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# An Algorithm for Minimal Generating Set of Parametric Homogeneous Polynomial Ideals

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**Abstract**

We introduce *comprehensive minimal generating systems (CMGS)*, a framework for computing minimal generating sets of homogeneous polynomial ideals with parametric coefficients, which is a non-trivial problem. While Schreyer's syzygy theorem (1980) solved this problem for constant coefficients, the parametric case was recently addressed by the author's MGSystem algorithm [6], which computes minimal generating sets for homogeneous parametric ideals forming Gröbner bases in the variables. In this paper, we extend these results to arbitrary homogeneous parametric polynomial ideals. Our approach combines completed Gröbner systems [7] with an extension of Schreyer's theorem to partition the parameter space into regions where minimal generators are uniformly computed. The CMGS algorithm utilizes syzygy computations to systematically eliminate redundant generators, resulting in significant reductions in generating set sizes under parameter constraints. A `Maple` implementation demonstrates practical efficiency, with examples showing how parameter specializations simplify ideals. This work bridges a 40-year gap in computational algebra, offering theoretical advances (e.g., structural invariants of parametric ideals) and practical algorithms for applications in algebraic geometry, homological algebra, and symbolic computation.

*Keywords:* Minimal generating set, Completed Gröbner systems, Schreyer's syzygy algorithm, Comprehensive minimal generating system, Schreyer's syzygy theorem.

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## 1. Introduction

Finding a minimal generating set for an ideal in commutative polynomial rings can be challenging, as determining a minimal length generating set remains an open problem. The process typically begins by identifying a generating set and then simplifying it by removing redundant generators, ensuring the remaining ones are essential. Tools like Gröbner bases can assist in this task. Determining a minimal set of generators for a polynomial ideal presents an intriguing and complex challenge in computer algebra. This problem is significant because it applies directly to various mathematical domains, such as homological

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algebra, where it enables the computation of minimal free resolutions. This allows us to derive numerical invariants that reflect the depth, regularity, Betti numbers, and graded Betti numbers of these ideals. The intricacy arises because omitting any polynomial from the set of generators naturally leads to a smaller set, which complicates the identification of the truly minimal generating set. A notable contribution to this field is made by Schreyer's syzygy theorem, formulated in 1980 [28], which provides a foundational method for computing the minimal generators specifically for homogeneous polynomial ideals. A simpler version of Schreyer's theorem can be found in [5, Theorem 3.2, page 223 and Theorem 3.3, page 224]. For homogeneous polynomial ideals, it is well established that every minimal generating subset of a set of polynomials $F$ can be identified, and all minimal generating sets of $F$ have the same cardinality. In this paper, we seek to build upon Schreyer's results and extend the approach to include parametric homogeneous polynomial ideals, thereby enhancing the applicability of these concepts (we shall note that the author recently described the MGSystem algorithm [6] to compute the minimal generating set of homogeneous parametric polynomial ideals that form a Gröbner basis w.r.t variables). Our goal is to explore the implications of this extension and provide a more comprehensive framework for understanding generators in both homogeneous and parametric scenarios. For this purpose, we utilize a powerful computer algebra tool called completed Gröbner systems [7]. To understand them, we'll begin with Gröbner bases, as discussed in [1, 4, 5].

This paper is organized as follows: Section 2 provides essential definitions and notation for Gröbner bases and Schreyer's Theorem. Section 3 briefly presents completed and extended Gröbner systems. Section 4 introduces comprehensive minimal generating systems. Finally, section 5 demonstrates the algorithm step-by-step through a straightforward example.

## 2. Gröbner bases and Schreyer's Theorem

In this paper, we consider $R = \mathbb{K}[\mathbf{x}]$, the polynomial ring in variables $\mathbf{x} = x_1, \ldots, x_n$ over the field $\mathbb{K}$. A monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in R$ is denoted as $\mathbf{x}^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ consists of non-negative integers. A monomial order $\prec$ on $R$ is a total order on all monomials with the following properties:
1) If $\mathbf{x}^\alpha \prec \mathbf{x}^\beta$, then $\mathbf{x}^{\alpha+\gamma} \prec \mathbf{x}^{\beta+\gamma}$ for all $\alpha, \beta, \gamma \in \mathbb{N}^n$.
2) The constant monomial is the smallest; specifically, $1 \prec \mathbf{x}^\alpha$ for all $\alpha \in \mathbb{N}^n$.
A common example is the lexicographical order $\prec_{lex}$, where $\mathbf{x}^\beta \prec_{lex} \mathbf{x}^\alpha$ if the left-most non-zero entry of $\alpha - \beta$ is positive.

For $f \in R$, the leading monomial, denoted $\mathrm{LM}(f)$, is the greatest monomial with respect to $\prec$. The leading monomial ideal of $I = \langle f_1, \ldots, f_m \rangle \subset R$ is defined as $\mathrm{LM}(I) = \langle \mathrm{LM}(f) | f \in I \rangle$.

**Definition 2.1.** Let $I = \langle f_1, \ldots, f_k \rangle \subset R$ denote the ideal generated by the polynomials $f_i$. A finite subset $\{g_1, \ldots, g_k\} \subset I$ is called a Gröbner basis of $I$ with respect to $\prec$ if

$$\mathrm{LM}(I) = \langle \mathrm{LM}(g_1), \ldots, \mathrm{LM}(g_k) \rangle.$$

Introduced by Bruno Buchberger in 1965 alongside Buchberger's algorithm [3], Gröbner bases are named after his advisor, Wolfgang Gröbner. Buchberger also developed criteria to eliminate unnecessary S-polynomials to improve the algorithm's efficiency [2]. However, the algorithm may not always be practical due to its complexity. In 1983, Lazard presented an algorithm for computing Gröbner bases using linear algebra techniques [22]. Faugère later introduced the $F_4$ and $F_5$ algorithms in 1999 and 2002, respectively [12, 13]. Subsequently, Gao et al. proposed the incremental G2V algorithm [14], an efficient variant of $F_5$, followed by the GVW algorithm [15] for computing Gröbner bases of ideals. Also, a Gröbner basis $G$ is called reduced if the leading coefficient of every polynomial in $G$ is one and no monomial of any polynomial

The S-polynomial of $f, g \in \mathbb{K}[\mathbf{x}]$ is given by:

$$S(f,g) = \left( \frac{\mathrm{LCM}(\mathrm{LT}(f), \mathrm{LT}(g))}{\mathrm{LT}(f)} \cdot f - \frac{\mathrm{LCM}(\mathrm{LT}(f), \mathrm{LT}(g))}{\mathrm{LT}(g)} \cdot g \right)$$

$p \in G$ lies in the ideal generated by the leading monomials of $G \setminus \{p\}$. Every nonzero ideal $I$ has a unique reduced Gröbner basis for any fixed monomial ordering.

In addition, the first syzygy module of a sequence $F = f_1, \dots, f_k$ is the set of $k$-tuples $(g_1, \dots, g_k) \in R^k$ satisfying $\sum_{i=1}^{k} g_i \cdot f_i = 0$. A syzygy matrix represents the syzygies among generators of a finitely generated module over a commutative ring, thus capturing the relationships essential to understanding the module's structure. This first syzygy module, $ker(\phi)$, is the kernel of a homomorphism and the initial step in resolving a module, as stated by Hilbert's syzygy theorem. Defined by the choice of generating set, it reflects the structure under consideration and is crucial for understanding polynomial rings and syzygy theory. Schreyer's syzygy theorem, introduced by Schreyer in 1980 [28], a simpler format [5, Theorem 3.2 and Theorem 3.3], focuses on syzygies in modules. It offers an algorithm for computing syzygies in polynomial rings and aids in understanding the projective dimension of modules and solving problems like Stillman's Question [27, Problem 3.14].

To prepare for Schreyer's Theorem, let $S(g_i, g_j)$ be the S-polynomial of $g_i$ and $g_j$. Assume $G = \{g_1, \dots, g_m\} \subset R^s$ be a Gröbner basis for a submodule $M \subset R^s$ with respect to a fixed monomial order $\prec$. The S*chreyer order* $\prec_S$ on $R^m$ is defined for monomials $\mathbf{x}^\alpha e_i, \mathbf{x}^\beta e_j \in R^m$ as:

$$\mathbf{x}^\alpha e_i \prec_S \mathbf{x}^\beta e_j \quad \text{if} \quad \begin{cases} \mathrm{LT}(g_i)\mathbf{x}^\alpha \prec \mathrm{LT}(g_j)\mathbf{x}^\beta \text{ in } R, \\ \text{or} \\ \mathrm{LT}(g_i)\mathbf{x}^\alpha = \mathrm{LT}(g_j)\mathbf{x}^\beta \text{ and } i > j \end{cases}$$

where $e_1, \dots, e_m$ are the standard basis vectors of $R^m$. Since $G$ is a Gröbner basis, the remainder of $S(g_i, g_j)$ when divided by $G$ is $\mathbf{0}$. The division algorithm [4, Theorem 3, Page 64] yields an expression: $S(g_i, g_j) = \sum_{k=1}^{m} a_{ijk} g_k$, where $a_{ijk} \in R$, and $\mathrm{LT}(a_{ijk} g_k) \prec \mathrm{LT}(S(g_i, g_j))$ for all $i, j, k$.

**Theorem 2.2** (Schreyer's Theorem [5])**.** *Let $G = \{g_1, \dots, g_m\} \subset R^s$ be a Gröbner basis with respect to $\prec$. For each pair $(i, j)$ with $1 \le i < j \le m$, let $\mathbf{a}_{ij} = \sum_{k=1}^{m} a_{ijk} e_k \in R^m$ be the vector of coefficients obtained from the division $S(g_i, g_j)$ by $G$:*

$$S(g_i, g_j) = \sum_{k=1}^{m} a_{ijk} g_k \quad \textit{with} \quad \mathrm{LT}(a_{ijk} g_k) \prec \mathrm{LT}(S(g_i, g_j)).$$

*Then the syzygies $\mathbf{s}_{ij}$ constructed as:*

$$\mathbf{s}_{ij} = \frac{m_{ij}}{\mathrm{LT}(g_i)} e_i - \frac{m_{ij}}{\mathrm{LT}(g_j)} e_j - \mathbf{a}_{ij}, \quad \textit{where} \quad m_{ij} = \mathrm{LCM}\big(\mathrm{LT}(g_i), \mathrm{LT}(g_j)\big),$$

*form a Gröbner basis for the syzygy module* $\mathrm{syz}(g_1, \dots, g_m)$ *with respect to the Schreyer order* $\prec_S$.

To compute $\mathrm{syz}(f_1, \dots, f_t)$ for a collection $F = \{f_1, \dots, f_t\}$ of non-zero polynomials in $R$ that may not form a Gröbner basis, first compute a Gröbner basis $G = \{g_1, \dots, g_m\}$ for $\langle f_1, \dots, f_t \rangle$. Since $F$ and $G$ generate the same ideal, there exist matrices $A_{t\times m}$ and $B_{m\times t}$ with entries in $R$ such that $G = FA$ and $F = GB$. A generating set $\{\mathbf{s}_1, \dots, \mathbf{s}_p\}$ for $\mathrm{syz}(G)$ can be computed using the previous theorem. Let $\mathbf{r}_1, \dots, \mathbf{r}_t$ be the columns of the matrix $I_t - AB$. Therefore, using the above notation, the following set is a generating set for $\mathrm{syz}(f_1, \dots, f_t)$:

$$\{A\mathbf{s}_1, \dots, A\mathbf{s}_p, \mathbf{r}_1, \dots, \mathbf{r}_t\}.$$

We present the following Syzygy-Schreyer algorithm for computing $\mathrm{syz}(f_1, \dots, f_t)$, summarizing the above content and consistent with [5, Proposition 3.8, page 227].

**Algorithm 1** SYZYGY-SCHREYER

**Require:** $F = \{f_1, \ldots, f_t\} \subset R = \mathbb{K}[\mathbf{x}]$ is a homogeneous polynomial ideal
**Ensure:** A syzygy matrix of $F$
  $G, A :=$ BASIS($F, \prec_{\mathbf{x},\mathbf{a}}, output = extended$); $G$ is the reduced Gröbner basis and $A$ is the matrix s.t. $G = F.A$
  $A :=$ Transpose $(A)$
  $IM :=$ Identity matrix $I_{t\times t}$
  $B :=$ Transpose $(B)$ where $B$ is the inverse transformation matrix s.t. $F = G.B$
  $IAB := IM - AB$
  S := syzygy matrix of $G$
  $AS :=$ Multiply $A$ and $S$
  $AS := [AS \mid IAB]$
  **Return** $(AS)$

If the polynomials $f_1, \ldots, f_t$ are included in the list $g_1, \ldots, g_m$, then $I_t - AB = 0$. Thus, $\text{syz}(f_1, \ldots, f_t) = \langle A\mathbf{s}_1, \ldots, A\mathbf{s}_p\rangle$. For simplicity, assume $f_1 = g_1, \ldots, f_t = g_t$ and $g_{t+i} = \sum_{j=1}^{t} a_{ij} f_j$ for $i = 1, \ldots, s - t$. If $(r_1, \ldots, r_m)$ is a syzygy for $(g_1, \ldots, g_m)$, then $(r_1, \ldots, r_t) + \sum_{i=t+1}^{s-t} r_i(a_{i1}, \ldots, a_{it})$ is a syzygy also for $(f_1, \ldots, f_t)$. By repeating this for all syzygies in the basis for $\text{syz}(g_1, \ldots, g_m)$, we can obtain a basis for $\text{syz}(f_1, \ldots, f_t)$. Furthermore, based on the previous explanation and according to Schreyer's theorem and algorithm, the following important corollary is concluded to compute a minimal generating set for a homogeneous polynomial ideal.

**Corollary 2.3.** *Let $G = \{g_1, \ldots, g_m\} \subset \mathbb{K}[\mathbf{x}]$ be a homogeneous reduced Gröbner basis. Then for each $1 \leq i \leq m$, the following are equivalent:*

1. $g_i \in \langle g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_m\rangle$
2. *The $i$-th row of the syzygy matrix of $G$ contains a non-zero constant $k \in \mathbb{K}$*

*Proof.* We prove both directions separately, using the graded structure of the syzygy matrix obtained from Schreyer's algorithm.
**(1 ⇒ 2):** Assume $g_i \in \langle g_j\rangle_{j\neq i}$. Then there exist homogeneous polynomials $\{a_j\}_{j\neq i}$ such that:

$$g_i = \sum_{j\neq i} a_j g_j$$

This yields the syzygy:

$$\mathbf{v} = (a_1, \ldots, a_{i-1}, -1, a_{i+1}, \ldots, a_m)^T \in \text{syz}(G)$$

By Schreyer's Theorem, the syzygy module $\text{syz}(G)$ has a graded basis $\{\mathbf{s}_{kl}\}$ constructed from S-polynomials $S(g_k, g_l)$. Since $G$ is homogeneous and reduced, $\mathbf{v}$ must be a linear combination:

$$\mathbf{v} = \sum c_{kl}\mathbf{s}_{kl}, \quad c_{kl} \in \mathbb{K}[\mathbf{x}]$$

The $-1$ in the $i$-th position of $\mathbf{v}$ forces at least one $\mathbf{s}_{kl}$ to have a non-zero constant in its $i$-th row to preserve homogeneity.
**(2 ⇒ 1):** Suppose the syzygy matrix has a column:

$$\mathbf{s} = (a_1, \ldots, a_{i-1}, k, a_{i+1}, \ldots, a_m)^T \quad (k \neq 0)$$

Then:

$$\sum_{j\neq i} a_j g_j + k g_i = 0 \implies g_i = -\frac{1}{k}\sum_{j\neq i} a_j g_j \in \langle g_j\rangle_{j\neq i}$$

□

*Remark* 2.4. The syzygy matrix in Corollary 2.3 is graded by construction via Schreyer's algorithm. This ensures the existence of minimal generators with constant entries when redundancy occurs, as shown in [5, Proposition 3.10] for the graded case.

Numerous engineering and scientific challenges can be modeled using a specific set of parametric polynomials. These problems often require iterative analysis with varying parameter values. This paper aims to determine the minimal generating set for parametric polynomial ideals that are homogeneous w.r.t. variables. In this direction, we introduce a new concept of *comprehensive minimal generating systems* for these ideals and present an algorithm for their computation. To achieve this, we utilize the concepts of completed Gröbner systems and extended Gröbner systems [7], which extend the Gröbner system framework to facilitate the computation of transformation matrices. So, in the next section, we will briefly overview completed Gröbner systems and extended Gröbner systems.

## 3. Completed and Extended Gröbner Systems

Gröbner systems generalize Gröbner bases from polynomial ideals with numerical coefficients to those with parametric coefficients. A Gröbner system is defined as a finite set of triples that include parametric constraints along with a set of parametric polynomials. For any specialization of the parameters, there exists a corresponding branch that satisfies the assignment, and upon substituting the parameters, the specialized polynomial set forms a Gröbner basis for the parametric ideal.

This extension is crucial for addressing practical challenges in systems of polynomial equations with parameters, significantly impacting fields such as parametric linear algebra [8, 10, 18], automated geometry theorem proving, and algebraic geometry [16, 24, 25, 30], as well as robotics and electrical networks [23, 25, 26]. These applications often necessitate repeated analysis with varying parameter values. Below, we will provide a brief overview of Gröbner systems and the relevant literature.

We consider $S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$, where $\mathbb{K}$ is any field, $\mathbf{a} = a_1, \ldots, a_m$ is a sequence of parameters, and $\mathbf{x} = x_1, \ldots, x_n$ is a sequence of variables. We define monomial orders $\prec_{\mathbf{x}}$ and $\prec_{\mathbf{a}}$ for the variables and parameters, respectively. These orders combine to establish a new ordering on $S$, denoted $\prec_{\mathbf{x},\mathbf{a}}$ for any $\alpha, \beta \in \mathbb{N}^n$ and $\gamma, \delta \in \mathbb{N}^m$, we have $\mathbf{x}^\alpha \mathbf{a}^\gamma \prec_{\mathbf{x},\mathbf{a}} \mathbf{x}^\beta \mathbf{a}^\delta$ if $\mathbf{x}^\alpha \prec_{\mathbf{x}} \mathbf{x}^\beta$, or if $\mathbf{x}^\alpha = \mathbf{x}^\beta$ and $\mathbf{a}^\gamma \prec_{\mathbf{a}} \mathbf{a}^\delta$.

**Definition 3.1.** Let $F \subset S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$ be a finite set, and let $G = \{(N_i, W_i, G_i)\}_{i=1}^{\ell}$ be a finite set of triples where $N_i, W_i \subset \mathbb{K}[\mathbf{a}]$ and $G_i \subset S$. The set $G$ is considered a Gröbner system for $\langle F \rangle$ with respect to $\prec_{\mathbf{x},\mathbf{a}}$ if, for any index $i$ and homomorphism $\sigma : \mathbb{K}[\mathbf{a}] \to \mathbb{K}' \supseteq \mathbb{K}$, the following conditions are satisfied:

- $\sigma(G_i) \subset \mathbb{K}'[\mathbf{x}]$ is a Gröbner basis for $\sigma(\langle F \rangle) \subset \mathbb{K}'[\mathbf{x}]$ with respect to $\prec_{\mathbf{x}}$
- $\sigma(p) = 0$ for all $p \in N_i$ and $\sigma(q) \neq 0$ for all $q \in W_i$

The concept of a Gröbner system was introduced by Weispfenning in 1992, demonstrating that every parametric polynomial ideal possesses a finite Gröbner system, along with an initial computation algorithm [30]. In 1997, M. Kalkbrenner established critical criteria regarding the specialization of parametric polynomial ideals and the stability of their Gröbner bases under specialization [20]. In 2002, Montes developed a more efficient algorithm, known as DisPGB, for computing Gröbner systems, building upon Buchberger's algorithm [25]. In 2013, Hashemi et al. enhanced DisPGB by integrating two Buchberger criteria to improve its efficiency further [17]. In 2006, Manubens and Montes refined the DisPGB algorithm by introducing the concept of discriminant ideals [23]. Following Kalkbrenner's stability criteria, Suzuki and Sato devised an effective algorithm for computing Gröbner systems, known as the Suzuki-Sato algorithm [29]. In 2010, Kapur, Sun, and Wang introduced the efficient PGBMain (KSW) algorithm, inspired by Weispfenning's and Suzuki-Sato's approaches [21]. In our recent work, we have made significant advancements in the computation of Gröbner systems by exploring parametric polynomial ideals and developing optimized algorithms in this area [10, 11, 16, 19]. In 2024, we presented the efficient GES-GVW-CGS algorithm for computing these systems [9].

**Example 3.2.** Consider the set $F = \{(b-a)xy - x - cy, cx^2 - 2xy\} \subset \mathbb{K}[a,b,c,x,y]$, where $x$ and $y$ are variables, and $a$, $b$, and $c$ are parameters. A Gröbner system for $F$ with respect to the product of monomial orderings $y \prec_{lex} x$ and $c \prec_{lex} b \prec_{lex} a$, computed using our `Maple` implementation of the GES-GVW-CGS algorithm, is as follows:

$$\begin{cases} ([c], & [\,], & [x]) \\ ([ac-bc], & [c, c^2+2], & [c^3y^2+2cy^2, cy+x]) \\ ([c^3+2c, ac-bc], & [c], & [cy+x]) \\ ([\,], & [2c^2(a-b)], & [2acy^3-2bcy^3+c^3y^2+2cy^2, \\ & & 2acy^2-2bcy^2+c^3y+c^2x]) \end{cases}$$

This Gröbner system consists of four triples. By setting $a = b = 2$ and $c = 1$, the second branch corresponds to these values, yielding $\{3y^2, x+y\}$ as a Gröbner basis for the ideal $\langle F\rangle|_{a=b=2,c=1}$.

Dehghani [7] introduced the concepts of completed Gröbner systems and extended Gröbner systems, expanding Gröbner system calculations to encompass both the computation of Gröbner systems and parametric transformation matrices. Parametric transformation matrices serve as fascinating computational objects in computer algebra, clearly illustrating the connection between parametric polynomial ideals and their related Gröbner systems. These matrices offer both theoretical insights and algorithmic utility, enabling efficient analysis of parameter-dependent algebraic structures by systematically tracking basis transformations during Gröbner system computations.

**Definition 3.3.** Let $F \subset S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$ be a finite set, and let $G = \{(N_i, W_i, G_i, M_i)\}_{i=1}^{\ell}$ be a finite set of quadruples, where $N_i, W_i \subset \mathbb{K}[\mathbf{a}]$, $G_i \subset S$, and $M_i$ is a parametric matrix over $S$. The set $G$ is defined as an *extended Gröbner system* for $\langle F\rangle$ with respect to $\prec_{\mathbf{x},\mathbf{a}}$ over $\mathcal{V} \subseteq \overline{\mathbb{K}}^m$ if, for any index $i$ and any homomorphism $\sigma : \mathbb{K}[\mathbf{a}] \to \mathbb{K}' \supseteq \mathbb{K}$, the following conditions hold:

- $\sigma(G_i)$ is a Gröbner basis for $\langle\sigma(F)\rangle$ with respect to $\prec_{\mathbf{x}}$.
- $\sigma(G_i) = \sigma(F) \cdot \sigma(M_i)$.
- $\mathcal{V} \subseteq \bigcup_{i=1}^{\ell} \mathcal{V}_i$, where $\mathcal{V}_i = \mathbb{V}(N_i) \setminus \mathbb{V}(W_i)$.

Like the definition of a Gröbner system, if $\mathcal{V} = \overline{\mathbb{K}}^m$, then $G$ is typically referred to as an extended Gröbner system of $F$. Consequently, throughout the remainder of this paper, the term "extended Gröbner system" will refer specifically to the extended Gröbner system over $\overline{\mathbb{K}}^m$.

Building on these definitions and drawing inspiration from the PGBMain algorithm [16, Algorithm 1], Dehghani introduced the Extended-PGB and Extended-PGBMain algorithms for computing an extended Gröbner system [7].

**Algorithm 2** Extended-PGB

**Require:** $N, W$; finite subsets of $\mathbb{K}[\mathbf{a}]$, $F \subset S = \mathbb{K}[\mathbf{a}, \mathbf{x}] = \mathbb{K}[a_1, \ldots, a_m, x_1, \ldots, x_n]$, $\prec_{\mathbf{x}}, \prec_{\mathbf{a}}$; two monomial orderings, and $I$; an arbitrary identity matrix
**Ensure:** PGB; an extended Gröbner system of $\langle F\rangle$ w.r.t. $\prec_{\mathbf{x},\mathbf{a}}$ on $\mathbb{V}(N) \setminus \mathbb{V}(W)$

```
PGB:= NULL
LIST:= [N, W, F, I]
while LIST is not empty do
  L:=Select the first quadruple of LIST
  Removing L from LIST
  PGB:= PGB, Extended-PGBMain(L)
end while
```

**Algorithm 3** EXTENDED-PGBMAIN

**Input:** $N, W$; finite subsets of $\mathbb{K}[\mathbf{a}]$, $F$; a finite subset of $\mathbb{K}[\mathbf{a},\mathbf{x}]$, and $MX$; a matrix
**Output:** an extended Gröbner system of $\langle F\rangle$ on $\mathbb{V}(N)\setminus\mathbb{V}(W)$

```
if (N, W) is inconsistent then
   Return (∅)
end if
t_n := number of elements in N
F' := [F, N]; a list with the last t_n polynomials from N.
G := Reduced Gröbner Basis(F', ≺_{x,a})
TM := Transformation Matrix(F')
TM := matrix obtained by eliminating the last t_n rows of TM.
if 1 ∈ G then
   Return ({(N, W, {1})})
end if
G_r := G ∩ K[a]
G' := G \ G_r
TM := Remove columns of TM corresponding to positions of G_r in G.
TM := Multiply MX by TM
if (G_r, W) is inconsistent then
   Return (PGB)
else
   G_m :=MDBasis(G')
   TM := Remove columns of TM corresponding to positions of G_m in G'.
   h =lcm{h_1, ..., h_k}, where h_i = LC_{≺_x}(g_i) for g_i ∈ G_m = {g_1, ..., g_k}
   if (G_r, W × {h}) is consistent then
      PGB:= PGB ∪ {G_r, W × {h}, G_m, TM}
   else
      LIST:= LIST, ⋃_{h_i ∈ {h_1,...,h_k}} {[G_r ∪ {h_i}, W × {h_1 h_2 ··· h_{i-1}}, G', TM]}
   end if
   while LIST ≠ ∅ do
      Select L; the first quadruple from LIST
      Update LIST by removing L
      Extended-PGBMain(L)
   end while
end if
Return PGB ∪{(Other cases, {1})}
```

**Definition 3.4.** Let $F \subset S = \mathbb{K}[\mathbf{a},\mathbf{x}]$ be a finite set and $\mathscr{G} = \{(N_i, W_i, G_i, M_i, B_i)\}_{i=1}^{\ell}$ be a set of finite quintuples where $N_i, W_i \subset \mathbb{K}[\mathbf{a}]$, $G_i \subset S$, and $M_i$ and $B_i$ are parametric matrices over $S$. The set $\mathscr{G}$ is a *completed Gröbner system* for $\langle F\rangle$ with respect to $\prec_{\mathbf{x},\mathbf{a}}$ over $\mathcal{V} \subseteq \overline{\mathbb{K}}^m$ if, for each $i$ and any homomorphism $\sigma : \mathbb{K}[\mathbf{a}] \to \mathbb{K}' \supseteq \mathbb{K}$, the following hold:

- $\sigma(G_i)$ is a Gröbner basis of $\langle \sigma(F)\rangle$ with respect to $\prec_{\mathbf{x}}$.
- $\sigma(G_i) = \sigma(F)\cdot\sigma(M_i)$.
- $\sigma(F) = \sigma(G_i)\cdot\sigma(B_i)$.
- $\mathcal{V} \subseteq \bigcup_{i=1}^{\ell} \mathcal{V}_i$ where $\mathcal{V}_i = \mathbb{V}(N_i)\setminus\mathbb{V}(W_i)$.

Also, a "completed Gröbner system" refers to the completed Gröbner system on $\overline{\mathbb{K}}^m$.

To compute the matrices $B_i$ that satisfy $F = G_i \cdot B_i$, we can divide the specialized polynomials of $F$ with respect to $(N_i, W_i)$ by $G$. The quotient from this division forms the entries of matrix $B_i$.

**Example 3.5.** Let $F = [xz^3 + ay + 1, bz^3 - cx - y, axz - xy - b]$ be a parametric polynomial ideal in $S = \mathbb{K}[a,b,c][x,y,z]$. Using our `Maple` implementation of the EXTENDED-PGBMAIN algorithm and based on the above explanation, we compute a completed Gröbner system (containing a Gröbner system and an extended Gröbner system) of $\langle F\rangle$ with respect to the order $c \prec_{\text{lex}} b \prec_{\text{lex}} a$ and $z \prec_{\text{tdeg}} y \prec_{\text{tdeg}} x$.

| $N_i, W_i$ | Gröbner Bases; $G_i$ | Transformation Matrices; $M_i$ and Inverse Transformation Matrices; $B_i$ |
|---|---|---|
| $[\ ], [abc]$ | $[ay^2 - a^2yz - az - cx,$ $-axz + xy + b,$ $aby + axz + cx^2,$ $bz^3 - cx - y]$ | $\begin{bmatrix} -az+y & 0 & b & 0 \\ 1 & 0 & -x & 1 \\ z^3 & -1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{b} & 0 & -1 \\ \frac{1}{b} & 0 & 0 \\ \frac{x}{b} & 1 & 0 \end{bmatrix}$ |
| $[a, c], [b]$ | $[xy + b, bz^3 - y]$ | $\begin{bmatrix} 0 & xz^3(-az+y)+cx \\ 0 & xz^3+1 \\ -1 & z^6x \end{bmatrix}, \begin{bmatrix} \frac{1}{b} & 0 & -1 \\ \frac{x}{b} & 1 & 0 \end{bmatrix}$ |
| $[b], [a, c]$ | $[-az + y, az + cx,$ $az^4 - a^2cz - c]$ | $\begin{bmatrix} -az+y & az-y & \left(-z^3+ac\right)(-az+y)-c \\ -a^2z+ay & a^2z-ay-1 & a^2z^4-a^3cz-ayz^3+a^2cy-z^3 \\ z^3-ac & -z^3+ac & \left(-z^3+ac\right)z^3-ac\left(-z^3+ac\right) \end{bmatrix}, \begin{bmatrix} a & -1 & -x \\ \frac{z^3}{c} & -1 & 0 \\ \frac{-1}{c} & 0 & 0 \end{bmatrix}$ |
| $[c], [a, b]$ | $[ab^2z^2 + ax, aby + axz,$ $-a^2yz + ay^2 - az,$ $aby + xy + b,$ $-ab^3yz + ax^2]$ | $\begin{bmatrix} -bxz^2+ax & b & -az+y & b & \left(-x^2z^2-b^2z\right)b+x^2a \\ x^2z^2+a^2x & -x & 1 & -x & -\left(-x^2z^2-b^2z\right)x+a^2x^2 \\ -bz^2a-xz^2 & 1 & z^3 & 0 & -abxz^2-x^2z^2-b^2z \end{bmatrix}, \begin{bmatrix} \frac{xz}{ab^2} & \frac{z}{ba} & 0 \\ \frac{-x}{ab^2} & \frac{-1}{b} & 1 \\ 0 & 0 & 0 \\ \frac{1}{b} & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ |
| $[a, b, c], [\ ]$ | $[y, xz^3 + 1]$ | $\begin{bmatrix} -xz^3(-az+y)-cx & 1 \\ -xz^3-1 & 0 \\ -z^6x & 0 \end{bmatrix}, \begin{bmatrix} a & -1 & -x \\ 1 & 0 & 0 \end{bmatrix}$ |

For example, with parameters $a = -4$, $b = \frac{1}{2}$, and $c = 0$, the fourth branch satisfies $(N_4, W_4)$. Thus, the Gröbner basis for $F|_{(a=-4,b=\frac{1}{2},c=0)}$ is

$$G_4|_{(a=-4,b=\frac{1}{2},c=0)} = [-z^2 - 4x, -4xz - 2y, -4y^2 - 16yz + 4z, xy - 2y + \frac{1}{2}, -4x^2 + \frac{yz}{2}]$$

and the transformation matrix is

$$M_4|_{(a=-4,b=\frac{1}{2},c=0)} = \begin{bmatrix} \frac{-xz^2}{2} - 4\,x & \frac{1}{2} & y + 4\,z & \frac{1}{2} & -\frac{x^2z^2}{2} - \frac{z}{8} - 4\,x^2 \\ x^2z^2 + 16\,x & -x & 1 & -x & -\left(-x^2z^2 - \frac{z}{4}\right)x + 16\,x^2 \\ -xz^2 + 2\,z^2 & 1 & z^3 & 0 & -x^2z^2 + 2\,xz^2 - \frac{z}{4} \end{bmatrix}.$$

In addition, the inverse transformation matrix for these parameters is

$$B_4|_{(a=-4,b=\frac{1}{2},c=0)} = \begin{bmatrix} -xz & \frac{z}{-2} & 0 \\ x & -2 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, for any specialization $\sigma$ fulfilling $(N_4, W_4)$, we have $\sigma(G) = \sigma(F)\sigma(M_4)$ and $\sigma(F) = \sigma(G)\sigma(B_4)$.

## 4. Comprehensive Minimal Generating Systems

In this section, we compute a minimal generating set for parametric homogeneous polynomial ideals, necessitating new notations to encompass all possible parameter values. Such ideals consist of homogeneous polynomials in their variables with parametric coefficients and hold significant importance in algebraic geometry and computer algebra (e.g., the implicitization problem [5]), both theoretically and practically. Identifying a minimal generating set enhances computational efficiency and provides deeper insights into their structural properties, thereby simplifying complex problems and organizing the representation of these ideals for further applications. We now introduce and elaborate on the *comprehensive minimal generating system (CMGS)* for these ideals, along with an algorithm for computing CMGS's. In this direction, we employ the completed Gröbner system, its transformation matrices, and their inverses under parametric constraints to develop this algorithm. The CMGS and its corresponding algorithm extend the concept of minimal generating sets and the Syzygy-Schreyer algorithm for polynomial ideals with constant coefficients.

**Definition 4.1.** Let $F \subset S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$ be a homogeneous parametric polynomial ideal. We define $M = \{(N_i, W_i, F_i)\}_{i=1}^{\ell}$ as a comprehensive minimal generating system of $F$ if, for each $i$ and any specialization $\sigma : \mathbb{K}[\mathbf{a}] \to \mathbb{K}' \supseteq \mathbb{K}$ fulfilling the parametric constraint $(N_i, W_i)$, the set $\sigma(F_i)$ constitutes a minimal generating set of $\sigma(F)$.

This definition extends [5, Proposition 3.8, page 227], which may be considered as an extension of Schreyer's syzygy theorem for all homogeneous parametric polynomial ideals.

**Theorem 4.2.** *Let $F = \{f_1, \ldots, f_t\} \subset S$ be a homogeneous parametric polynomial ideal, and let the set $G = \{(N_i, W_i, G_i, A_i, B_i)\}_{i=1}^{\ell}$ be a completed Gröbner system for $\langle F \rangle$ with respect to any compatible monomial ordering $\prec_{x,a}$. For each $(N_i, W_i, G_i, A_i, B_i) \in G$, where $G_i = \{g_1, \ldots, g_s\}$, let $\{\mathbf{s}_1, \ldots, \mathbf{s}_p\}$ be a basis for* $\operatorname{syz}(g_1, \ldots, g_s)$*. Define $\mathbf{r}_1, \ldots, \mathbf{r}_t$ as the columns of the matrix $I_t - \sigma(A_i)\sigma(B_i)$. Then, the following set constitutes a generating set for* $\operatorname{syz}(\sigma(f_1), \ldots, \sigma(f_t))$ *that satisfies $(N_i, W_i)$:*

$$\{A_i\mathbf{s}_1, \ldots, A_i\mathbf{s}_p, \mathbf{r}_1, \ldots, \mathbf{r}_t\}.$$

*Proof.* It is evident that $\langle A_i\mathbf{s}_1, \ldots, A_i\mathbf{s}_p, \mathbf{r}_1, \ldots, \mathbf{r}_t \rangle$ is a submodule of $\operatorname{syz}(f_1, \ldots, f_t)$. Thus, to prove the theorem, it suffices to show that every syzygy on $F$ can be expressed as an $R$-linear combination of the elements $\{A_i\mathbf{s}_1, \ldots, A_i\mathbf{s}_p, \mathbf{r}_1, \ldots, \mathbf{r}_t\}$.

Let $\mathbf{p}$ be any element of $\operatorname{syz}(\sigma(f_1), \ldots, \sigma(f_t))$. By the definition of a completed Gröbner system, we have $\sigma(F) = \sigma(G_i)\sigma(B_i)$ for $(N_i, W_i)$. Multiplying on the right by the column vector $\mathbf{p} \in \operatorname{syz}(\sigma(F))$, we obtain the equations: $\sigma(F)\mathbf{p} = \sigma(G_i)\sigma(B_i)\mathbf{p}$. Since $\sigma(F)\mathbf{p} = 0$, the relation $\sigma(G_i)\sigma(B_i)\mathbf{p} = 0$ holds. Noting that $\sigma(B_i)\mathbf{p} = \sigma(B_i\mathbf{p})$ (by linearity of $\sigma$), we conclude

$$\sigma(G_i)\sigma(B_i\mathbf{p}) = 0.$$

Therefore, we have $\sigma(B_i\mathbf{p}) \in \operatorname{syz}(\sigma(G_i)) = \operatorname{syz}(\sigma(g_1), \ldots, \sigma(g_s))$.

Since $\{\mathbf{s}_1, \ldots, \mathbf{s}_p\}$ is a basis for $\operatorname{syz}(g_1, \ldots, g_s)$, there exist coefficients $q_i \in S$ such that

$$\sigma(B_i\mathbf{p}) = \sum_i \sigma(q_i)\mathbf{s}_i.$$

Multiplying both sides by $\sigma(A_i)$ on the left gives us

$$\sigma(A_i)\sigma(B_i)\mathbf{p} = \sum_i \sigma(q_i)\sigma(A_i)\mathbf{s}_i.$$

Furthermore, $\mathbf{p}$ can be rewritten as follows:

$$\begin{aligned}\mathbf{p} &= \Big((I_t - \sigma(A_i)\sigma(B_i)) + \sigma(A_i)\sigma(B_i)\Big)\mathbf{p} \\ &= (I_t - \sigma(A_i)\sigma(B_i))\,\mathbf{p} + \sigma(A_i)\sigma(B_i)\mathbf{p} \\ &= (I_t - \sigma(A_i)\sigma(B_i))\,\mathbf{p} + \sum_i \sigma(q_i)\sigma(A_i)\mathbf{s}_i.\end{aligned}$$

The first term on the right-hand side is an $R$-linear combination of the columns $\mathbf{r}_1, \ldots, \mathbf{r}_t$ from $(I_t - \sigma(A_i)\sigma(B_i))$, implying that $\mathbf{p}$ can be expressed as an $R$-linear combination of the elements of $\{A_i\mathbf{s}_1, \ldots, A_i\mathbf{s}_p, \mathbf{r}_1, \ldots, \mathbf{r}_t\}$. Consequently, the set $\{A_i\mathbf{s}_1, \ldots, A_i\mathbf{s}_p, \mathbf{r}_1, \ldots, \mathbf{r}_t\}$ forms a generating set for $\operatorname{syz}(\sigma(f_1), \ldots, \sigma(f_t))$ under the parameters $(N_i, W_i)$. □

We now present the efficient CMGS algorithm for computing a comprehensive minimal generating system based on the aforementioned theorem and Corollary 2.3. This algorithm inputs a parametric homogeneous polynomial ideal w.r.t. the variables $\mathbf{x}$ and outputs its comprehensive minimal generating system. Initially, the variable LIST is an empty set, ultimately becoming the comprehensive minimal generating system. The REMOVE function also takes a set of polynomials $F \subset S$. It returns $F_i \subset F$, obtained by removing the $i^{\text{th}}$ element of $F$ if the $i^{\text{th}}$ row of the syzygy matrix of $F$ contains a non-zero polynomial $q \in \mathbb{K}[\mathbf{a}]$. Furthermore, at the start of the algorithm, the Homogenize function uses a new variable $h$ to transform a homogeneous polynomial ideal w.r.t. variables into a homogeneous polynomial ideal concerning both variables and parameters. For instance, $[ax^2 - y^2, by^3 - axy^2]$ is homogeneous in $x, y$, while $[ax^2 - hy^2, by^3 - axy^2]$ is homogeneous in $a, b, h, x, y$.

**Algorithm 4** CMGS (Comprehensive Minimal Generating System)

**Require:** $F = \langle f_1, \dots, f_m \rangle \subset S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$ is a homogeneous polynomial ideal w.r.t. variables $\mathbf{x}$
**Ensure:** A comprehensive minimal generating system of $F$
  $I :=$ Homogenize$(F, h)$
  LIST := { }
  $G := \{(N_i, W_i, G_i, A_i, B_i)\}_{i=1}^{\ell}$ a Completed Gröbner system of $I$ with respect to $\prec_{\mathbf{x},\mathbf{a}}$
  **for** $(N_i, W_i, G_i, A_i, B_i) \in G$ **do**
    $A :=$ Transpose $(A_i)$
    $B :=$ Transpose $(B_i)$
    $r :=$ row dimension of $A$
    $IM :=$ Identity matrix $I_{r\times r}$
    $IAB := IM - AB$
    Dehomogenize all polynomials by setting $h = 1$ in all computations involving $h$
    S := syzygy matrix of $G_i$
    $AS :=$ Multiply $A$ and $S$
    $AS := [AS \mid IAB]$
    $AS :=$ delete all zero columns from the matrix $AS$
    $F_i := G_i$
    **while** there exists $0 \neq q \in \mathbb{K}[\mathbf{a}]$ in $j^{\text{th}}$ row of $AS$ **do**
      $F_i :=$REMOVE$(F_i)$
      $AS :=$ SYZYGY-SCHREYER$(F_i)$
    **end while**
    LIST := LIST $\bigcup\{(N_i, W_i, F_i)\}$
  **end for**
  **Return** (LIST)

**Theorem 4.3.** *The* CMGS *algorithm terminates in a finite number of steps and accurately computes a comprehensive minimal generating system.*

*Proof.* The termination of the CMGS algorithm is guaranteed by the termination of the completed Gröbner system computations, while its correctness is established by theorem 4.2. Let $(N_i, W_i, G_i, A_i, B_i)$ denote the $i^{th}$ branch of the completed Gröbner system of $F$. In the $i^{th}$ iteration of the **for**-loop, since all leading coefficients in $G_i$ are non-zero under $(N_i, W_i)$, we can apply the SYZYGY-SCHREYER algorithm to compute the syzygy matrix of $G_i$. Consequently, based on corollary 2.3 and theorem 4.2, we derive a minimal generating set of $\sigma(F)$ that meets the parametric conditions $(N_i, W_i)$. The inner **while** loop iteratively checks whether redundant generators can be removed from the current set $F_i$. In each iteration, if the syzygy matrix of $F_i$ contains a non-zero polynomial $q \in \mathbb{K}[\mathbf{a}]$ in its $j$th row, then the $j$th generator is redundant (by corollary 2.3) and is removed. Otherwise, if no such row exists, the loop terminates because $F_i$ is already minimal. Since each removal reduces the size of $F_i$ by at least one element, and the polynomial ring $\mathbb{K}[\mathbf{a}, \mathbf{x}]$ is Noetherian, the strictly descending chain of ideals

$$\langle F_i^{(0)} \rangle \supset \langle F_i^{(1)} \rangle \supset \langle F_i^{(2)} \rangle \supset \cdots$$

(if removals occur) must stabilize after finitely many steps. Hence, the while-loop always terminates. It is worth noting that in the while loop, we invoke SYZYGY-SCHREYER$(F_i)$ under the key observation that, under the given parametric constraints (all $a_j \neq 0$ for parameters $\mathbf{a} = (a_1, \ldots, a_m)$), we localize $\mathbb{K}[\mathbf{a}]$ yielding the field $\mathbb{K}(\mathbf{a})$ of rational functions in $\mathbf{a}$. This allows us to treat parametric coefficients as invertible scalars during syzygy computation. Furthermore, while $G_i$ is indeed a Gröbner basis, recomputing syzygies via Syzygy-Schreyer($F_i$) ensures correctness. Ultimately, for each branch $(N_i, W_i, G_i, A_i, B_i)$ the algorithm produces a set $F_i$ that is a minimal generating set of $\sigma(F)$ for every specialization $\sigma$ satisfying $(N_i, W_i)$. Therefore the union of all $(N_i, W_i, F_i)$ forms a comprehensive minimal generating system of $F$. □

*Complexity remarks*

The CMGS algorithm relies on two computationally intensive steps: the construction of a completed Gröbner system and the syzygy computations over each branch. The first step, in the worst case, can be doubly exponential in the number of variables, a known complexity for comprehensive Gröbner systems. The second step, for each branch, involves computing a syzygy matrix of a Gröbner basis, which is polynomial in the size of the basis and the degrees of the generators. In practice, the number of branches is often moderate, and the algorithm benefits from the fact that the syzygy matrix needs only to be computed once per branch. Compared to the non-parametric case, the overhead is proportional to the number of branches, but this is necessary to obtain a uniform description of the minimal generators over the whole parameter space. Table 1 highlights the key differences between MGSystem and CMGS regarding input, output, and computational cost. CMGS handles arbitrary homogeneous parametric ideals, offering greater generality than MGSystem, which is limited to ideals that are already Gröbner bases. However, this broader applicability comes at the cost of increased computational complexity, as CMGS employs completed Gröbner systems and transformation matrices.

Table 1: Comparison of the MGSystem and CMGS algorithms

| **Aspect** | MGSystem | **CMGS** |
|---|---|---|
| Input ideal | Homogeneous parametric ideal that is already a Gröbner basis w.r.t. $\prec_{\mathbf{x}}$ | Arbitrary homogeneous parametric ideal |
| Parameter handling | Uses known Gröbner system | Uses a *completed* Gröbner system (includes transformation matrices) |
| Output | Minimal generating system for the given Gröbner basis | Comprehensive minimal generating system (uniform over parameter regions) |
| Generality | Special case of CMGS when input is a Gröbner basis w.r.t. variables | Generalization of MGSystem to arbitrary homogeneous ideals |
| Computational cost | Lower (no need for completed system) | Higher (requires completed system and more syzygy computations) |

## 5. Computation of a Straightforward Example

This section demonstrates the algorithm steps with the following simple example.

**Example 5.1.** Consider the parametric polynomial ideal $I \subset \mathbb{K}[a,b,c][x,y,z]$, which is homogeneous in the variables $x,y,z$:

$$F = [ayz + bxy, bx^2 - cyz - yz, -byz^2 - yz^2 + y^2z, -cyz^2 + x^2y + xyz].$$

Using our `Maple` implementation of the CMGS algorithm, we aim to compute a minimal generating set for $F$. We first homogenize $F$ with a new variable $h$, resulting in:

$$I = [ayz + bxy, bx^2 - cyz - hyz, -byz^2 - hyz^2 + y^2z, -cyz^2 + x^2y + xyz].$$

Next, we compute a completed Gröbner system of $I$ with respect to any compatible monomial ordering $\prec_{\mathbf{x},\mathbf{a}}$, denoted as $G = \{(N_i, W_i, G_i, A_i, B_i)\}_{i=1}^{6}$.

– – – – – – – – – – – – – – – – – – Branch 1 – – – – – – – – – – – – – – – – – –

$$[h-1], [b, a-b-c-1], \left[ayz^2 - byz^2 - cyz^2 - yz^2, -byz^2 + y^2z - yz^2, ayz + bxy, bx^2 - cyz - yz\right]$$

$$\begin{bmatrix} z & y & c+1 & -b \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{h}{a-b-c-1} + (a-b-c-1)^{-1} & 1 & 0 & 0 \\ -\frac{c}{a-b-c-1} + \frac{a^2}{b^2(a-b-c-1)} - \frac{a}{b(a-b-c-1)} & 0 & -\frac{za}{b^2} + \frac{x}{b} + \frac{z}{b} & 0 \end{bmatrix}$$

– – – – – – – – – – – – – – – – – – Branch 2 – – – – – – – – – – – – – – – – – –

$$[b, h-1], [a], \left[yza, x^2y + xyz + yz^2\right]$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -z & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -\frac{c}{a} - \frac{h}{a} & 0 \\ -\frac{bz}{a} - \frac{hz}{a} + \frac{y}{a} & 0 \\ -\frac{cz}{a} - \frac{z}{a} & 1 \end{bmatrix}$$

– – – – – – – – – – – – – – – – – – Branch 3 – – – – – – – – – – – – – – – – – –

$$[a-b-c-1, h-1], \left[b, b^2c - bc - c^2 - b - 2c - 1\right],$$

$$\left[b^2cyz^2 - bcyz^2 - c^2yz^2 - byz^2 - 2cyz^2 - yz^2, -byz^2 + y^2z - yz^2, bxy + byz + cyz + yz, bx^2 - cyz - yz\right]$$

$$\begin{bmatrix} bx + bz + cz + z & ay + cy + y & ac + c^2 + a + 2c + 1 & -ab - b^2 - bc - b \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1-h}{b^2c - bc - c^2 - b - 2c - 1} & 1 & 0 & 0 \\ -b^{-2} & 0 & \frac{x}{b} - \frac{cz}{b^2} - \frac{z}{b^2} & 0 \end{bmatrix}$$

– – – – – – – – – – – – – – – – – – Branch 4 – – – – – – – – – – – – – – – – – –

$$[b, a, h-1], [c+1], \left[cyz + yzx^2y + xyz + yz^2\right]$$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & -z & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{a}{c+1} & 0 \\ -\frac{c}{c+1} - \frac{h}{c+1} & 0 \\ -\frac{bz}{c+1} - \frac{hz}{c+1} + \frac{y}{c+1} & 0 \\ -z & 1 \end{bmatrix}$$

– – – – – – – – – – – – – – – – – – Branch 5 – – – – – – – – – – – – – – – – – –

$$\left[b^2c - bc - c^2 - b - 2c - 1, a - b - c - 1, h - 1\right], [b], \left[-byz^2 + y^2z - yz, bxy + byz + cyz + yzbx^2 - cyz - yz\right]$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \frac{x}{b} - \frac{cz}{b^2} - \frac{z}{b^2} & 0 \end{bmatrix}$$

– – – – – – – – – – – – – – – – – – Branch 6 – – – – – – – – – – – – – – – – – –

$$[c+1, b, a, h-1].[1], \left[y^2z - yz^2, x^2y + xyz + yz^2\right]$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -z & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The calculation follows six branches as per the algorithm's structure; however, we will focus on the second branch, with others calculated similarly. From branch 2, we obtain the following matrices based on the CMGS algorithm's trace:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -z \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -\frac{c}{a} - \frac{h}{a} & -\frac{zb}{a} - \frac{hz}{a} + \frac{y}{a} & -\frac{zc}{a} - \frac{z}{a} \\ 0 & 0 & 0 & 1 \end{bmatrix}, IM = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$IAB = IM - AB = \begin{bmatrix} 0 & \frac{c}{a} + \frac{h}{a} & \frac{zb}{a} + \frac{hz}{a} - \frac{y}{a} & \frac{zc}{a} + \frac{z}{a} \\ 0 & 1 & 0 & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Next, we compute a syzygy matrix $S$ of $G_2$, followed by $AS = A \times S$, and then we combine $AS$ and $IAB$ to obtain a generating set for $\mathrm{syz}(\sigma(F))$ satisfying $(N_2, W_2)$:

$$S = \begin{bmatrix} \frac{zx}{a} + \frac{z^2}{a} + \frac{x^2}{a} \\ -z \end{bmatrix}, A \times S = \begin{bmatrix} \frac{zx}{a} + \frac{z^2}{a} + \frac{x^2}{a} \\ z^2 \\ 0 \\ -z \end{bmatrix}, AS = [AS \mid IAB] = \begin{bmatrix} \frac{x^2+zx+z^2}{a} & \frac{c+1}{a} & -\frac{y-z}{a} & \frac{z(c+1)}{a} \\ z^2 & 1 & 0 & z \\ 0 & 0 & 1 & 0 \\ -z & 0 & 0 & 0 \end{bmatrix}$$

The first three rows of $AS$ contain non-zero entries in $\mathbb{K}[\mathbf{a}]$, leading to the removal of the first polynomial from $F$. Thus, we compute the NormalForm of the remaining elements on $N_2 = [b, h-1]$, yielding:

$$F_2 = [-cyz - yz, y^2z - yz^2, -cyz^2 + (x^2 + xz)y].$$

Since all coefficients in $F_2$ are non-zero according to $(N_2, W_2)$, we apply the Syzygy-Schreyer algorithm to determine if $F_2$ can be minimized further. The resulting syzygy matrix is:

$$AS = \begin{bmatrix} \frac{cz^2-x^2-zx}{c+1} & 0 & \frac{y-z}{c+1} & 0 \\ 0 & 0 & 1 & 0 \\ -z & 0 & 0 & 0 \end{bmatrix}.$$

The second row of $AS$ has a non-zero entry, necessitating the removal of the second polynomial from $F_2$, leading to an updated form:

$$F_2 = [-cyz - yz, -cyz^2 + (x^2 + xz)y].$$

Repeating the Syzygy-Schreyer algorithm gives the following syzygy matrix for $F_2$:

$$AS = \begin{bmatrix} \frac{cz^2-x^2-zx}{c+1} & 0 & 0 \\ -z & 0 & 0 \end{bmatrix}.$$

Since there are no non-zero elements of $\mathbb{K}[\mathbf{a}]$ in $AS$, we conclude that a minimal generating set for the original $F$ under the parametric conditions $(N_2, W_2) = ([b, h-1], [a])$ is:

$$F_2 = [-cyz - yz, -cyz^2 + (x^2 + xz)y].$$

Applying the same procedure to the other five ignored branches of the completed Gröbner system $G$ yields a comprehensive minimal generating system for input $F$. Since the variable $h$ does not appear in the original polynomial ideal $F$, we can eliminate $h-1$ from all null sets $N_i$.

$$\begin{cases} ([\,],[b,a-b-c-1], & [ayz+bxy,bx^2-cyz-yz,-cyz^2+(x^2+xz)y]) \\ ([b],[a], & [-cyz-yz,-cyz^2+(x^2+xz)y]) \\ ([a-b-c-1],[b,b^2c-bc-c^2-b-2c-1], & [(x+z)yb+cyz+yz,bx^2-cyz-yz,-cyz^2+(x^2+xz)y]) \\ ([b,a],[c+1], & [-cyz-yz,-cyz^2+(x^2+xz)y]) \\ ([b^2c-bc-c^2-b-2c-1,a-b-c-1],[b], & [(x+z)yb+cyz+yz,bx^2-cyz-yz,-cyz^2+(x^2+xz)y]) \\ ([c+1,b,a],[1], & [y^2z-yz^2,x^2y+xyz+yz^2]). \end{cases}$$

For instance, if $a=\frac{3}{2}, b=-\frac{1}{2}$ and $c=1$ then the third branch corresponds to these values of parameters (these values hold $(N_3,W_3)$). Therefore,

$$F_3|_{a=\frac{3}{2},b=-\frac{1}{2},c=1}=[-1/2(x+z)y+2yz,-2yz-(1/2)x^2,-yz^2+(x^2+xz)y]$$

will be a minimal generating set for the ideal

$$F\,|_{a=\frac{3}{2},b=-\frac{1}{2},c=1}=[-(1/2)xy+(3/2)yz,-2yz-(1/2)x^2,y^2z-(1/2)yz^2,x^2y+xyz-yz^2].$$

This means that for any $1\le i\le 6$ and for each specialization $\sigma$ satisfying $(N_i,W_i)$, $\sigma(F_i)$ constitutes a minimal generating set for $\sigma(F)$. Furthermore, considering all branches, under the parametric constraint $(N_i,W_i)$, the minimal generating set $F_i$ contains fewer than 4 generators, the number of generators of $F$, which indicates a substantial reduction in the size of the generating set for $F$.

## 6. Applications and Future Works

By computing the minimal generating set for parametric homogeneous polynomial ideals, we can analyze their minimal free resolution, allowing us to derive numerical invariants that reflect the *depth, regularity, Betti numbers, and graded Betti numbers* of these ideals viewed as a $\mathbb{K}[\mathbf{a}][\mathbf{x}]$ module. This approach aims to clarify the structure of these polynomial ideals, enabling further research into the relationships between their numerical invariants and algebraic properties, ultimately deepening our understanding of these mathematical objects.

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