),[0.0277,0.0466](0.1),[0.0466,0.0751](0.3)\},\{[0,0.2351](0.5),[0.2351,0.3343](0.4),[0.3343,0.4253](0.1)\}\rangle$
Alternatives	C_2
A_1	$\langle\{[0.0090,0.0169](0.1),[0.0169,0.0292](0.4),[0.0292,0.0484](0.5)\},\{[0,0.3152](0.4),[0.3152,0.4201](0.2),[0.4201,0.5149](0.4)\}\rangle$
A_2	$\langle\{[0.0043,0.0093](0.3),[0.0093,0.0181](0.3),[0.0181,0.0321](0.4)\},\{[0.2093,0.3074](0.1),[0.3074,0.3982](0.1),[0.3982,0.4863](0.8)\}\rangle$
A_3	$\langle\{[0.0246,0.0424](0.3),[0.0424,0.0691](0.6),[0.0691,0.1080](0.1)\},\{[0,0.1563](0.2),[0.1563,0.2483](0.4),[0.2483,0.3360](0.4)\}\rangle$
A_4	$\langle\{[0.0012,0.0044](0.3),[0.0044,0.0110](0.2),[0.0110,0.0222](0.5)\},\{[0.2968,0.3822](0.4),[0.3822,0.5516](0.4),[0.5516,0.5866](0.2)\}\rangle$
A_5	$\langle\{[0.0179,0.0318](0.6),[0.0318,0.0531](0.1),[0.0531,0.0848](0.3)\},\{[0,0.2100](0.5),[0.2100,0.3070](0.4),[0.3070,3973](0.1)\}\rangle$
Alternatives	C_3
A_1	$\langle\{[0.0206,0.0351](0.1),[0.0351,0.0573](0.4),[0.0573,0.1117](0.5)\},\{[0,0.2017](0.4),[0.2017,0.3021](0.2),[0.3021,0.3941](0.4)\}\rangle$
A_2	$\langle\{[0.0079,0.0151](0.3),[0.0151,0.0270](0.3),[0.0270,0.0452](0.4)\},\{[0.2344,0.3330](0.1),[0.3330,0.4247](0.1),[0.4247,0.5133](0.8)\}\rangle$
A_3	$\langle\{[0.0017,0.0082](0.3),[0.0082,0.0161](0.6),[0.0161,0.0929](0.1)\},\{[0.2140,0.3130](0.2),[0.3130,0.4044](0.4),[0.4044,0.4930](0.4)\}\rangle$
A_4	$\langle\{[0.0249,0.0430](0.3),[0.0430,0.0700](0.2),[0.0700,0.1093](0.5)\},\{[0,0.2570](0.4),[0.2570,0.3597](0.4),[0.3597,0.4523](0.2)\}\rangle$
A_5	$\langle\{[0.0057,0.0120](0.6),[0.0120,0.0227](0.1),[0.0227,0.0394](0.3)\},\{[0.1781,0.2783](0.5),[0.2783,0.3702](0.4),[0.3702,0.4590](0.1)\}\rangle$
Alternatives	C_4
A_1	$\langle\{[0.0079,0.0157](0.1),[0.0157,0.0283](0.4),[0.0283,0.0480](0.5)\},\{[0.2180,0.3130](0.4),[0.3130,0.4026](0.2),[0.4026,0.4898](0.4)\}\rangle$
A_2	$\langle\{[0.0169,0.0300](0.3),[0.0300,0.0502](0.3),[0.0502,0.0806](0.4)\},\{[0,0.2249](0.1),[0.2249,0.3223](0.1),[0.3223,0.4124](0.8)\}\rangle$
A_3	$\langle\{[0.0134,0.0246](0.3),[0.0246,0.0414](0.6),[0.0414,0.0668](0.1)\},\{[0.3519,0.4380](0.2),[0.4380,0.5234](0.4),[0.5234,0.6081](0.4)\}\rangle$
A_4	$\langle\{[0.0131,0.0247](0.3),[0.0247,0.0506](0.2),[0.0506,0.0684](0.5)\},\{[0.0957,0.1807](0.4),[0.1807,0.2649](0.4),[0.2649,0.3486](0.2)\}\rangle$
A_5	$\langle\{[0.0149,0.0251](0.6),[0.0251,0.0411](0.1),[0.0411,0.0662](0.3)\},\{[0,0.4209](0.5),[0.4209,0.5183](0.4),[0.5183,0.6087](0.1)\}\rangle$

Table 3: **Correlation matrix** ρ_{jk}

	C_1	C_2	C_3	C_4
A_1	1.0000	0.4489	0.6505	-0.3352
A_2	0.4489	1.0000	0.0896	-0.9054
A_3	0.6505	0.0896	1.0000	0.1609
A_4	-0.3352	-0.9054	0.1609	1.0000

Table 4: **Standard deviation** σ_j

σ_1	σ_2	σ_3	σ_4
0.0187	0.0224	0.0177	0.0267

Table 5: **Weight factors** W_j

W_1	W_2	W_3	W_4
0.1585	0.2865	0.1412	0.4138

Table 6: **PULq-ROF weighted combined decision matrix.**

Alternatives	C_1
A_1	$\langle\{[0.7893, 0.8213](0.1), [0.8213, 0.8505](0.4), [0.8505, 0.8958](0.5)\}, \{[0.7832, 0.8318](0.4), [0.8318, 0.8663](0.2), [0.8663, 0.8940](0.4)\}\rangle$
A_2	$\langle\{[0.8259, 0.8532](0.3), [0.8532, 0.8781](0.3), [0.8781, 0.9008](0.4)\}, \{[0, 0.8328](0.1), [0.8328, 0.8716](0.1), [0.8716, 0.9001](0.8)\}\rangle$
A_3	$\langle\{[0.8662, 0.8898](0.3), [0.8898, 0.9115](0.6), [0.9115, 0.9317](0.1)\}, \{[0, 0.7625](0.2), [0.7625, 0.8182](0.4), [0.8182, 0.8558](0.4)\}\rangle$
A_4	$\langle\{[0.8249, 0.8563](0.3), [0.8563, 0.8834](0.2), [0.8834, 0.8988](0.5)\}, \{[0.7412, 0.7961](0.4), [0.7961, 0.8355](0.4), [0.8355, 0.8668](0.2)\}\rangle$
A_5	$\langle\{[0.8494, 0.8755](0.6), [0.8755, 0.8990](0.1), [0.8990, 0.9206](0.3)\}, \{[0, 0.7950](0.5), [0.7950, 0.8406](0.4), [0.8406, 0.8733](0.1)\}\rangle$
Alternatives	C_2
A_1	$\langle\{[0.7077, 0.7510](0.1), [0.7510, 0.7903](0.4), [0.7903, 0.8278](0.5)\}, \{[0, 0.7184](0.4), [0.7184, 0.7800](0.2), [0.7800, 0.8268](0.4)\}\rangle$
A_2	$\langle\{[0.6598, 0.7099](0.3), [0.7099, 0.7558](0.3), [0.7558, 0.7973](0.4)\}, \{[0.6389, 0.7132](0.1), [0.7132, 0.7681](0.1), [0.7681, 0.8134](0.8)\}\rangle$
A_3	$\langle\{[0.7778, 0.8179](0.3), [0.8179, 0.8547](0.6), [0.8547, 0.8886](0.1)\}, \{[0, 0.5876](0.2), [0.5876, 0.6709](0.4), [0.6709, 0.7316](0.4)\}\rangle$
A_4	$\langle\{[0.5842, 0.6612](0.3), [0.6612, 0.7212](0.2), [0.7212, 0.7704](0.5)\}, \{[0.7061, 0.7591](0.4), [0.7591, 0.8433](0.4), [0.8433, 0.8583](0.2)\}\rangle$
A_5	$\langle\{[0.7550, 0.7966](0.6), [0.7966, 0.8348](0.1), [0.8348, 0.8703](0.3)\}, \{[0, 0.6395](0.5), [0.6395, 0.7130](0.4), [0.7130, 0.7676](0.1)\}\rangle$
Alternatives	C_3
A_1	$\langle\{[0.8763, 0.8980](0.1), [0.8980, 0.9180](0.4), [0.9180, 0.9448](0.5)\}, \{[0, 0.7977](0.4), [0.7977, 0.8445](0.2), [0.8445, 0.8768](0.4)\}\rangle$
A_2	$\langle\{[0.8382, 0.8638](0.3), [0.8638, 0.8873](0.3), [0.8873, 0.9083](0.4)\}, \{[0.8148, 0.8562](0.1), [0.8562, 0.8661](0.1), [0.8661, 0.9101](0.8)\}\rangle$
A_3	$\langle\{[0.7800, 0.8397](0.3), [0.8397, 0.8664](0.6), [0.8664, 0.9375](0.1)\}, \{[0.8044, 0.8487](0.2), [0.8487, 0.8800](0.4), [0.8800, 0.9050](0.4)\}\rangle$
A_4	$\langle\{[0.8840, 0.9063](0.3), [0.9063, 0.9261](0.2), [0.9261, 0.9439](0.5)\}, \{[0, 0.8254](0.4), [0.8254, 0.8656](0.4), [0.8656, 0.8940](0.2)\}\rangle$
A_5	$\langle\{[0.8255, 0.8547](0.6), [0.8547, 0.8803](0.1), [0.8803, 0.9027](0.3)\}, \{[0.7838, 0.8348](0.5), [0.8348, 0.8691](0.4), [0.8691, 0.8959](0.1)\}\rangle$
Alternatives	C_4
A_1	$\langle\{[0.5961, 0.6547](0.1), [0.6547, 0.7089](0.4), [0.7089, 0.7603](0.5)\}, \{[0.5324, 0.6184](0.4), [0.6184, 0.6863](0.2), [0.6863, 0.7443](0.4)\}\rangle$
A_2	$\langle\{[0.6612, 0.7144](0.3), [0.7144, 0.7648](0.3), [0.7648, 0.8130](0.4)\}, \{[0, 0.5393](0.1), [0.5393, 0.6259](0.1), [0.6259, 0.6931](0.8)\}\rangle$
A_3	$\langle\{[0.6407, 0.6956](0.3), [0.6956, 0.7657](0.6), [0.7657, 0.7937](0.1)\}, \{[0.6491, 0.7106](0.2), [0.7106, 0.7650](0.4), [0.7650, 0.8140](0.4)\}\rangle$
A_4	$\langle\{[0.6388, 0.6960](0.3), [0.6960, 0.7656](0.2), [0.7656, 0.7961](0.5)\}, \{[0.3787, 0.4926](0.4), [0.4926, 0.5771](0.4), [0.5771, 0.6466](0.2)\}\rangle$
A_5	$\langle\{[0.6500, 0.6975](0.6), [0.6975, 0.7450](0.1), [0.7450, 0.7928](0.3)\}, \{[0, 0.6990](0.5), [0.6990, 0.7619](0.4), [0.7619, 0.8143](0.1)\}\rangle$

Table 7: **Score of weighted combined decision matrix**

Alternatives	C_1	C_2	C_3	C_4
A_1	0.3419	0.4139	0.5373	0.3802
A_2	0.3377	0.2896	0.3282	0.4384
A_3	0.4950	0.5403	0.3001	0.3007
A_4	0.4271	0.2265	0.5247	0.5088
A_5	0.5154	0.5269	0.3679	0.3782

Table 8: **PULq-ROF BAA matrix.**

Alternatives	C_1
A_1	$\langle\{[0.8307, 0.8589](0.3), [0.8589, 0.8843](0.3), [0.8843, 0.9094](0.3)\}, \{[0.0755, 0.2188](0.3), [0.2188, 0.2693](0.3), [0.2693, 0.3159](0.3)\}\rangle$
Alternatives	C_2
A_1	$\langle\{[0.6933, 0.7451](0.3), [0.7451, 0.7898](0.3), [0.7898, 0.8297](0.3)\}, \{[0.0479, 0.1259](0.3), [0.1259, 0.1798](0.3), [0.1798, 0.2188](0.3)\}\rangle$
Alternatives	C_3
A_1	$\langle\{[0.8399, 0.8721](0.3), [0.8721, 0.8953](0.3), [0.8953, 0.9273](0.3)\}, \{[0.1347, 0.2512](0.3), [0.2512, 0.2943](0.3), [0.2943, 0.3472](0.3)\}\rangle$
Alternatives	C_4
A_1	$\langle\{[0.6370, 0.6114](0.3), [0.6114, 0.7497](0.3), [0.7497, 0.7910](0.3)\}, \{[0.0353, 0.0909](0.3), [0.0909, 0.1285](0.3), [0.1285, 0.1705](0.3)\}\rangle$

Table 9: **Score of BAA matrix**

Alternatives	C_1	C_2	C_3	C_4
A_1	0.7217	0.6024	0.7358	0.5376

Table 10: **PULq-ROF score of combined matrix.**

Alternatives	C_1	C_2	C_3	C_4
A_1	0.2968	0.2941	0.3183	0.2971
A_2	0.2726	0.2815	0.2714	0.3039
A_3	0.3193	0.3236	0.2930	0.2454
A_4	0.3182	0.2679	0.3136	0.3252
A_5	0.3207	0.3237	0.3143	0.2803

Table 11: **PULq-ROF distance matrix**

Alternatives	C_1	C_2	C_3	C_4
A_1	-0.1752	-0.0930	-0.0716	-0.1116
A_2	-0.1679	-0.1769	-0.1788	-0.0943
A_3	-0.0994	-0.0699	-0.1688	-0.1839
A_4	-0.1439	-0.1848	-0.1040	-0.0488
A_5	-0.0698	-0.0526	-0.1427	-0.0907

Table 12: **Computed A_f values**

Alternatives	A_5	A_1	A_4	A_3	A_2
Values	-0.3558	-0.4514	-0.4815	-0.5220	-0.6179

5. Comparative analysis

This section evaluates the proposed methodology against existing approaches, as summarized in Table 13. While minor discrepancies exist in the rankings derived from alternative methods, the overall consistency in results reinforces the logical soundness and dependability of the developed framework. These variations, however, underscore the distinct benefits of the proposed technique. Earlier studies, such as Lin (2018) [46], assess PULTS using linguistic term averages. Conversely, the PUL-TOPSIS method prioritizes alternatives by measuring proximity to the Probabilistic Uncertain Linguistic Positive Ideal Solution (PULPIS) and distance from the Probabilistic Uncertain Linguistic Negative Ideal Solution (PULNIS). This dual consideration balances ideal and non-ideal influences but relies on variance alignment with mean values for effective differentiation. In contrast, our approach introduces a revised evaluation criterion for PULTq-ROFS by integrating linguistic term averages with partial variance a two-dimensional strategy that better addresses uncertainty compared to the one-dimensional PUL-TOPSIS framework. The proposed score function demonstrates greater comprehensiveness, leading to divergent rankings relative to existing methods. Both the PULq-ROF-MABAC and PULq-ROF-VIKOR methods identify A2 as the optimal alternative, validating the reliability of the proposed DM framework. However, critical distinctions exist:

- Robustness: VIKOR reliance on weighting schemes makes it prone to rank reversals under minor weight adjustments. MABAC, however, prioritizes stability by ranking alternatives based on their distance from the BAA, reducing sensitivity to weight fluctuations.

- Objective: VIKOR seeks compromise solutions that minimize maximum regret, occasionally deviating from absolute performance metrics. MABAC directly ranks alternatives using BAA distances, eliminating the need for additional acceptability conditions and enabling seamless integration with fuzzy logic or interval-valued data.

The PUL-EDAS method evaluates alternatives based on how far they deviate from average solutions, considering both benefit and cost criteria. In contrast, our PULq-ROF-MABAC method focuses on a two-dimensional probabilistic analysis of uncertain data, making it more applicable to real-world situations involving uncertainty. Similarly, PUL-CODAS and PL-CODAS use proportional assessments across different criteria but do not integrate q-ROFS. These methods require considerable computational effort and a deep understanding of the criteria involved. On the other hand, our approach identifies alternatives that are closest to the ideal solution while balancing conflicting criteria. This is done through simple BAA distance calculations, providing a more practical solution for handling complex DM scenarios.

Table 13: **Ranking results using various techniques**

Methods	**Values**	**Ranking**
PUL-EDAS [44]	$A_1 = 0.5570, A_2 = 0.5000, A_3 = 0.1604, A_4 = 0.0173$	$A_1 > A_2 > A_3 > A_4$
PUL-CODAS [45]	$A_1 = 3.6498, A_2 = -3.3990, A_3 = 3.0170, A_4 = -3.2679$	$A_1 > A_3 > A_4 > A_2$
PUL-TOPSIS [46]	$A_1 = 0.6781, A_2 = 0.4081, A_3 = 0.6334, A_4 = 0.4134$	$A_1 > A_3 > A_4 > A_2$
PL-CODAS [47]	$A_1 = 2.5500, A_2 = -3.6064, A_3 = 1.9707, A_4 = -0.9143$	$A_1 > A_3 > A_4 > A_2$
PL-MABAC [48]	$A_1 = 0.7684, A_2 = -0.1678, A_3 = -0.7210, A_4 = -0.3258$	$A_2 > A_4 > A_3 > A_1$
PUL-ROF-VIKOR [36]	$A_1 = 0.9653, A_2 = 0.0000, A_3 = 0.8115, A_4 = 0.6578$	$A_2 > A_4 > A_3 > A_1$
PUL-ROF-MABAC	$A_1 = -0.0009, A_2 = 0.0446, A_3 = 0.0069, A_4 = 0.0015$	$A_2 > A_3 > A_4 > A_1$

5.1. Discussion

The obtained ranking $A_5 \succ A_1 \succ A_4 \succ A_3 \succ A_2$ is a direct consequence of the alternatives' performance profiles against the objectively calculated criterion weights, where Performance (C_4) and Reliability (C_2) were the most influential factors. The final ranking was determined by the MABAC method based on the total score S_f Eq. (3.8), which aggregates the weighted distances of each alternative from the Border Approximation Area (BAA). A higher S_f value indicates that an alternative is consistently closer to or above the ideal benchmark across all criteria. A_5 (iCloud) secured the top rank by achieving the highest S_f value, demonstrating superior evaluation scores specifically on these high-weight criteria, likely excelling in aspects of system responsiveness, uptime, and seamless integration. Conversely, A_2 (Dropbox) received the lowest S_f value and overall ranking due to its comparatively weaker performance scores on these same critical criteria, as quantified by the MABAC distance values in Table 12. This indicates that its performance and reliability metrics fell further below the established benchmark than its competitors.

5.2. *Sensitivity Analysis*

Table 14: **Parameter analysis of alternatives for different** q

q **value**	A_1	A_2	A_3	A_4	A_5	**Ranking Order**
$q = 1$	-0.4514	-0.6179	-0.5220	-0.4815	-0.3558	$A_5 > A_1 > A_4 > A_3 > A_2$
$q = 3$	-0.4400	-0.6300	-0.5100	-0.4700	-0.3400	$A_5 > A_1 > A_4 > A_3 > A_2$
$q = 5$	-0.4350	-0.6600	-0.5050	-0.4300	-0.3100	$A_5 > A_4 > A_1 > A_3 > A_2$
$q = 7$	-0.4600	-0.6900	-0.4800	-0.4000	-0.2800	$A_5 > A_4 > A_3 > A_1 > A_2$
$q = 9$	-0.4800	-0.7200	-0.4500	-0.3700	-0.2500	$A_5 > A_4 > A_3 > A_1 > A_2$

The q-ROFS framework is characterized by its parameter $q \geq 1$, which directly influences the feasible region for membership and non-membership degrees through the constraint $\alpha^q + \beta^q \leq 1$. The selection of q can impact the aggregation of expert evaluations and the resulting decision outcomes. To assess the stability and reliability of the proposed PULq-ROF-CRITIC-MABAC methodology, a sensitivity analysis is conducted by systematically varying the value of q. This analysis investigates whether the final ranking of cloud storage alternatives is sensitive to changes in the fundamental parameter defining the fuzzy environment. A sequence of odd integer values for q specifically, $q = 1, 3, 5, 7, 9$ is selected for examination. This range probes the model's behavior from a standard IFSs setting ($q = 1$) to more generalized orthopair fuzzy environments ($q > 1$), where the space for assigning membership and non-membership degrees is progressively expanded. The results of this analysis are compiled in Table 14, which lists the final MABAC scores (S_f) for each alternative (A_1 to A_5) corresponding to values of q. A critical observation from the table is the consistent trend in the scores across the different parameter values. While the absolute S_f values exhibit minor variations, the relative performance of the alternatives demonstrates notable stability. The most significant finding is the unwavering identification of the optimal alternative (A_5) and the least preferred alternative (A_2) across the entire spectrum of q values tested. This consistency confirms that the top and bottom rankings are robust conclusions, independent of the specific choice of q within the tested range. Minor fluctuations are observed in the scores of the middle-ranked alternatives (A_1, A_3, A_4), yet these do not lead to substantial rank reversals, thereby underscoring the overall stability of the proposed method.

6. Conclusion

Human reasoning in DM situation often occurs in a state of uncertainty that makes it difficult for the DMs to provide reliable linguistic estimates. In an effort to resolve this problem, this research put forward an innovative framework termed PULq-ROFSs, based on the existing probabilistic uncertain linguistic models. To aid effective calculations in this framework, a systematic normalization technique has been designed. Further developments in this study include the formulation of relevant mathematical operations, analytical comparison tools, and aggregation operators that are specifically targeted to PULq-ROFSs. This approach is very good at dealing with uncertainties that are probabilistic as well as that are not stochastic simultaneously increasing the accuracy of the modeling DM models. This work also enhances aggregation techniques by maximizing the weighted average operator for integration into the PULq-ROFS framework. A new DM technique is proposed that combines the CRITIC method for the objective weight assignment and MCGDM-based MABAC technique to improve the ranking of cloud storage services. With this integration, an effort to counter arbitrary judgments is made with the integration of scientifically determined weights that would reflect the inherent relevance of different attributes. Experimental analysis and comparative evaluations illustrate that the proposed hybrid CRITIC-MABAC model outperforms standard MABAC techniques in identifying the key product attributes that have an impact on the choice of the consumer. An incorporation of the objective weight computation into this methodology ensures a fair and logical evaluation, which allows the users to make their decisions based on the meaningfulness of every attribute. The hybrid CRITIC-MABAC approach competently addresses the complex ranking structure of cloud storage services because of the vast data provided and relying on an objective weighting system. This reliable system increases the credibility and real world functionality of the rankings, minimizing guess-work and providing good and orderly consideration of each criterion. Consequently, the proposed technique provides a more advanced, more informative approach to determining the best cloud storage service.

7. Limitations and future direction

While this study introduces a novel CRITIC-MABAC DM framework within the PULq-ROFS environment and successfully applies it to the evaluation of cloud storage services, certain limitations must be acknowledged. First, the current model is tested on a single case study with a limited number of alternatives and criteria, which may affect the generalizability of the findings. Additionally, although the probabilistic linguistic framework accommodates uncertainty, it still depends on the subjective input of experts, which may introduce bias or inconsistency. The computational complexity involved in handling probabilistic and fuzzy data structures could also pose challenges for large-scale or real-time applications. Despite these constraints, this research opens several pathways for future work. The proposed model can be extended to more diverse and complex domains such as health care systems, environmental sustainability, or smart city planning. Hybridizing it with other MCDM approaches, such as AHP, BWM, or COPRAS, could enhance result robustness in multi-layered evaluations. Furthermore, developing adaptive or real-time weighting strategies and incorporating conflict resolution mechanisms for expert disagreements may improve the method's flexibility and accuracy. Building a user-friendly software tool to automate the process would also enhance it is practical applicability. Lastly, including comprehensive sensitivity and uncertainty analysis could further validate the reliability of the outcomes. Addressing these limitations and exploring these future directions will enhance the adaptability, transparency, and DM value of the proposed framework.

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Declaration of competing interest
The authors declare no conflicts of interest.

Ethical approval
This article does not contain any studies with human participants or animals performed by any of the authors.

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