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Limit Cycles for a Class of Planar Differential Systems with a Degenerate Singular Point

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Abstract

This paper investigates a class of planar differential systems characterized by a degenerate singular point. We demonstrate that this class is Liouville integrable and explicitly derive a first integral using an Abel equation of the second kind. Moreover, through an analysis based on the Poincaré return map, we establish the existence of two non-algebraic limit cycles or a single algebraic limit cycle arising near the degenerate point. The occurrence of these limit cycles is shown to depend sensitively on the system's parameters.

Keywords: Polynomial differential systems; Integrability; Degenerate Singular Point; Abel equation; Limit Cycle.

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1. Introduction

In the qualitative theory of planar polynomial vector fields, key open problems include determining the number and spatial distribution of limit cycles, distinguishing between centers and foci, commonly referred to as the center problem and identifying corresponding first integrals. These challenges are central to the field and remain the subject of active research (see, for example [2], [4], [6], [7], [11], [20], [21]). Notably, these issues are closely related to Hilbert's 16^{th} problem [13], which concerns planar polynomial differential systems of degree $m = \max\{\deg P_{m_1}, \deg Q_{m_2}\}$,

$$\begin{cases} \dot{x} = P_{m_1}(x, y), \\ \dot{y} = Q_{m_2}(x, y), \end{cases} \tag{1.1}$$

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where P_{m_1} and Q_{m_2} are real polynomials in the variables x and y. As usual, the dot in (1.1) denotes the derivative concerning the independent variable t. Hilbert's 16^{th} problem presents considerable challenges; therefore, in this paper, we restrict our investigation to a specific class of polynomial differential systems, with particular focus on their integrability and the existence of hyperbolic limit cycles.
For the polynomial differential system (1.1), the polynomial vector field associated with it is

$$\gamma = P_{m_1}(x,y)\frac{\partial}{\partial x} + Q_{m_2}(x,y)\frac{\partial}{\partial y}. \tag{1.2}$$

The system (1.1) is said to be integrable on an open set $\Theta \subset \mathbb{R}^2$ if there exists a non-constant, continuously differentiable function $\mathcal{F} : \Theta \to \mathbb{R}$ that remains constant along the trajectories of system (1.1) contained in Θ; that is, if

$$\gamma(\mathcal{F})\,|_{\Theta} \equiv 0. \tag{1.3}$$

The function $\mathcal{F}$ is called the first integral of the system on Θ. Moreover, $\mathcal{F} = f$ is the general solution of the equation (1.3), where f is an arbitrary constant. The existence of the first integrals are of the utmost importance for a vector field since they help to obtain the phase portrait of the system and reduce the dimension of the system by its number of independent first integrals, for more details see for instance ([10], [18], [22]) and references therein.

A limit cycle of system (1.1) is defined as an isolated periodic solution among the family of all periodic solutions of the same system. If such a limit cycle lies entirely on an invariant algebraic curve, it is referred to as an algebraic limit cycle; otherwise, it is called a non-algebraic limit cycle.

In general, the explicit expressions of limit cycles for polynomial differential systems are not known, except in certain special cases (see, [8], [9], [17]). However, in recent years, several studies have introduced planar nonlinear differential systems that admit explicitly known non-algebraic limit cycles. The earliest such examples were presented in the work of A. Gasull et al. [5]. Since then, considerable attention has been devoted to investigating the existence and coexistence of both algebraic and non-algebraic limit cycles; see, for instance, [1], [12], [15] and [16].
This article is devoted to the study of integrability and the existence of limit cycles in a class of polynomial differential systems, where the origin becomes a non-elementary singular point of the form

$$\begin{cases} \dot{x} = xQ_2(x,y) + \left(x^3 + xy^2\right) Q_4(x,y) - 2yQ_6(x,y), \\ \dot{y} = yQ_2(x,y) + \left(yx^2 + y^3\right) Q_4(x,y) + 2xQ_6(x,y), \end{cases} \tag{1.4}$$

where

$$\begin{aligned} Q_2(x,y) &= -\mu\left(\alpha^2 y^2 + x^2\right), \\ Q_4(x,y) &= \left(\delta - x^2 - y^2\right)\left((\sigma - \alpha\beta)\, x^2 - \alpha\,(\beta - \sigma\alpha)\, y^2\right), \\ Q_6(x,y) &= \left(\alpha^2 y^2 + x^2\right)\left(x^2 + y^2\right)\left(\delta - 2x^2 - 2y^2\right), \end{aligned}$$

in which α, β, δ, μ, and σ are real constants. We prove the integrability of the system by converting it into polar coordinates, which yields an Abel differential equation of the second kind. This approach allows us to derive an explicit expression for a first integral. In addition, we establish sufficient conditions under which the differential system (1.4) possesses either two non-algebraic limit cycles or a single algebraic one. These limit cycles are explicitly constructed.

2. Main result

The main result is stated in the following theorem, which will be proved in this section.

Theorem 2.1. *Let us consider the multi-parameter polynomial differential system* (1.4)*. Then, the following statements are valid:*

(1) The system (1.4) *has the first integral:*

$$F(x,y)=\frac{(x^2+y^2)\left(x^2+y^2-\delta\right)}{\exp\left(\int_0^{\arctan\frac{y}{x}}\frac{2f_0(s)}{g(s)}ds\right)}-\int_0^{\arctan\frac{y}{x}}\frac{2f_0(s)}{g(s)h(s)}ds.$$

where

$$h(\theta) = \exp\left(\int_0^{\theta}\frac{2f_0(s)}{g(s)}ds\right) \tag{2.1}$$

$$f_0(\theta) = -\frac{1}{2}\mu\left(1+\alpha^2+\left(1-\alpha^2\right)\cos 2\theta\right) \tag{2.2}$$

$$g(\theta) = \left(\alpha^2-1\right)\cos 2\theta-\alpha^2-1 \tag{2.3}$$

(2) If $-\frac{1}{2}\mu\left(1+\alpha^2\right)-\frac{1}{2}\left|\mu\left(1-\alpha^2\right)\right|>0,\ \mu<0, \delta>0,\ \alpha\notin\{-1,1\}$ *and* $1-\frac{\delta^2}{4}<0$ *Then, the system*(1.4) *possesses two explicit non-algebraic limit cycles, expressed in polar coordinates* (r,θ) *as follows*

$$r_1^*(\theta) = \frac{\sqrt{2}}{2}\left(\delta^2+\left(\delta^2+4h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{2}},$$

$$r_2^*(\theta) = \frac{\sqrt{2}}{2}\left(\delta^2-\left(\delta^2+4h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{2}},$$

where $\phi(\theta)=\int_0^{\theta}\frac{2f_0(s)}{g(s)h(s)}ds.$

3) If $\alpha\in\{-1,1\},\ \delta>0,\ \mu<0$ *and* $1-\frac{\delta^2}{4}<0$, *then system* (1.4) *admits explicit algebraic limit cycles, given in Cartesian coordinates* (x,y), *by*

$$(x^2+y^2)^2-\delta(x^2+y^2)+\frac{2}{\mu}=0.$$

Proof. First, to determine the equilibrium points of system (1.4), we note that if (x,y) is an equilibrium point, then it must satisfy the system of equations:

$$\left\{(x,y)\in\mathbb{R}^2,\ y\dot{x}-x\dot{y}=-2\left(\alpha^2y^2+x^2\right)\left(x^2+y^2\right)^2\left(2x^2+2y^2-\delta\right)\right\},$$

includes this point, we conclude that the origin, also known as an equilibrium point is a degenerate non-elementary singular point of the system (1.4), for the reason that the linear part of this system is identically zero, and any other equilibrium points, if they exist are present in the equation curve's

$$2x^2+2y^2-\delta=0. \tag{2.4}$$

Proof of statement (1).

The differential system (1.4), when expressed in polar coordinates, takes the following form

$$\begin{cases}\dot{r}=f_2(\theta)r^7+f_1(\theta)r^5+f_0(\theta)r^3\\ \dot{\theta}=-r^4\left(\delta-2r^2\right)g(\theta),\end{cases} \tag{2.5}$$

where

$$f_2(\theta) = -\frac{1}{2}\left(\sigma+\sigma\alpha^2-2\alpha\beta+\left(\sigma-\sigma\alpha^2\right)\cos 2\theta\right)$$

$$f_1(\theta) = -\frac{1}{2}\left(2\alpha\delta\beta-\sigma\alpha^2\delta-\sigma\delta+\left(\sigma\alpha^2\delta-\sigma\delta\right)\cos 2\theta\right)$$

$$f_0(\theta) = -\frac{1}{2}\mu\left(1+\alpha^2+\left(1-\alpha^2\right)\cos 2\theta\right)$$

$$g(\theta) = \left(\alpha^2-1\right)\cos 2\theta-\alpha^2-1$$

The differential system (2.5) where $r^2 \neq \frac{\delta}{2}$, can be written as the equivalent differential equation

$$\frac{dr}{d\theta} = \frac{f_2(\theta)r^4 + f_1(\theta)r^2 + f_0(\theta)}{2r\left(\delta - 2r^2\right)g(\theta)}. \tag{2.6}$$

Via the change of variables $\rho = r^2$, the differential equation (2.6) is transformed into the Abel equation of the second kind

$$\left(\rho - \frac{\delta}{2}\right)\frac{d\rho}{d\theta} = \frac{f_2(\theta)}{g(\theta)}\rho^2 + \frac{f_1(\theta)}{g(\theta)}\rho + \frac{f_0(\theta)}{g(\theta)}. \tag{2.7}$$

With the aid of the change of variables

$$w = \left(\rho - \frac{\delta}{2}\right)\exp\left(\int_0^\theta \frac{f_2(s)}{f_1(s)}ds\right),$$

the differential equation (2.7) is reducible to a Abel equation of the second kind

$$ww' = E_1(\theta)w + E_0(\theta), \tag{2.8}$$

where

$$E_1(\theta) = \frac{-1}{2g(\theta)}\left(f_1(\theta) + \delta f_2(\theta)\right)\exp\left(\int_0^\theta \frac{f_2(s)}{f_1(s)}ds\right),$$

and

$$E_0(\theta) = \frac{-1}{2g(\theta)}\left[f_0(\theta) + \frac{\delta}{2}f_1(\theta) + \frac{\delta^2}{4}f_2(\theta)\right]\exp\left(2\int_0^\theta \frac{f_2(s)}{f_1(s)}ds\right).$$

The coefficients of (2.7) satisfy the functional relation

$$f_1(\theta) + \delta f_2(\theta) = 0,$$

then the general solution of Abel equation (2.7) is

$$\rho^2(\theta) - \delta\rho(\theta) = h(\theta)\left(k + \int_0^\theta \frac{2f_0(s)}{g(s)h(s)}ds\right), \tag{2.9}$$

where $h(\theta) = \exp\left(\int_0^\theta \frac{2f_0(s)}{g(s)}ds\right)$ and k is an integration constant for more details see ([3], [19]).

Thus, the implicit solution of differential equation (2.6) is given by

$$H(r,\theta) = r^4 - \delta r^2 - h(\theta)\left(k + \int_0^\theta \frac{2f_0(s)}{g(s)h(s)}ds\right). \tag{2.10}$$

By passing to Cartesian coordinates, we deduce that the first integral takes the form

$$F(x,y) = \frac{\left(x^2+y^2-\delta\right)\left(x^2+y^2\right)}{\exp\left(\int_0^{\arctan\frac{y}{x}} \frac{2f_0(s)}{g(s)}ds\right)} - \int_0^{\arctan\frac{y}{x}} \frac{2f_0(s)}{g(s)h(s)}ds. \tag{2.11}$$

Therefore, system (1.4) is Liouville integrable, since this first integral is a function that can be expressed through quadratures of elementary functions.

Proof of statement (2). System (1.4) exhibits a periodic orbit precisely when equation (2.6) admits a positive solution that is 2π-periodic. This condition is equivalent to the existence of a function $r(\theta, r_*)$ such that $r(0, r_*) = r(2\pi, r_*)$ and $r(\theta, r_*) > 0$ for all $\theta \in [0, 2\pi]$.

The implicit form of the solution $r(\theta, r_0)$ of the differential equation (2.6) such as $r(0, r_0) = r_0 > 0$, is

$$H(r,\theta) = r^4 - \delta r^2 - h(\theta)\left(r_0^4 - \delta r_0^2 + \phi(\theta)\right).$$

where $\phi(\theta)=\int_0^\theta \frac{2f_0(s)}{g(s)h(s)}ds$. To ensure periodicity of a solution to system (2.5), the condition $r(2\pi,r_0)=r(0,r_0)$ must hold. This requirement leads to the equation:

$$r_0^4-\delta r_0^2=\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}. \tag{2.12}$$

This corresponds to the value $k=r_0^4-\delta r_0^2$, and allows a reformulation of the implicit solution to the differential equation in terms of k. (2.6) as

$$H(r,\theta)=r^4-\delta r^2-h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right)=0. \tag{2.13}$$

Next we prove that

$$0<-h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right)<\frac{\delta^2}{4} \text{ for all } \theta\in\mathbb{R}. \tag{2.14}$$

Let $\omega(\theta)-\phi(\theta)=\frac{\delta^2}{4}\widetilde{h}(\theta)$, where $\widetilde{h}(\theta)=\exp\left(-\int_0^\theta \frac{2f_0(s)}{g(s)}ds\right)$.

The function ω is differentiable for all $\theta\in[0,2\pi[$, then we have

$$\begin{aligned}\frac{d\omega(\theta)}{d\theta} &= -\frac{\delta^2}{4}\frac{2f_0(\theta)}{g(\theta)}\widetilde{h}(\theta)+\frac{2f_0(\theta)}{g(\theta)h(\theta)}\\ &= \frac{2\widetilde{h}(\theta)f_0(\theta)}{g(\theta)}\left(1-\frac{\delta^2}{4}\right).\end{aligned}$$

Since $-\frac{1}{2}\mu\left(\alpha^2+1\right)-\frac{1}{2}\left|\mu\left(1-\alpha^2\right)\right|>0$, $\mu<0$ and $\left|\alpha^2-1\right|-\alpha^2-1<0$, we see that

$$f_0(\theta)=-\frac{1}{2}\mu\alpha^2-\frac{1}{2}\mu-\frac{1}{2}\left(\mu-\mu\alpha^2\right)\cos 2\theta>0$$

and

$$g(\theta)=\left(\alpha^2-1\right)\cos 2\theta-\alpha^2-1<0.$$

Since $1-\frac{\delta^2}{4}<0$ and $\widetilde{h}(\theta)>0$ for all $\theta\in\mathbb{R}$, then $\frac{2\widetilde{h}(\theta)f_0(\theta)}{g(\theta)}\left(-\frac{\delta^2}{4}+1\right)>0$, and the function $\theta\mapsto\omega(\theta)$ is strictly increasing with

$$\begin{aligned}\omega(0) &= \frac{\delta^2}{4}<\widetilde{h}(\theta)\frac{\delta^2}{4}+\phi(\theta) &(2.15)\\ &= \omega(\theta)<\omega(2\pi)=\widetilde{h}(2\pi)\frac{\delta^2}{4}+\phi(2\pi) &(2.16)\end{aligned}$$

for all $\theta\in[0,2\pi[$.

According to (2.15), we have $\frac{\delta^2}{4}<\widetilde{h}(2\pi)\frac{\delta^2}{4}+\phi(2\pi)$, then

$$-\frac{\delta^2}{4}\left(1-h(2\pi)\right)<h(2\pi)\phi(2\pi).$$

Taking into account (2.15) and since $1-h(2\pi)>0$, it follows that

$$-\widetilde{h}(\theta)\frac{\delta^2}{4}-\phi(\theta)<-\frac{\delta^2}{4}<\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)},$$

thus

$$h(\theta)\left(-\widetilde{h}(\theta)\frac{\delta^2}{4}-\phi(\theta)+\phi(\theta)\right)<h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right),$$

and

$$-h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right) < -h(\theta)\left(-\widetilde{h}(\theta)\frac{\delta^2}{4}-\phi(\theta)+\phi(\theta)\right).$$

Then $-h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right) < \frac{\delta^2}{4}$ for all $\theta\in\mathbb{R}$.

On the other hand, it can be shown that $\phi(\theta)<0$, $h(\theta)>0$ and for all $\theta\in\mathbb{R}$ and $1-h(2\pi)>0$ we have

$$-h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right) > 0.$$

From equation (2.13), we obtain the following expression:

$$r^4-\delta r^2-h(\theta)\left(\frac{h(2\pi)\,\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right)=0.$$

By applying the substitution $\rho=r^2$, this equation reduces to:

$$\rho^2(\theta)-\delta\rho(\theta)-h(\theta)\left(\frac{h(2\pi)\,\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right)=0. \tag{2.17}$$

The discriminant of (2.17) is $\triangle=\delta^2+4h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right)$.

Since

$$-h(\theta)\left(\frac{h(2\pi)\,\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right) < \frac{\delta^2}{4},$$

it follows that the discriminant $\Delta>0$ for all $\theta\in\mathbb{R}$. Therefore, equation (2.17) has two distinct real solutions. The roots are given by:

$$\rho=\frac{1}{2}\left(\delta+\sqrt{\delta^2+4h(\theta)\left(\frac{h(2\pi)\,\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right)}\right).$$

or

$$\rho=\frac{1}{2}\left(\delta^2-\left(\delta^2+4h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right)\right)^{\frac{1}{2}}\right).$$

Returning to the change of variables, we deduce

$$r_1^*(\theta) = \frac{\sqrt{2}}{2}\left(\delta^2+\left(\delta^2+4h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}, \tag{2.18}$$

$$r_2^*(\theta) = \frac{\sqrt{2}}{2}\left(\delta^2-\left(\delta^2+4h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}, \tag{2.19}$$

where $\phi(\theta)=\int_0^\theta \frac{2f_0(s)}{g(s)h(s)}ds$.

From equation (2.12), there exist two distinct values for which the condition $r(0,r_0)=r_0$ holds,

$$\tau_1^* = \frac{\sqrt{2}}{2}\left(\delta^2+\left(\delta^2+4\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{2}},$$

$$\tau_2^* = \frac{\sqrt{2}}{2}\left(\delta^2-\left(\delta^2+4\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{2}},$$

which are solution of $r_0^4-\delta r_0^2=\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}$.

Next we prove that $r_i^*(\theta) > 0$, for $i = 1, 2$, for all $\theta \in \mathbb{R}$. Indeed since $0 < -h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)} + \phi(\theta)\right) < \frac{\delta^2}{4}$, and $\delta > 0$ and we have

$$-\delta^2 - 4h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)} + \phi(\theta)\right) > -\delta^2,$$

which implies $\delta > \left(\delta^2 + 4h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)} + \phi(\theta)\right)\right)^{\frac{1}{2}}$, hence

$$\delta - \left(\delta^2 + 4h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)} + \phi(\theta)\right)\right)^{\frac{1}{2}} > 0,$$

and

$$\delta + \left(\delta^2 + 4h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)} + \phi(\theta)\right)\right)^{\frac{1}{2}} > 0.$$

Therefore $r_i^*(\theta) > 0$, $i = 1, 2$, one can see that $r_i^*(\theta)$, $i = 1, 2$ are $2\pi-$periodic, since ϕ and h are $2\pi-$periodic
To demonstrate that this periodic solution constitutes a limit cycle, we consider the following:

$$\begin{aligned} r_1^*(\theta,\xi) &= \frac{\sqrt{2}}{2}\left(\delta^2 + \left(\delta^2 + 4h(\theta)\left(\xi^4 - \delta\xi^2 + \phi(\theta)\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}, \\ r_2^*(\theta,\xi) &= \frac{\sqrt{2}}{2}\left(\delta^2 - \left(\delta^2 + 4h(\theta)\left(\xi^4 - \delta\xi^2 + \phi(\theta)\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}, \end{aligned}$$

where $\xi^4 - \delta\xi^2 = \frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}$, and introduce the Poincaré return map $\xi \mapsto \Pi(2\pi, \xi) = r_{\underset{i}{*}}(2\pi, \xi)$, $i = 1, 2$, for more details see [18].

We compute $\left.\dfrac{dr_i^*(2\pi,\xi)}{d\xi}\right|_{\xi=\tau_{\underset{i}{*}}}$, $i = 1, 2$ at the value $\xi = \tau_{\underset{i}{*}}$, $i = 1, 2$, we find that

$$\left.\frac{dr_1^*(2\pi,\xi)}{d\xi}\right|_{\xi=\tau_{\underset{1}{*}}} = \frac{h(2\pi)\left(4\tau_{\underset{1}{*}}^3 - 2\delta\tau_{\underset{1}{*}}\right)\left(\delta^2 + 4h(2\pi)\left(\tau_{\underset{1}{*}}^4 - \delta\tau_{\underset{1}{*}}^2 + \phi(2\pi)\right)\right)^{\frac{-1}{2}}}{\left(\delta^2 + \left(\delta^2 + 4h(2\pi)\left(\tau_{\underset{1}{*}}^4 - \delta\tau_{\underset{1}{*}}^2 + \phi(2\pi)\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}},$$

and

$$\left.\frac{dr_2^*(2\pi,\xi)}{d\xi}\right|_{\xi=\tau_{\underset{2}{*}}} = \frac{h(2\pi)\left(2\delta\tau_{\underset{2}{*}} - 4\tau_{\underset{2}{*}}^3\right)\left(\delta^2 + 4h(2\pi)\left(\tau_{\underset{2}{*}}^4 - \delta\tau_{\underset{2}{*}}^2 + \phi(2\pi)\right)\right)^{\frac{-1}{2}}}{\left(\delta^2 - \left(\delta^2 + 4h(2\pi)\left(\tau_{\underset{2}{*}}^4 - \delta\tau_{\underset{2}{*}}^2 + \phi(2\pi)\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}},$$

where

$$\begin{aligned} \tau_{\underset{1}{*}} &= \frac{\sqrt{2}}{2}\left(\delta^2 + \left(\delta^2 + 4\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}. \\ \tau_{\underset{2}{*}} &= \frac{\sqrt{2}}{2}\left(\delta^2 - \left(\delta^2 + 4\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}. \end{aligned}$$

Taking into account (2.14), we deduce that

$$\left.\frac{dr_i^*(2\pi,\xi)}{d\xi}\right|_{\xi=\tau_{\underset{i}{*}}} \neq 1, i = 1, 2.$$

It follows that the limit cycles of differential equation (2.6) are hyperbolic. For additional details, refer to [18].

It remains to establish that the limit cycles are non-algebraic. In the phase plane, all trajectories of system (1.4) are described by the level curves of the first integral $F(x, y)$. In particular, the limit cycles correspond to the level sets $F(x, y) = r_0$, where r_0 is determined by condition (2.12). Suppose, by contradiction, that a limit cycle is algebraic. Then $F(x, y)$ would necessarily be a polynomial. Consequently, there must exist an integer n such that the partial derivative $\frac{\partial^n F}{\partial x^n}$ vanishes identically. However, this is not the case. The function $F(x, y)$ contains transcendental terms, such as

$$\exp\left(\int_0^{\arctan\left(\frac{y}{x}\right)} \frac{2f_0(s)}{g(s)}\,ds\right),$$

which persist under differentiation of any order. As a result, $F(x, y)$ cannot be a polynomial, implying that the limit cycles are non-algebraic.
Clearly the curve $(r(\theta)\cos\theta, r(\theta)\sin\theta)$ in the (x, y) plane with

$$F(r,\theta) = r^4 - \delta r^2 - h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)} + \phi(\theta)\right), \tag{2.20}$$

is not algebraic, due to the expression $\theta \to h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)} + \phi(\theta)\right)$. More precisely, in Cartesian coordinates the curve defined by this limit cycle is

$$F(x,y) = \left(x^2+y^2\right)^2 - \delta\left(x^2+y^2\right) - h(\arctan\frac{y}{x})\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)} + \phi(\arctan\frac{y}{x})\right) = 0.$$

If the limit cycle is algebraic this curve must be given by a polynomial, but a polynomial $F(x, y)$ in the variables x and y satisfies that there is a positive integer m such that $\frac{\partial^m F}{\partial x^m} = 0$ and this is not the case because in the derivative $\frac{\partial F}{\partial x}$ appears again the expression

$$h(\arctan\frac{y}{x})\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)} + \phi(\arctan\frac{y}{x})\right),$$

which is already present in $F(x, y)$, and continues to appear in its partial derivatives of all orders.
Proof of statement (3). If we take $\alpha \in \{-1, 1\}$ in (2.17), we obtain $\rho^2(\theta) - \delta\rho(\theta) = -\frac{2}{\mu}$. Reverting to the original variables, we obtain the relation

$$r^4(\theta) - \delta r^2(\theta) = -\frac{2}{\mu}.$$

By transforming back to Cartesian coordinates (x, y), this yields the expression

$$(x^2+y^2)^2 - \delta(x^2+y^2) + \frac{2}{\mu} = 0.$$

This concludes the proof of statement(3). □

Example 2.2. When $\alpha = \frac{1}{4}$, $\mu = \sigma = -1, \delta = 4, \beta = -2$, system (1.4) reads

$$\begin{cases} \dot{x} = xQ_2(x,y) + \left(x^3+xy^2\right)Q_4(x,y) - 2yQ_6(x,y), \\ \dot{y} = yQ_2(x,y) + \left(yx^2+y^3\right)Q_4(x,y) + 2xQ_6(x,y), \end{cases} \tag{2.21}$$

where

$$
\begin{aligned}
Q_2(x,y) &= \left(\frac{1}{16}y^2 + x^2\right).\\
Q_4(x,y) &= (4-x^2-y^2)\left(-\frac{1}{2}x^2+\frac{7}{16}y^2\right).\\
Q_6(x,y) &= \left(\frac{1}{16}y^2 + x^2\right)(x^2+y^2)\left(4-2x^2-2y^2\right).
\end{aligned}
$$

The assumptions in statements (1) and (2) of Theorem 1 are readily satisfied. Consequently, system (2.21) admits two non-algebraic limit cycles, as illustrated in Figure 1. Their explicit expressions in polar coordinates (r,θ) are given by:

$$
\begin{aligned}
r_1^*(\theta) &= \frac{\sqrt{2}}{2}\left(16+\left(16+4h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}.\\
r_2^*(\theta) &= \frac{\sqrt{2}}{2}\left(16-\left(16+4h(\theta)\left(\frac{h(2\pi)\phi(2\pi)}{1-h(2\pi)}+\phi(\theta)\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}.
\end{aligned}
$$

where $\phi(\theta)=\int_0^\theta \frac{2f_0(s)}{g(s)h(s)}ds$, $f_0(\theta)=\frac{1}{2}\left(\frac{17}{16}+\frac{15}{16}\cos 2\theta\right)$, $g(\theta)=\frac{15}{16}\cos 2\theta-\frac{17}{16}$, and $h(\theta)=\exp\left(\int_0^\theta \frac{2f_0(s)}{g(s)}ds\right)$.

3. Conclusions

In this work, we introduced and studied a class of planar polynomial differential systems exhibiting a degenerate singularity. By appropriately choosing the system parameters under the conditions stated in Theorem 1, we derived an explicit expression for a first integral, thereby enabling a qualitative description of the system's trajectories an essential step in the global analysis of dynamical systems. Furthermore, we established sufficient conditions under which the system admits either two distinct non-algebraic limit cycles or a single algebraic limit cycle. These limit cycles were presented in explicit form.

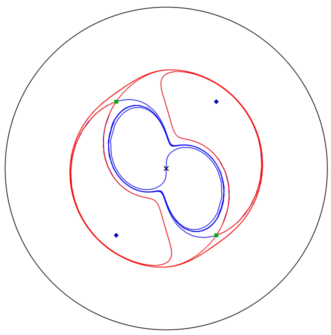

Figure 1: The phase portrait in the Poincaré disc of the system (2.21).

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