Available online at http://jprm.sms.edu.pk/
Journal of Prime Research in Mathematics, 21(2) (2025), 25–36

Journal of Prime Research in Mathematics

New Results on Singularly Perturbed Fractional KdV and KdV-Burgers Equations

Iqbal Jebril[a], Zoubir Dahmani[b,*]

[a] *Department of Mathematics, Al-Zaytoonah University of Jordan, Amman, 11733, Jordan.*
[b] *Laboratory of Discrete and Applied Mathematical Analysis and Operations Research (LAMDA-RO), University of Blida 1, 09000, Blida, Algeria.*

Abstract

This paper investigates traveling wave solutions for singularly perturbed fractional KdV equation and KdV–Burgers equation with a linear confinement term using Caputo fractional derivatives of order $\alpha \in (3,4)$. Using the tanh method, we first derive exact solutions for the classical cases, recovering the well-known hyperbolic secant soliton for the KdV equation and a tanh-profile solution for the KdV–Burgers equation with confinement term. For the singularly perturbed cases, we employ matched asymptotic expansions to construct solutions that capture both the localized wave core and the algebraic decay in the outer region induced by the fractional derivatives. The inner case retains the classical soliton structure, and the outer solution exhibits a power-law tail of order $\xi^{3-\alpha}$, reflecting the nonlocal effects of fractional derivatives. Our main results reveal that the fractional perturbation modifies the wave's decay behavior, giving rise to nonlocal effects absent from the classical formulation.

Keywords: Caputo derivative, KdV equation, KdV–Burgers equation with confinement, Singular perturbations, Tanh method, Matched asymptotic expansions.
2010 MSC: 35C07, 35Q53, 34E10, 70K70, 93C95.

1. Introduction

Nonlinear wave equations such as the KdV and KdV–Burgers equations are foundational models in dispersive and dissipative wave dynamics. The standard KdV equation

$$w_t + d\,ww_x + \gamma w_{xxx} = 0$$

describes solitons in shallow water and plasma systems [30], and the KdV–Burgers equation

$$w_t + d\,ww_x + \beta w_{xx} + \gamma w_{xxx} = 0$$

*Corresponding author
Email addresses: `iqbal501@hotmail.com` (Iqbal Jebril), `zzdahmani@yahoo.fr` (Zoubir Dahmani)

Received : 01 August 2025*; Accepted:* 24 September 2025*; Published Online:* 30 September 2025.

includes viscous dissipation and models phenomena such as shock waves in gas dynamics and traffic flow [18].
Analytical techniques such as the Tanh method and inverse scattering yield explicit traveling wave solutions: the classical squared hyperbolic secant soliton for KdV [29], and tanh-type or kink profiles for KdV–Burgers [11]. These results demonstrate a rich wave structure formed by the interaction of non-linearity, dispersion, and dissipation.
To capture memory effects and nonlocal interactions, fractional problems have been developed across a broad range of differential equations, including but not limited to the KdV and KdV–Burgers equations. These models replace classical integer-order derivatives with the Caputo fractional derivative of order $\alpha > 0$, enabling the description of anomalous diffusion, long-range dependence, and history effects; see [5, 6, 17]. In the context of nonlinear wave dynamics, fractional versions of the KdV and KdV–Burgers equations extend the classical models by incorporating anomalous dispersion and dissipation, with applications in viscoelastic materials, porous media, and anomalous diffusion processes [10, 13]. Meanwhile, fractional differential equations of other types, not necessarily wave-based, also benefit from this fractional framework to model complex physical and engineering systems governed by nonlocal and memory-dependent processes.
Several studies have investigated fractional KdV-type equations using different analytical and computational methods. H. Thabet et al. [27] introduced an approximate-analytical method to obtain traveling wave solutions of the following fractional modified KdV equations in the Caputo sense:

$$\begin{cases} D_t^q u(x,t) - \alpha\big(u(x,t)\big)^2 u_x(x,t) + u_{xxx}(x,t) = 0, \quad x \in \mathbb{R},\, t > 0, \\ u(x,0) = \dfrac{\sqrt{6}}{2\sqrt{\alpha}} \tanh\Big(\dfrac{x}{2} + 1\Big), \end{cases}$$

where α is a constant.
Then, K. Shehzada et al.[25] studied the generalized perturbed fractional KdV equation with a power-law kernel:

$${}_0^C D_t^\eta \Upsilon(x,t) + a\,\Upsilon_x(x,t) + b\,\Upsilon(x,t)\Upsilon_x(x,t) + c\,\Upsilon_{xxx}(x,t) = 0, \quad \eta \in (0,1],$$

where, ${}_0^C D_t^\eta$ denotes the Caputo fractional derivative of order η. By employing the Shehu transform and decomposition approach, the authors derived bright, dark-bright, and other soliton-type solutions.
In parallel, M.M. AlBaidani et al. [3] focused on the fractional forced KdV equation with non singular Caputo–Fabrizio and Atangana–Baleanu–Caputo derivatives, using the Natural decomposition and Natural transform methods to obtain convergent series solutions that highlight the influence of nonsingular kernels on wave propagation.
C. Pawar et al. [22] applied two different decomposition techniques to study a mixed fractional KdV–Burgers equation. Some comparisons of approximate solutions have also been presented.
Very recently, N.S. Alharthi [4] examined the following fractional KdV–Burgers equation in the Caputo sense:

$${}^C D_\tau^\varepsilon w(\kappa,\tau) = 6w(\kappa,\tau)w_\kappa(\kappa,\tau) - w_{\kappa\kappa\kappa}(\kappa,\tau), \quad 0 < \varepsilon \le 1.$$

Combining hybrid integral transforms, the author derived series-type solutions validated by numerical simulations.
In contrast to the above cited works, the present study treats the singularly perturbed case for $\alpha \in (3,4)$, where the appearance of algebraic tails requires a matched asymptotic expansion that explicitly separates an internal core from an external nonlocal contribution.
A particular scenario arises when fractional terms are singularly perturbed. Such problems involve layers or multiple spatial scales and are addressed via matched asymptotic expansions (MAE), which construct inner and outer approximations and match them in an overlapping region [16]. The MAE method is established in the classical singular perturbation theory [9, 19], and its application in fractional settings, especially for space-fractional problems with $\alpha \in (3,4)$, is still emerging. Most works focus on integer-order or time-fractional cases, or rely on numerical/series methods without an analytical matched expansion [1, 2].

Despite increasing interest in fractional models, the analytical study of singularly perturbed fractional wave equations is still lacking. This is a significant gap in nonlinear dynamics, where nonlocality and memory effects have a major impact on wave propagation and structure. Classical KdV-type models do not capture these features, and traditional tools are not well equipped to handle the complexities introduced by fractional derivatives. This study addresses that gap by extending the MAE technique to fractional dispersive problems, offering new insights into how wave structures behave in the presence of strong nonlocal effects.
In this paper, we analyze traveling waves for the singularly perturbed space-fractional KdV equation and the KdV–Burgers equation with linear confinement term, involving Caputo derivatives of order $\alpha \in (3,4)$:

$$\varepsilon\, {}^{C}D_x^{\alpha} w + \delta w_t + \gamma w w_x + \beta w_{xxx} = 0, \quad \alpha \in (3,4) \tag{1.1}$$

and

$$\varepsilon\, {}^{C}D_x^{\alpha} w + w_t + d\, w w_x + \beta w_{xx} + \gamma w_{xxx} + \mu w = 0, \quad \alpha \in (3,4). \tag{1.2}$$

Here, ${}^{C}D_x^{\alpha}$ denotes the fractional Caputo derivative of order $\alpha \in (3,4)$ with respect to space [23], and ε is a small positive parameter.
We first recover classical hyperbolic secant and tanh-type solutions via the tanh method when $\varepsilon = 0$. For $\varepsilon > 0$, we apply the MAE to derive inner solutions, outer solutions, and composite solutions that connect the two regions. Our work shows that when using fractional dispersion, the solutions impose long-range algebraic tails that decay slowly.

2. First Results: Singular Perturbed KdV

2.1. Tanh Method for Reduced Problem

In this section, we focus on (1.1) for the classical case$\varepsilon = 0$. We have the following equation:

$$\delta w_t + \gamma w w_x + \beta w_{xxx} = 0, \tag{2.1}$$

where, δ, γ and β are real constants.
We study traveling wave solutions using the tanh method.
For this method, we assume a solution of the form:

$$w(x,t) = W(\xi), \quad \xi = x - ct, \tag{2.2}$$

where, c is the wave speed.
Substituting (2.2) into the classical KdV equation (2.1), we get:

$$\delta(-cW') + \gamma W W' + \beta W''' = 0 \quad \Rightarrow \quad -c\delta W' + \gamma W W' + \beta W''' = 0.$$

This is the ordinary differential equation to solve for $W(\xi)$.
We express $W(\xi)$ as follows:

$$W(\xi) = \sum_{k=0}^{n} a_k \tanh^k(\xi). \tag{2.3}$$

To determine n from (2.3), we balance the nonlinear term WW' (which gives a term of order $2n+1$) and the highest derivative W''' (which gives a term of order $n+3$):

$$2n + 1 = n + 3 \quad \Rightarrow \quad n = 2.$$

Thus, we have the following

$$W(\xi) = a_0 + a_1 \tanh(\xi) + a_2 \tanh^2(\xi). \tag{2.4}$$

Using the identities:

$$\frac{d}{d\xi}\tanh(\xi) = \operatorname{sech}^2(\xi), \quad \frac{d}{d\xi}\operatorname{sech}^2(\xi) = -2\operatorname{sech}^2(\xi)\tanh(\xi),$$

and by (2.4), we obtain the following derivatives:

$$\begin{aligned} W'(\xi) &= a_1 \operatorname{sech}^2(\xi) + 2a_2 \tanh(\xi) \operatorname{sech}^2(\xi), \\ W''(\xi) &= -2a_1 \operatorname{sech}^2(\xi) \tanh(\xi) + 2a_2 \operatorname{sech}^4(\xi) - 4a_2 \operatorname{sech}^2(\xi) \tanh^2(\xi), \\ W'''(\xi) &= -2a_1 \operatorname{sech}^2(\xi)(1 - 2\tanh^2(\xi)) - 8a_2 \operatorname{sech}^2(\xi) \tanh(\xi) \\ &\quad + 8a_2 \operatorname{sech}^4(\xi) \tanh(\xi) + 8a_2 \operatorname{sech}^2(\xi) \tanh^3(\xi). \end{aligned}$$

Substituting W, W', and W''', we get:

$$-c\delta W' + \gamma W W' + \beta W''' = 0.$$

Coefficient of $\tanh^3(\xi)$:

$$\gamma a_2^2 - 8\beta a_2 = 0 \quad \Rightarrow \quad a_2 = \frac{8\beta}{\gamma}.$$

Coefficient of $\tanh^2(\xi)$:

$$\gamma a_1 a_2 - 2\beta a_1 = a_1(\gamma a_2 - 2\beta) = 0 \quad \Rightarrow \quad a_1 = 0.$$

Coefficient of $\tanh(\xi)$:

$$-c\delta a_1 + \gamma a_0 a_1 - 2\beta a_1 = 0 \quad \text{(trivially satisfied since } a_1 = 0\text{)}.$$

To obtain a localized solution, we require:

$$a_0 + a_2 = 0 \quad \Rightarrow \quad a_0 = -\frac{8\beta}{\gamma}.$$

Thus, the final form is given by:

$$W(\xi) = -\frac{8\beta}{\gamma} + \frac{8\beta}{\gamma} \tanh^2(\xi) = \frac{8\beta}{\gamma}(\tanh^2(\xi) - 1) = -\frac{8\beta}{\gamma}\operatorname{sech}^2(\xi). \tag{2.5}$$

For completeness and to compare with the classical soliton form commonly found in the literature, we note the equivalent expression:

$$w(x,t) = \frac{12\beta}{\gamma}\operatorname{sech}^2\left(\sqrt{\frac{c\delta}{4\beta}}(x - ct)\right), \quad c = \frac{4\beta}{\delta}. \tag{2.6}$$

2.2. Perturbed Problem

We analyze the singular perturbed KdV equation (1.1).
The form (2.2) leads to the equation:

$$\varepsilon\, {}^C D_\xi^\alpha W - c\delta W' + \gamma W W' + \beta W''' = 0. \tag{2.7}$$

We use matched asymptotic expansions [20] to construct a uniform approximation. So, we have the following steps:

2.2.1. Inner Region Expansion

In the inner region, (i.e. near the core of the wave), we use a regular asymptotic expansion as follows:

$$W(\xi) = W_0(\xi) + \varepsilon W_1(\xi) + O(\varepsilon^2). \tag{2.8}$$

Leading Order:. The leading-order equation is obtained by taking $\varepsilon = 0$ in (2.7). So,

$$-c\delta W_0' + \gamma W_0 W_0' + \beta W_0''' = 0. \tag{2.9}$$

Equation (2.9) is the classical KdV equation in traveling wave form. As shown earlier, one solution is the expression derived using the tanh method, given in (2.5). Alternatively, a well-known solitary wave solution can be written as:

$$W_0(\xi) = \frac{12\beta}{\gamma} \sec h^2 \left(\sqrt{\frac{c\delta}{4\beta}}\, \xi \right), \quad \text{with } c = \frac{4\beta}{\delta}, \tag{2.10}$$

which corresponds to (2.6) expressed in terms of traveling wave variable ξ.

First-Order Correction:. The equation for $W_1(\xi)$ introduced in (2.8) is given by:

$$L[W_1] = -{}^C D_\xi^\alpha W_0,$$

where,

$$L[W_1] := -c\delta W_1' + \gamma(W_0 W_1' + W_1 W_0') + \beta W_1'''.$$

The source term $-{}^C D_\xi^\alpha W_0$ presents the following asymptotic behavior:

$${}^C D_\xi^\alpha W_0(\xi) \sim \begin{cases} \xi^{4-\alpha}, & \xi \to 0^+, \\ \xi^{-\alpha+3}, & \xi \to +\infty, \end{cases}$$

which induces algebraic tails in $W_1(\xi)$.

2.2.2. Outer Region Expansion

In the far field, the leading-order solution W_0 is negligible, so we assume the form:

$$W(\xi) \sim \varepsilon^2 \kappa \varphi(\xi), \quad |\xi| \gg 1.$$

Substituting into (2.7) and retaining the leading-order terms yields the following:

$$\varepsilon^{1+\kappa C} D_\xi^\alpha \varphi - c\delta \varepsilon^\kappa \varphi' + \cdots = 0.$$

So we have $\kappa = 1$, and then we can write:

$${}^C D_\xi^\alpha \varphi \approx 0.$$

Motivated by the decay of the source term in the inner region, we adopt:

$$\varphi(\xi) \sim C\xi^{-\alpha}, \quad \text{so that } W_{\text{outer}}(\xi) \sim \varepsilon \xi^{-\alpha}.$$

2.2.3. Composite Approximation and Matching

Since the first-order inner correction W_1 has matching asymptotics:

$$W_1^{\text{match}}(\xi) \sim \begin{cases} \xi^{4-\alpha}, & \xi \to 0, \\ \xi^{-\alpha+3}, & \xi \to \infty, \end{cases}$$

so, we define the composite approximation as:

$$W_{\text{comp}}(\xi) = W_0(\xi) + \varepsilon W_1^{\text{match}}(\xi). \tag{2.11}$$

One possible form for $W_1^{\text{match}}(\xi)$ can be given as follows:

$$W_1^{\text{match}}(\xi) \approx C_\alpha \xi^{4-\alpha} \sec h(\xi), \tag{2.12}$$

which satisfies both asymptotic limits and decays at infinity.
Therefore, using (2.10), (2.11) and (2.12), the final composite solution is:

$$W_{\text{comp}}(\xi) \approx \frac{12\beta}{\gamma} \sec h^2(\xi) + \varepsilon C_\alpha \xi^{4-\alpha} \sec h(\xi), \quad C_\alpha = \frac{6\delta c}{\gamma \Gamma(4-\alpha)}. \tag{2.13}$$

Interpretation

The singular perturbation generates algebraic tails in the wave profile due to the nonlocal contribution of the Caputo fractional derivative. The composite solution (2.13) captures both the localized soliton structure in the inner region and the algebraic decay in the outer region. The presence of the fractional term introduces a long-range memory effect that alters the tail structure of the classical KdV soliton.

2.3. Discussion on the Existence of an Exact Solution

The matched asymptotic expansion conducted in Section 2.2 presents a strong heuristic tool for the structure of the solution to (2.7), characterized by a soliton core and algebraic tails.
A rigorous and natural question arises: Does an exact solution $W_{\text{exact}}(\xi)$ to problem (2.7) exist, and under what conditions? This discussion presents the main mathematical challenges, such as the need for function spaces to handle the algebraic tails, the properties of the fractional source term, and the invertibility of the linearized operator for a rigorous existence proof, thereby providing a solid conceptual foundation for our asymptotic results.
The literature on singular perturbation theory for differential equations [8, 20, 21] suggests that the existence of such a solution can be established under the following considerations:
Perturbation parameter and connection to Tikhonov's theorem: The asymptotic expansion (2.8) is formal. For classical singularly perturbed ordinary differential equations, the theorem of Tikhonov [28] provides the conceptual framework for establishing the existence of a solution and its convergence to the reduced solution. Extending this framework to fractional differential equations, where the non-local nature of the derivative alters the stability analysis and the layer phenomena, is a non-trivial challenge and an active area of research. Thus, for our fractional problem, ε must be sufficiently small for a prospective generalization of such theorems to apply, or for alternative methods like fixed-point theorems to be valid.
Decay and regularity of the leading-order solution: The construction of W_0 and the properties of the Caputo derivative require that W_0 is sufficiently smooth and decays rapidly. The solution (2.10) satisfies these requirements, being a C^∞ function with exponential decay.
Behavior of the fractional source term: The existence theory for W_1 relies on the properties of $-{}^C D_\xi^\alpha W_0$. Its complete asymptotic behavior is given by:

$$
\begin{aligned}
{}^C D_\xi^\alpha W_0(\xi) &\sim \xi^{4-\alpha}, \quad \xi \to 0^+, \\
{}^C D_\xi^\alpha W_0(\xi) &\sim \xi^{-\alpha+3}, \quad \xi \to +\infty.
\end{aligned}
$$

This dual behavior necessitates working in appropriate weighted function spaces to handle both the singularity at zero and the slow algebraic decay at infinity [15].
Functional setting and linearized operator: A rigorous proof of existence would involve working in a function space that accommodates both the exponential decay of the soliton core and the algebraic decay of the tails. The invertibility of the linearized operator L around W_0, defined by:

$$L[V] = -c\delta V' + \gamma(W_0 V' + V W_0') + \beta V''',$$

in such a space is a critical step that is often addressed using spectral theory or Lyapunov-Schmidt reduction [24, 26]. The non-local nature of the fractional derivative adds significant complexity to this analysis.
Under these conditions, the solution $W_{\text{comp}}(\xi)$ constructed in (2.13) is expected to be the first-order approximation of a true solution $W_{\text{exact}}(\xi)$. A rigorous analysis, which is the objective of our ongoing work, has two main thrusts: establishing existence for a fixed $\varepsilon > 0$, most naturally via a fixed-point argument applied to the equivalent integral formulation of (2.7); and extending the framework of Tikhonov theorem to justify the convergence to the reduced solution as $\varepsilon \to 0$ in this fractional context. The core challenges lie in defining the appropriate weighted function spaces and proving the required properties of the corresponding non-local operators for both approaches.
The same rigorous considerations apply to the existence of a solution for the singularly perturbed KdV-Burgers equation with confinement term discussed in Result 2. While the specific governing equation and

the resulting linearized operator differ, the comprehensive framework to proving existence remains identical.

2.4. Stability Considerations

The dynamical stability of the constructed traveling waves is a crucial aspect of their physical relevance. Our asymptotic framework provides a basis for a stability argument for both the fractional KdV soliton (Result 1) and the fractional dissipative wave (Result 2).
For the fractional KdV soliton $W_{\text{comp}}(\xi)$ in (2.13), the argument is based on regular perturbation theory. This solution constitutes a small, regular perturbation of the classical KdV soliton $W_0(\xi)$, which is well-known to be spectrally stable [12, 14]. The linearized operator around W_{comp} is a perturbation of the operator linearized around this stable core. Since the stability of the KdV soliton is robust under sufficiently regular and small perturbations [7], this suggests that for a very small ϵ, the essential spectrum is only mildly altered and remains in the left half-plane, and then no new unstable point spectrum emerges from the origin.
For the wave in (3.8) (of page 12), the stability argument is more nuanced due to the presence of dissipation (βw_{xx}) and confinement (μw) in the considered equation. The dissipative term generally enhances stability by suppressing perturbations. The leading-order wave $W_0(\xi)$ in this case is often a stable traveling wave. The small fractional correction $\varepsilon W_1(\xi)$ is thus a perturbation to a already stable solution. The combination of dissipative and confining mechanisms, supplemented by the small nonlocal term, suggests the overall stability of the composite wave.
In both cases, the primary effect of the fractional term is not to destabilize the wave but to modify its asymptotic structure, replacing exponential decay with a power-law tail. This long range interaction is a fundamental feature of the fractional problems. A rigorous spectral analysis is non trivial due to the nonlocal nature of the fractional operator and the need to work in function spaces that account for algebraic decay. Such an analysis represents a significant and separate challenge in the field of nonlocal PDEs. However, the perturbative considerations presented here provide a heuristic argument for the dynamical relevance of the asymptotic solutions we have constructed.

3. Second Result: Singular Perturbed KdV-Burgers with Term Confinement

3.1. Tanh Method for Reduced Problem

We consider the singularly perturbed fractional KdV-Burgers equation (1.2) with a confinement term, in the case $\varepsilon = 0$:

$$w_t + d\,w_x + \beta w_{xx} + \gamma w_{xxx} + \mu w = 0. \tag{3.1}$$

We look for traveling wave solutions of the form:

$$w(x,t) = W(\xi), \quad \xi = x - ct,$$

which transforms equation (3.1) into the ODE:

$$-cW' + dWW' + \beta W'' + \gamma W''' + \mu W = 0. \tag{3.2}$$

As before, we can write:

$$W(\xi) = a_0 + a_1 \tanh(\xi) + a_2 \tanh^2(\xi),$$

which implies:

$$\begin{aligned}
W'(\xi) &= a_1\ \sec h^2(\xi) + 2a_2\ \tanh(\xi) \sec h^2(\xi),\\
W''(\xi) &= -2a_1\ \tanh(\xi) \sec h^2(\xi) + 2a_2 \sec h^4(\xi) - 4a_2 \tanh^2(\xi) \sec h^2(\xi),\\
W'''(\xi) &= -2a_1 \sec h^4(\xi) + 4a_1 \tanh^2(\xi) \sec h^2(\xi) - 8a_2 \tanh(\xi) \sec h^4(\xi) + 8a_2 \tanh^3(\xi) \sec h^2(\xi).
\end{aligned}$$

Substituting into (3.2), and collecting terms, we obtain the following algebraic system:

$$\begin{aligned}
&-ca_1 + 2da_0a_1 - 2\gamma a_1 + \mu a_1 = 0,\\
&-2ca_2 + da_1^2 + 2da_0a_2 - 2\beta a_1 - 8\gamma a_2 + \mu a_2 = 0,\\
&3da_1a_2 - 6\beta a_2 + 6\gamma a_1 = 0,\\
&2da_2^2 + 16\gamma a_2 = 0.
\end{aligned}$$

3.1.1. Special Case: $\gamma = 0$

When $\gamma = 0$, some straightforward calculations lead to the following form of the solution

$$W(\xi) = \frac{c-\mu}{2d} + a_1 \tanh(\xi), \tag{3.3}$$

where a_1 solves the quadratic:

$$a_1 = \frac{\beta \pm \sqrt{\beta^2 - \frac{\mu(c-\mu)}{2}}}{d}.$$

3.2. Perturbed Problem

We investigate the singularly perturbed fractional KdV-Burgers equation with a confinement term (1.2) to obtain the ODE of fractional order:

$$\varepsilon\, {}^C D_\xi^\alpha W - cW' + dWW' + \beta W'' + \gamma W''' + \mu W = 0. \tag{3.4}$$

This equation models traveling waves under the combined effects of nonlinearity, dispersion, dissipation, fractional memory, and confinement.

3.2.1. Inner Region Expansion

We try to postulate a regular expansion:

$$W(\xi) = W_0(\xi) + \varepsilon W_1(\xi) + O(\varepsilon^2). \tag{3.5}$$

Leading Order:
In leading order, the quantity W_0 of (3.5) satisfies the following:

$$-cW_0' + dW_0W_0' + \beta W_0'' + \gamma W_0''' + \mu W_0 = 0.$$

This is the ODE associated to the classical KdV-Burgers equation with a confinement term.
We consider the quantity:

$$W_0(\xi) = a_0 + a_1 \tanh(\xi).$$

This form is motivated by the structure of solitary waves and yields algebraically solvable coefficients (see reduced problem analysis).

First-order Correction
The correction $W_1(\xi)$ of (3.5) satisfies:

$$L[W_1] = -{}^C D_\xi^\alpha W_0,$$

where

$$L[W_1] = -cW_1' + d(W_0W_1' + W_1W_0') + \beta W_1'' + \gamma W_1''' + \mu W_1.$$

The term $-{}^C D_\xi^\alpha W_0$ is singular due to the fractional nature of the derivative.
For $\alpha \in (3,4)$, the Caputo derivative of W_0 verifies:

$${}^C D_\xi^\alpha W_0(\xi) \sim \begin{cases} \xi^{4-\alpha}, & \xi \to 0^+,\\ \xi^{-\alpha}, & \xi \to +\infty. \end{cases}$$

This result follows from the following definition:

$$^{C}D_{\xi}^{\alpha}W_0(\xi) = \frac{1}{\Gamma(4-\alpha)}\int_0^{\xi}\frac{W_0^{(4)}(s)}{(\xi-s)^{\alpha-3}}\,ds,$$

where

- Near $\xi = 0^+$, $W_0^{(4)}(s) \sim \text{const}$, leading to

$$\int_0^{\xi}(\xi-s)^{-(\alpha-3)}ds = \int_0^{\xi}u^{3-\alpha}du = \frac{\xi^{4-\alpha}}{4-\alpha}.$$

- As $\xi \to +\infty$, $W_0^{(4)}(\xi-\eta) \sim e^{-2(\xi-\eta)}$, and

$$^{C}D_{\xi}^{\alpha}W_0(\xi) \sim e^{-2\xi}\int_0^{\xi}\frac{e^{2\eta}}{\eta^{\alpha-3}}d\eta \sim \xi^{-\alpha}.$$

These behaviors explain the algebraic tails in W_1. They reflect the memory effects of the Caputo derivative.

3.3. Outer Region Expansion

Far from the wave core, W is small. We assume that:

$$W(\xi) \sim \varepsilon^{\kappa}\phi(\xi),$$

and substitute in (3.4), we obtain the following:

$$\varepsilon^{1+\kappa C}D_{\xi}^{\alpha}\phi - c\varepsilon^{\kappa}\phi' + \mu\varepsilon^{\kappa}\phi + \cdots = 0.$$

Attempting to directly balance the leading order terms by equating $\varepsilon^{1+\kappa} = \varepsilon^{\kappa}$ implies $1+\kappa = \kappa$, which is clearly impossible. Therefore, a different dominant balance must be sought.
Instead, we focus on balancing the fractional derivative with the dispersion term $\gamma\phi'''$ or the confinement term $\mu\phi$. Choosing to balance

$$\varepsilon\,^{C}D_{\xi}^{\alpha}\phi \sim \gamma\phi'''$$

leads to the scaling law:

$$\phi(\xi) \sim \xi^{3-\alpha} \quad \Rightarrow \quad W_{\text{outer}}(\xi) \sim \varepsilon\xi^{3-\alpha}.$$

This reveals that the outer solution decays algebraically rather than exponentially. Such slow decay is characteristic of fractional differential operators and reflects the nonlocal behavior they induce. These tails signify that the influence of the wave extends far from its core, a fundamental feature of systems governed by fractional dynamics.

3.4. Composite Approximation and Matching

The inner solution W_0 dominates near $\xi = 0$, where $^{C}D_{\xi}^{\alpha}W_0 \sim \xi^{4-\alpha}$ due to the behavior of the integral kernel. However, the outer tail is governed by the decay of W_1, which matches with $\xi^{3-\alpha}$ to balance the far-field behavior of the Caputo derivative.

$$^{C}D_{\xi}^{\alpha}\phi(\xi) \sim \phi'''(\xi) \quad \Rightarrow \quad \phi(\xi) \sim \xi^{3-\alpha}.$$

We construct the composite solution:

$$W_{\text{comp}}(\xi) = W_0(\xi) + \varepsilon W_{\text{match}}(\xi), \tag{3.6}$$

where the matching correction satisfies:

$$W_{\text{match}}(\xi) \sim C_\alpha \xi^{3-\alpha} e^{-\lambda|\xi|}. \tag{3.7}$$

It combines both the algebraic tail and a decaying exponential envelope (due to matching or smoothing the transition between regions). Thus, substituting (3.3) and (3.7) in (3.6), the final composite approximation is:

$$W_{\text{comp}}(\xi) \approx \frac{c-\mu}{2d} + a_1 \tanh(\xi) + \varepsilon C_\alpha \xi^{3-\alpha} e^{-\lambda|\xi|}, \tag{3.8}$$

where C_α is determined by asymptotic matching.

Interpretation The approximation (3.8) reveals the interaction of confinement with fractional dispersion and dissipation. The algebraic tail $\xi^{3-\alpha}$ is a signature of the Caputo memory effect at infinity, and the local behavior ${}^C D_\xi^\alpha W_0 \sim \xi^{4-\alpha}$ near $\xi = 0$ arises from the singularity of the Caputo kernel.

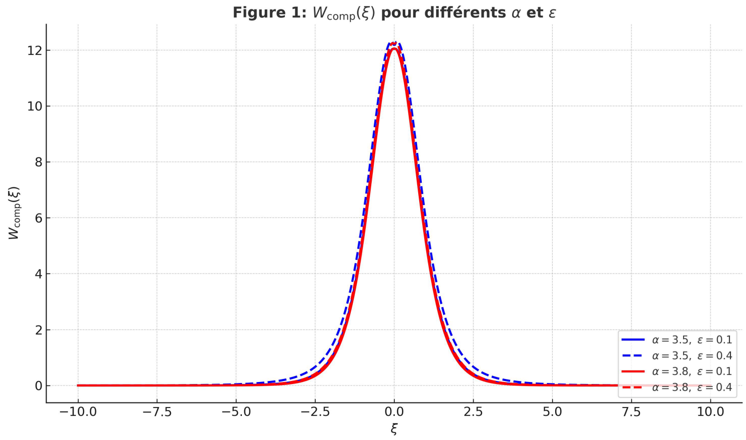

Figure 1: Comparison of velocity profiles $U(\zeta)$ for different ε.

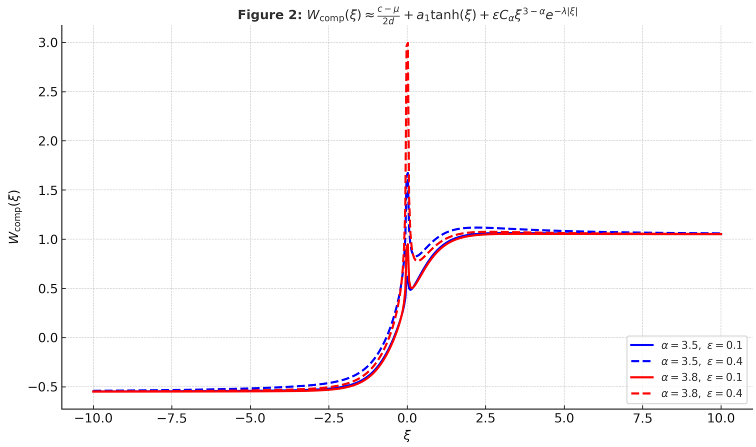

Figure 2: Comparison of velocity profiles $U(\zeta)$ for different ε.

4. Numerical Results and Interpretation

Interpretation of Figure 1:

Figure 1 shows the transformation of soliton behavior under fractional Caputo effects. The exponential decay of KdV soliton is replaced by algebraic tails that decay as $\xi^{4-\alpha}$. For $\alpha = 3.5$, the slower decay

($\xi^{-0.5}$) indicates stronger long-range interactions, where wave energy persists over greater distances due to the memory effects of the Caputo fractional derivative. In contrast, $\alpha = 3.8$ presents faster decay ($\xi^{-0.2}$), approaching classical behavior as the fractional order approaches the integer limit. This transition shows how α can contros the degree of non locality in the problem.
The parameter ε governs the amplitude of fractional corrections while preserving the soliton core structure. Smaller values ($\varepsilon = 0.1$) maintain proximity to the classical soliton, but larger values ($\varepsilon = 0.4$) enhance the algebraic tails and modify the wave profile. This behavior confirms that fractional derivatives introduce wave interactions.

Interpretation of Figure 2:

Figure 2 presents the interplay between fractional effects, dissipation, and confinement. The solution exhibits a tanh-profile transitioning between different background levels, dominated by $\frac{c-\mu}{2d} + a_1 \tanh(\xi)$. The fractional contribution $\varepsilon C_\alpha \xi^{3-\alpha} e^{-\lambda|\xi|}$ introduces algebraic corrections that compete with the exponential localization from the confining potential. For $\alpha = 3.5$, nonlocal effects create more extended wave profiles, but $\alpha = 3.8$ yields better localized solutions due to weaker fractional interactions.
The combination of dissipation and confinement produces asymmetric wave profiles distinct from the pure KdV case. This behavior shows how Caputo derivative modifies wave propagation .

5. Conclusion

This work presented an analytical construction of traveling wave solutions for singularly perturbed space-fractional KdV and KdV–Burgers equations with Caputo derivatives of order $\alpha \in (3, 4)$. By extending the method of matched asymptotic expansions (MAE) to these nonlocal problems, we derived new composite solutions that revealed a clear separation between the classical soliton core and fractional-induced algebraic tails.
The novelty was showing that fractional derivatives modified the asymptotic decay of wave solutions, replacing the exponential decay seen in classical models with persistent power-law tails ($\xi^{3-\alpha}$ or $\xi^{4-\alpha}$), which reflected nonlocal memory effects. This dual structure, with a localized inner soliton and an extended outer tail, could not be captured by traditional KdV-type models.
In conclusion, our results provided a new analytical framework for understanding how fractional dispersion and dissipation reshaped nonlinear wave propagation, with implications for physical systems exhibiting anomalous diffusion and nonlocal transport.

References

[1] R. Akhter and M. S. Alquran, “Fractional KdV–Burgers equation via modified residual power series method,” *Alex. Eng. J.*, vol. 59, pp. 99–108, 2020. 1

[2] M. Akbarzade and S. Soleimani, “Approximate analytical solutions of KdV–Burgers equations using an iterative method,” *Int. J. Numer. Methods Heat Fluid Flow*, vol. 26, no. 1, pp. 80–90, 2016. 1

[3] M. M. AlBaidani, F. Aljuaydi, N. S. Alharthi, A. Khan, and A. H. Ganie, “Study of fractional forced KdV equation with Caputo–Fabrizio and Atangana–Baleanu–Caputo differential operators,” *AIP Advances*, vol. 14, no. 1, p. 015340, Jan. 2024. 1

[4] N. S. Alharthi, “Dynamical and computational analysis of fractional Korteweg–de Vries–Burgers and Sawada–Kotera equations in terms of Caputo fractional derivative,” *Fractal and Fractional*, vol. 9, no. 411, pp. 1–22, 2025. 1

[5] I. M. Batiha, M. S. Hijazi, A. Hioual, A. Ouannas, M. Odeh, and S. Momani, “Stability analysis and numerical simulations of a discrete-time epidemic model,” *Partial Differential Equations in Applied Mathematics*, vol. 13, p. 101118, 2025. 1

[6] O. K. Benhamouda, M. E. Smakdji, A. Derbazi, A. Boudjedour, M. Dalah, K. Zennir, and A. Zarour, “On stability and convergence of a fractional convection reaction-diffusion model,” *Nonlinear Dynamics and Systems Theory*, vol. 24, no. 6, pp. 575–581, 2024. 1

[7] J. L. Bona, P. E. Souganidis, and W. A. Strauss, “Stability and instability of solitary waves of Korteweg–de Vries type,” *Proc. R. Soc. Lond. A*, vol. 411, no. 1841, pp. 395–412, 1987. 2.4

[8] K. W. Chang and F. A. Howes, *Nonlinear Singular Perturbation Phenomena: Theory and Applications.* Springer-Verlag, 1984. 2.3
[9] W. Eckhaus, *Asymptotic Analysis of Singular Perturbations.* Amsterdam, Netherlands: North-Holland, 1979. 1
[10] A. El-Ajou, O. Abu Arqub, and S. Momani, "Approximate analytical solution of the nonlinear fractional KdV–Burgers equation: A new iterative algorithm," *J. Comput. Phys.*, vol. 293, pp. 81–95, 2015. 1
[11] S. A. Elwakil, "Exact solutions for the KdV–Burgers equation using the tanh-function method," *Appl. Math. Comput.*, vol. 131, pp. 517–529, 2002. 1
[12] S. Hakkaev and K. Şamdanlı, "On the spectral stability of periodic waves of the dispersive systems of modified KdV equations," *Nonlinear Anal. Real World Appl.*, vol. 82, p. 104250, Apr. 2025. 2.4
[13] M. Jleli and B. Samet, "Analysis of a fractional KdV–Burgers equation and some exact solutions," *Electron. J. Differ. Equ.*, vol. 2017, no. 73, pp. 1–10, 2017. 1
[14] M. A. Johnson, P. Noble, L. M. Rodrigues, and K. Zumbrun, "Spectral stability of periodic wave trains of the Korteweg–de Vries/Kuramoto–Sivashinsky equation in the Korteweg–de Vries limit," *Trans. Amer. Math. Soc.*, vol. 367, no. 3, pp. 2159–2212, Mar. 2015. 2.4
[15] V. I. Karpman, "Soliton evolution in the presence of perturbation," *Physica Scripta*, vol. 20, no. 3–4, pp. 462–478, 1979. 2.3
[16] J. Kevorkian and J. D. Cole, *Perturbation Methods in Applied Mathematics.* New York: Springer-Verlag, 1981. 1
[17] A. Lamamri, Y. Gouari, Z. Dahmani, M. Rakah, and M. Z. Sarıkaya, "A class of nonlinear systems with new boundary conditions: existence of solutions, stability and travelling waves," *Math. Model. Anal.*, vol. 30, no. 2, pp. 233–253, 2025. 1
[18] M. B. A. Mansour, "Travelling wave solutions for a singularly perturbed Burgers–KdV equation," *Pramana J. Phys.*, vol. 73, no. 5, pp. 799–806, 2009. 1
[19] A. H. Nayfeh, *Introduction to Perturbation Techniques.* New York: Wiley, 1981. 1
[20] R. E. O'Malley, *Asymptotic Analysis and Singular Perturbation Theory.* Univ. of California, Davis, Lecture Notes, 2004. 2.2, 2.3
[21] R. E. O'Malley, "Singular perturbation methods for ordinary differential equations," *Appl. Math. Sci.*, vol. 89, Springer, New York, 2012. 2.3
[22] C. Pawar, K. Takale, and S. Gaikwad, "Comparative study of solutions of fractional order mixed KdV–Burgers equation," *Indian J. Sci. Technol.*, vol. 17, no. 25, pp. 2591–2598, 2024. 1
[23] I. Podlubny, *Fractional Differential Equations.* San Diego, CA: Academic Press, 1999. 1
[24] G. Schneider, "Error estimates for the Ginzburg-Landau approximation," *Z. Angew. Math. Phys.*, vol. 45, pp. 433–457, 1994. 2.3
[25] K. Shehzada, A. Ullah, S. Saifullah, and A. Akgül, "Fractional generalized perturbed KdV equation with a power law kernel: A computational study," *Results Control Optim.*, vol. 12, p. 100298, 2023. 1
[26] M. Taniguchi, "A remark on singular perturbation methods via the Lyapunov–Schmidt reduction," *Publ. RIMS, Kyoto Univ.*, vol. 31, pp. 1001–1010, 1995. 2.3
[27] H. Thabet, S. Kendre, and J. Peters, "Travelling wave solutions for fractional Korteweg–de Vries equations via an approximate-analytical method," *AIMS Math.*, vol. 4, no. 4, pp. 1203–1222, Aug. 2019. 1
[28] A. N. Tikhonov, "Systems of differential equations containing small parameters in the derivatives," *Mat. Sb.*, vol. 31(73), no. 3, pp. 575–586, 1952. 2.3
[29] M. Wadati, "Exact solution of the modified Korteweg–de Vries equation," *J. Phys. Soc. Jpn.*, vol. 32, no. 6, pp. 1681–1687, 1972. 1
[30] G. B. Whitham, *Linear and Nonlinear Waves.* New York: Wiley, 1984. 1