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Solvability of Optimization Problems for Nonlinear Integral Equations with Singularities at the Boundary of Integration

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Abstract

This article investigates an optimization problem for objects described by nonlinear integral equations with a singularity at the boundary of integration. Within the framework of the theorem, possibly weaker conditions were imposed, which allow for the verification of the kernel of the integrand and the nonlinear part. The results obtained enable a direct approximation of the solution to the singular nonlinear integral equation and, among the set of solutions, the identification of a control that ensures the minimal value of the function.

Keywords: Optimization problem, nonlinear equations, boundary singularity, boundedness, continuity, uniform continuity, Holder's inequality, problem solvability, contraction mapping method.

2010 Mathematics Subject Classification: 45G10, 47H10.

1. Introduction

The fundamental principles of optimal control theory were laid down in the 1960s by L.S. Pontryagin and his students [16], and since then the theory has been actively developed and applied to various practical problems in science, engineering, and computational technology [7, 10, 15]. Many natural and engineered processes are goal-oriented, requiring the effective management of parameters that govern their dynamics. Mathematical models of such processes are commonly described by differential, integral, and integro-differential equations, including inequalities and inclusions [2, 3].

Integral equations, particularly those with singularities at the boundary of integration, appear naturally in boundary value problems, inverse problems, and systems with memory effects. Analytical methods for studying such equations include fixed-point theorems, regularization techniques, kernel decomposition, and successive approximations, which help establish existence and uniqueness results for nonlinear and singular equations [1, 5].

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For practical computations, numerical methods have been extensively developed. Quadrature methods, collocation and projection techniques, B-spline approximations, spectral collocation using Jacobi or Chebyshev polynomials, and fractional spline methods have been successfully applied to nonlinear Volterra-Fredholm and Fredholm integral equations with weak or strong singularities [4, 6, 8, 9]. These methods provide accurate approximations and stable convergence, even in the presence of singular kernels or ill-posed problems.

This article focuses on the optimization of systems described by nonlinear integral equations with singularities at the boundary of integration. In particular, it explores both analytical solvability frameworks and contemporary numerical approaches, providing strategies for handling singular behavior at integration boundaries and achieving reliable computational solutions.

2. Problem Statement

The following problem is considered:

$$J\left(x\left(t\right),u\left(t\right)\right)=\|x(t)\|_{L_p[a,b]}+\|u(t)\|_{L_p[a,b]}\to\inf, \tag{2.1}$$

$$x(t)+\int_a^b\frac{K(t,\tau)}{\tau-a}f\left(\tau,x(\tau),u(\tau)\right)d\tau=\phi(t). \tag{2.2}$$

Here: $x(t)\in L_p[a,b]$ is the unknown state function of the process, $u(t)\in L_p[a,b]$ is the control function, $K(t,\tau)$ is a known bounded function, i.e., $|K(t,\tau)|\le k=\text{const}$.

The problem (2.1), (2.2) can be rewritten in the following form:

$$J\left(x(t),u(t)\right)=\|x(t)\|_{L_p[a,b]}+\|u(t)\|_{L_p[a,b]}\to\inf, \tag{2.3}$$

$$x(t)+\int_a^{a+\varepsilon}\frac{K(t,\tau)}{\tau-a}f\left(\tau,x(\tau),u(\tau)\right)d\tau+\int_{a+\varepsilon}^b\frac{K(t,\tau)}{\tau-a}f\left(\tau,x(\tau),u(\tau)\right)d\tau=\phi(t), \tag{2.4}$$

or equivalently,

$$J\left(x(t),u(t)\right)=\|x(t)\|_{L_p[a,b]}+\|u(t)\|_{L_p[a,b]}+\frac{1}{\varepsilon}\left|\int_a^{a+\varepsilon}\frac{K(t,\tau)}{\tau-a}f\left(\tau,x(\tau),u(\tau)\right)d\tau\right|\to\inf, \tag{2.5}$$

$$x(t)+\int_{a+\varepsilon}^b\frac{K(t,\tau)}{\tau-a}f\left(\tau,x(\tau),u(\tau)\right)d\tau=\phi(t), \tag{2.6}$$

where the term $\frac{1}{\varepsilon}\left|\int_a^{a+\varepsilon}\frac{K(t,\tau)}{\tau-a}f\left(\tau,x(\tau),u(\tau)\right)d\tau\right|$ is called a penalty function [10, 15].

If we assume that

$$\left|\int_a^{a+\varepsilon}\frac{K(t,\tau)}{\tau-a}f\left(\tau,x(\tau),u(\tau)\right)d\tau\right|\le\varepsilon^2, \tag{2.7}$$

then the problem (2.1), (2.2) becomes equivalent to problem (2.5), (2.6), where $\varepsilon>0$ is a small number and $\varepsilon\to0$.

3. Main Results

For condition (2.7) to hold, the function $f(\tau,x(\tau),u(\tau))$ must be monotonically decreasing in a small ε-neighborhood of the boundary point a of the integration interval $[a,b]$.

The following theorem holds.

Theorem 3.1. *Let the function $f(\tau, \xi_0, \xi_1)$ satisfy the following conditions:*

1) For all $\tau \in [a, b] \subset \mathbb{R}$ and for all $(\xi_0, \xi_1) = \xi \in \Omega \subset \mathbb{R}^2$, the inequality

$$|f(\tau, \xi_0, \xi_1)| \leq g(\tau) + C\left(\sum_{i=0}^{1} |\xi_i|\right)^{p-1} < \infty$$

holds, where $g(\tau) \in L_q[a, b]$, $\frac{1}{p} + \frac{1}{q} = 1$, with natural numbers $p, q > 1$, and $C = const$;

2) The function is uniformly continuous with respect to the variables $(\xi_0, \xi_1) = \xi \in \Omega \subset \mathbb{R}^2$;

3) The function is monotonically decreasing in a small ε-neighborhood of the boundary point a of the integration domain $[a, b]$, furthermore, for all $\tau_1, \tau_2 \in [a, a + \varepsilon]$ satisfying $\tau_1 > \tau_2$, the following holds

$$|f(\tau_1, \xi_0, \xi_1) - f(\tau_2, \xi_0, \xi_1)| < \varepsilon^2, \quad \forall (\xi_0, \xi_1) = \xi \in \Omega \subset \mathbb{R}^2;$$

4) For all $\tau_1, \tau_2 \in [a + \varepsilon, b]$, with fixed variable ξ_1, and for all $\xi_0^{(1)}, \xi_0^{(2)} \in \mathbb{R}$, the function satisfies the condition

$$|f(\tau_1, \xi_0^{(1)}, \xi_1) - f(\tau_2, \xi_0^{(2)}, \xi_1)| < \alpha|\xi_0^{(1)} - \xi_0^{(2)}|, \quad \text{where } 0 < \alpha < 1.$$

Then,

a) The mapping $F(x, u) = f(\tau, x(\tau), u(\tau)) : L_p[a, b] \times L_p[a, b] \to L_q[a, b]$;

b) It possesses the properties of continuity and boundedness with respect to the functions $x(\tau)$ and $u(\tau)$;

c) The equation (2.6) has a solution for every fixed $u(\tau)$, and this solution is unique;

d) The problem (2.5), (2.6) is solvable, and its solution is equivalent to the solution of the original problem (2.1), (2.2) under the conditions of the theorem.

Proof. Part *a)* of the theorem is proved as follows.

If condition *4)* is satisfied, then

$$\begin{aligned}
\int_a^b [f(\tau, x(\tau), u(\tau))]^q d\tau &\leq \left|\int_a^b [f(\tau, x(\tau), u(\tau))]^q d\tau\right| \leq \int_a^b |f(\tau, x(\tau), u(\tau))|^q d\tau \leq \\
&\leq \int_a^b |g(\tau)|^q d\tau + C\int_a^b |x(\tau)|^{(p-1)q} d\tau + C\int_a^b |u(\tau)|^{(p-1)q} d\tau = \\
&= \left(\int_a^b |g(\tau)|^q d\tau\right) + C\left(\int_a^b |x(\tau)|^p d\tau + \int_a^b |u(\tau)|^p d\tau\right) = \text{const} < \infty,
\end{aligned} \tag{3.1}$$

where we used the identity $\frac{1}{p} + \frac{1}{q} = 1$, or equivalently, $(p - 1)q = p$.

From inequality (3.1), it follows that the following relations hold: $f(\tau, x(\tau), u(\tau)) \in L_q[a, b]$ and in operator form: $F(x, u) = f(\tau, x(\tau), u(\tau)) : L_p[a, b] \times L_p[a, b] \to L_q[a, b]$.

Now we proceed to the proof of part *b)* of the statement, i.e., the **continuity** of this mapping.

Thus, under condition *2)*, i.e., from the uniform continuity of the function $f(\tau, \xi_0, \xi_1)$, it follows that for all $\xi^{(1)}, \xi^{(2)} \in [a, b] \times [a, b] = \Omega \subset \mathbb{R}^2$, satisfying the conditions $|\xi_0^{(1)} - \xi_0^{(2)}| < \delta_0$ and $|\xi_1^{(1)} - \xi_1^{(2)}| < \delta_1$, with $\delta = \min\{\delta_0, \delta_1\}$, there exists a sufficiently small number $\varepsilon_{(\delta)} > 0$ such that the following inequality holds: $|f(\tau, \xi_0^{(1)}, \xi_1^{(1)}) - f(\tau, \xi_0^{(2)}, \xi_1^{(2)})| < \varepsilon$. From this property of uniform continuity, the continuity of the operator $F(\cdot, \cdot) : L_p[a, b] \times L_p[a, b] \to L_q[a, b]$ follows: for every small number $\varepsilon > 0$, there exists a number $\delta > 0$ such that for all functions $x_1(\tau), x_2(\tau), u_1(\tau), u_2(\tau) \in L_p[a, b]$, satisfying the conditions $\|x_1(\tau) - x_2(\tau)\|_{L_p[a,b]} < \delta_1$ and $\|u_1(\tau) - u_2(\tau)\|_{L_p[a,b]} < \delta_2$, the following inequality holds:

$$\|F(x_1, u_1) - F(x_2, u_2)\|_{L_q[a,b]} = \|f(\tau, x_1(\tau), u_1(\tau)) - f(\tau, x_2(\tau), u_2(\tau))\|_{L_q[a,b]} < \varepsilon, \tag{3.2}$$

for all $x_1, x_2, u_1, u_2 \in L_p[a, b]$ such that $\|x_1(\tau) - x_2(\tau)\|_{L_p[a,b]} < \delta$ and $\|u_1(\tau) - u_2(\tau)\|_{L_p[a,b]} < \delta$, where $\delta = \min\{\delta_0, \delta_1\}$. This proves the continuity of the operator.

The **boundedness** of this operator follows directly from relation (3.2). It is noted that the mapping $F(\cdot,\cdot): L_p[a,b]\times L_p[a,b]\to L_q[a,b]$ is called bounded if it maps a bounded set in the space $L_p[a,b]\times L_p[a,b]$ to a bounded set in the space $L_q[a,b]$.

Since $f(\tau,x(\tau),u(\tau))\in L_q[a,b]$, then for any bounded functions $x_1(\tau),x_2(\tau),u_1(\tau),\ u_2(\tau)\in L_p[a,b]$, the difference $f(\tau,x_1(\tau),u_1(\tau))-f(\tau,x_2(\tau),u_2(\tau))\in L_q[a,b]$, and the relation

$$\int_a^b |f(\tau,x_1(\tau),u_1(\tau))-f(\tau,x_1(\tau),u_2(\tau))|^q\,d\tau\le$$

$$\le\int_a^b |g(\tau)|^q d\tau + C\left(\int_a^b |x_1(\tau)-x_2(\tau)|^p d\tau+\int_a^b |u_1(\tau)-u_2(\tau)|^p d\tau\right)=\text{const}<\infty$$

holds. Thus, part *b)* of the theorem is completely proven.

Now we proceed to the proof of part *c)* of the theorem - the **existence and uniqueness** of the solution.

If for some fixed control function $u=u(t)\in U\subset L_r[a,b]$, where U is a bounded set, and there is an initial condition $x(t_0)=x_0=x_0(t)$, then from equality (2.4) we compute

$$x_1(t)=\phi(t)-\int_a^{a+\varepsilon}\frac{K(t,\tau)}{\tau-a}f(\tau,x(\tau_0),u(\tau))\,d\tau-\int_{a+\varepsilon}^b\frac{K(t,\tau)}{\tau-a}f(\tau,x(\tau_0),u(\tau))\,d\tau,$$

or, according to equality (2.6), we compute

$$x_1(t)=\phi(t)-\int_{a+\varepsilon}^b\frac{K(t,\tau)}{\tau-a}f(\tau,x(\tau_0),u(\tau))\,d\tau.$$

Subsequent states are computed using recurrence relations, i.e.,

$$x_{n+1}(t)=\phi(t)-\int_a^{a+\varepsilon}\frac{K(t,\tau)}{\tau-a}f(\tau,x_n(\tau),u(\tau))\,d\tau-\int_{a+\varepsilon}^b\frac{K(t,\tau)}{\tau-a}f(\tau,x_n(\tau),u(\tau))\,d\tau, \tag{3.3}$$

or

$$x_{n+1}(t)=\phi(t)-\int_{a+\varepsilon}^b\frac{K(t,\tau)}{\tau-a}f(\tau,x_n(\tau),u(\tau))\,d\tau,\quad n=0,1,2,\ldots. \tag{3.4}$$

Hence, for all $t,\tau\in[a,b]$, the following difference holds

$$|x_{n+1}(t)-x_{m+1}(t)|=$$

$$=\left|\phi(t)-\int_a^{a+\varepsilon}\frac{K(t,\tau)}{\tau-a}f(\tau,x_n(\tau),u(\tau))d\tau-\int_{a+\varepsilon}^b\frac{K(t,\tau)}{\tau-a}f(\tau,x_n(\tau),u(\tau))d\tau-\right.$$

$$\left.-\phi(t)+\int_a^{a+\varepsilon}\frac{K(t,\tau)}{\tau-a}f(\tau,x_m(\tau),u(\tau))\,d\tau+\int_{a+\varepsilon}^b\frac{K(t,\tau)}{\tau-a}f(\tau,x_m(\tau),u(\tau))\,d\tau\right|=$$

$$=\left|\int_a^{a+\varepsilon}\frac{K(t,\tau)}{\tau-a}\left(f(\tau,x_n(\tau),u(\tau))-f(\tau,x_m(\tau),u(\tau))\right)\,d\tau\right|+$$

$$+\left|\int_{a+\varepsilon}^b\frac{K(t,\tau)}{\tau-a}\left(f(\tau,x_n(\tau),u(\tau))-f(\tau,x_m(\tau),u(\tau))\right)\,d\tau\right|\le$$

$$\le\int\limits_a^{a+\varepsilon}\left|\frac{K\left(t,\tau\right)}{\tau-a}\left[f\left(\tau,x_n\left(\tau\right),u\left(\tau\right)\right)-f\left(\tau,x_m\left(\tau\right),u\left(\tau\right)\right)\right]\right|d\tau+$$

$$+\int\limits_{a+\varepsilon}^b\left|\frac{K\left(t,\tau\right)}{\tau-a}\left[f\left(\tau,x_n\left(\tau\right),u\left(\tau\right)\right)-f\left(\tau,x_m\left(\tau\right),u\left(\tau\right)\right)\right]\right|d\tau.$$

Alternatively, for all $t, \tau \in [a+\varepsilon, b]$, we have

$$|x_{n+1}(t) - x_{m+1}(t)| = \int_{a+\varepsilon}^{b} \left| \frac{K(t,\tau)}{\tau - a} \left[f(\tau, x_n(\tau), u(\tau)) - f(\tau, x_m(\tau), u(\tau)) \right] \right| d\tau.$$

Using condition *3)* of the theorem, there exists a neighborhood ε-neighborhood of the boundary point a, such that for all $\tau \in [a, a+\varepsilon]$ the following inequality holds

$$\int_{a}^{a+\varepsilon} \left| \frac{K(t,\tau)}{\tau - a} \left[f(\tau, x_n(\tau), u(\tau)) - f(\tau, x_m(\tau), u(\tau)) \right] \right| d\tau \leq$$
$$\leq \int_{a}^{a+\varepsilon} \left| \frac{K(t,\tau)}{\tau - a} \right| \cdot \left| f(\tau, x_n(\tau), u(\tau)) - f(\tau, x_m(\tau), u(\tau)) \right| d\tau \leq \tag{3.5}$$
$$\leq k \int_{a}^{a+\varepsilon} \frac{1}{\tau - a} \cdot \varepsilon^2 d\tau \leq k\gamma \int_{a}^{a+\varepsilon} \frac{1}{\varepsilon} \cdot \varepsilon^2 d\tau = k\gamma\varepsilon(a + \varepsilon - a) = k\gamma\varepsilon^2,$$

since $|K(t,\tau)| \leq k = \text{const.}$, and the value $(\tau - a)$ depends linearly on ε, i.e., $(\tau - a) = \frac{\varepsilon}{\gamma}$, where $\gamma \geq 1$ is a positive number.

Using Holder's integral inequality [11], for all $t, \tau \in [a+\varepsilon, b]$, the following holds:

$$\int_{a+\varepsilon}^{b} \left| \frac{K(t,\tau)}{\tau - a} \left[f(\tau, x_n(\tau), u(\tau)) - f(\tau, x_m(\tau), u(\tau)) \right] \right| d\tau \leq$$
$$\leq \left(\int_{a+\varepsilon}^{b} \left| \frac{K(t,\tau)}{\tau - a} \right|^p d\tau \right)^{\frac{1}{p}} \left(\int_{a+\varepsilon}^{b} \left| f(\tau, x_n(\tau), u(\tau)) - f(\tau, x_m(\tau), u(\tau)) \right|^q d\tau \right)^{\frac{1}{q}} \leq$$
$$\leq \left(\left(\frac{k}{\varepsilon} \right)^p (b - a - \varepsilon) \right)^{\frac{1}{p}} \left(\int_{a+\varepsilon}^{b} \left| f(\tau, x_n(\tau), u(\tau)) - f(\tau, x_m(\tau), u(\tau)) \right|^q d\tau \right)^{\frac{1}{q}} =$$
$$= \frac{k}{\varepsilon} (b - a - \varepsilon)^{\frac{1}{p}} \left(\int_{a+\varepsilon}^{b} \left| f(\tau, x_n(\tau), u(\tau)) - f(\tau, x_m(\tau), u(\tau)) \right|^q d\tau \right)^{\frac{1}{q}} =$$
$$= \frac{k}{\varepsilon} (b - a - \varepsilon)^{\frac{1}{p}} \| f(\tau, x_n(\tau), u(\tau)) - f(\tau, x_m(\tau), u(\tau)) \|_{L_q[a+\varepsilon, b]} =$$
$$= B \| f(\tau, x_n(\tau), u(\tau)) - f(\tau, x_m(\tau), u(\tau)) \|_{L_q[a+\varepsilon, b]},$$

where the constant is denoted by $B = \frac{k}{\varepsilon}(b - a - \varepsilon)^{1/p}$.

It is known that a metric can be defined via a norm, but not every metric satisfies the properties of a norm [11]. In the problem under consideration, the metric is introduced via the norm in the following form

$$\rho_{L_p[a+\varepsilon, b]}(x_n, x_m) = \frac{k}{\varepsilon} (b - a - \varepsilon)^{\frac{1}{p}} \| f(\tau, x_n(\tau), u(\tau)) - f(\tau, x_m(\tau), u(\tau)) \|_{L_q[a+\varepsilon, b]} =$$
$$= B \| f(\tau, x_n(\tau), u(\tau)) - f(\tau, x_m(\tau), u(\tau)) \|_{L_q[a+\varepsilon, b].} \tag{3.6}$$

For any fixed function $u(\tau) \in L_p[a+\varepsilon, b]$, and taking into account condition *4)* of the theorem, and using the contraction mapping principle, we have

$$B \| f(\tau, x_n(\tau), u(\tau)) - f(\tau, x_m(\tau), u(\tau)) \|_{L_q[a+\varepsilon, b]} \leq B\alpha \| x_n(\tau) - x_m(\tau) \|_{L_p[a+\varepsilon, b]} =$$

$$= B\alpha \|x_{m+l}(\tau) - x_m(\tau)\|_{L_p[a+\varepsilon,b]} =$$

$$= B\alpha \left\|x_{m+l}(\tau) - x_{m+(l-1)}(\tau) + x_{m+(l-1)}(\tau) - x_{m+(l-2)}(\tau) + ...\right.$$

$$\left. ... - x_{m+1}(\tau) + x_{m+1}(\tau) - x_m(\tau)\right\|_{L_p[a+\varepsilon,b]} \leq$$

$$\leq B\alpha \left(\left\|x_{m+l}(\tau) - x_{m+(l-1)}\right\|_{L_p[a+\varepsilon,b]} + \left\|x_{m+(l-1)}(\tau) - x_{m+(l-2)}\right\|_{L_P[a+\varepsilon,b]} + ...\right.$$

$$\left. ... + \|x_{m+1}(\tau) - x_m(\tau)\|_{L_p[a+\varepsilon,b]}\right).$$

Assuming $m > n$, which is possible, we obtain the following relation

$$\rho_{L_p[a+\varepsilon,b]}(x_n, x_m) \leq \rho_{L_p[a+\varepsilon,b]}(F^n x_0, F^m x_0) \leq \alpha^n \rho_{L_p[a+\varepsilon,b]}\left(x_0, F^{m-n}x_0\right) =$$

$$= \alpha^n \rho_{L_p[a+\varepsilon,b]}(x_0, x_{m-n}) \leq$$

$$\leq \alpha^n [\rho_{L_p[a+\varepsilon,b]}(x_0, x_1) + \rho_{L_p[a+\varepsilon,b]}(x_1, x_2) + ... + \rho_{L_P[a+\varepsilon,b]}(x_{m-n-1}, x_{m-n})] \leq$$

$$\leq \alpha^n \left[\rho_{L_p[a+\varepsilon,b]}(x_0, x_1) + \rho_{L_p[a+\varepsilon]}(Fx_0, Fx_1) + \rho_{L_p[a+\varepsilon,b]}\left(F^2x_0, F^2x_1\right) + ...\right.$$

$$\left. ... + \rho_{L[a+\varepsilon,b]}\left(F^{m-n-1}x_0, F^{m-n-1}x_1\right)\right] \leq$$

$$\leq \alpha^n \rho_{L_P[a+\varepsilon,b]}(x_0, x_1)\left[1 + \alpha + \alpha^2 + ... + \alpha^{m-n-1}\right] \leq$$

$$\leq \alpha^n \rho_{L_p[a+\varepsilon,b]}(x_0, x_1)\frac{1}{1-\alpha} \leq \frac{\alpha^n}{1-\alpha} B\alpha \|x_1(\tau) - x_0(\tau)\|_{L_p[a+\varepsilon,b]} =$$

$$= \frac{Br}{1-\alpha}\alpha^{n+1} = \frac{kr(b-a-\varepsilon)^{\frac{1}{p}}}{\varepsilon(1-\alpha)}\alpha^{n+1}, \tag{3.7}$$

since the expression in the square brackets is the sum of the terms of a monotonically decreasing geometric series, and we denote $r = \|x_1(\tau) - x_0(\tau)\|_{L_p[a+\varepsilon,b]}$, where $\varepsilon > 0$ is a fixed sufficiently small positive number. Hence, taking into account that $\alpha < 1$, we conclude: $\rho_{L_p[a+\varepsilon,b]}(x_n, x_m) = \rho_{L_p[a+\varepsilon,b]}(x_n, x_{n+l}) \to 0 \quad$ as $n \to \infty$.

Thus, the sequence obtained from equation (3.4), $\{x_n(\tau)\} \subset L_p[a+\varepsilon, b]$, is a Cauchy (fundamental) sequence. Since the space $L_p[a+\varepsilon, b]$ is complete, this sequence converges to some limit: $\lim_{n\to\infty} x_n(\tau) = x^0(\tau) \in L_p[a+\varepsilon, b]$.

In the problem under consideration, for any fixed small number $\varepsilon > 0$, the following integral inequality holds:

$$\int_{a+\varepsilon}^{b} \frac{K(t,\tau)}{\tau - a}\, d\tau \leq \int_{a+\varepsilon}^{b} \left|\frac{K(t,\tau)}{\tau - a}\right| d\tau \leq \frac{k}{\varepsilon}(b-a-\varepsilon).$$

This means that the linear bounded operator $I(\cdot) = \int_{a+\varepsilon}^{b} \frac{K(t,\tau)}{\tau - a}(\cdot)\, d\tau$ is continuous. It follows that the composition of continuous operators is also continuous

$$IF(\cdot, u) : L_p[a+\varepsilon, b] \to L_p[a+\varepsilon, b].$$

Then it follows that

$$IF(x^0, u) = \lim_{n\to\infty} IF(x_n, u) = IF\left(\lim_{n\to\infty} x_n, u\right) = \lim_{n\to\infty} x_{n+1} = x^0.$$

That is, starting from some index $N_1(\varepsilon_1)$, for all $n > N_1(\varepsilon_1)$, the following relation holds:

$$\|x_n - x^0\|_{L_p[a+\varepsilon,b]} < \varepsilon_1 \to 0, \quad \text{as } n \to \infty. \tag{3.8}$$

Now, let us establish that this solution $x^0(\tau) \in L_p[a+\varepsilon, b]$ is unique for a fixed function $u(\tau) \in L_p[a+\varepsilon, b]$.

Assume that under the conditions of the theorem, there exist two solutions $x^0(\tau),\ \ x^1(\tau) \in L_p[a+\varepsilon, b]$ such that $x^0(\tau) \neq x^1(\tau)$, and the following equalities hold

$$x^0 = IF(x^0, u) \quad \text{and} \quad x^1 = IF(x^1, u),$$

or explicitly,

$$x^0(t) = \phi(t) - \int_{a+\varepsilon}^{b} \frac{K(t,\tau)}{\tau - a} f(\tau, x^0(\tau), u(\tau))\, d\tau,$$

$$x^1(t) = \phi(t) - \int_{a+\varepsilon}^{b} \frac{K(t,\tau)}{\tau - a} f(\tau, x^1(\tau), u(\tau))\, d\tau.$$

That is, as mentioned above, starting from some indices $N_1(\varepsilon_1)$ and $N_2(\varepsilon_1)$, for all $n > N(\varepsilon_1) = \max\{N_1(\varepsilon_1), N_2(\varepsilon_1)\}$, the following inequalities hold:

$$\|x_n - x^0\|_{L_p[a+\varepsilon,b]} < \varepsilon_1 \to 0 \quad \text{and} \quad \|x_n - x^1\|_{L_p[a+\varepsilon,b]} < \varepsilon_1 \to 0, \quad \text{as } n \to \infty. \tag{3.9}$$

From this it follows that

$$\|x^1 - x^0\|_{L_p[a+\varepsilon,b]} = \|x^1 - x_n + x_n - x^0\|_{L_p[a+\varepsilon,b]} \leq$$

$$\leq \|x_n - x^1\|_{L_p[a+\varepsilon,b]} + \|x_n - x^0\|_{L_p[a+\varepsilon,b]} < 2\varepsilon_1 \to 0, \quad \text{as } n \to \infty. \tag{3.10}$$

Moreover, from equality (3.6), we have:

$$0 \leq \rho_{L_p[a+\varepsilon,b]}(x^0, x^1) = \frac{k}{\varepsilon}(b-a-\varepsilon)^{\frac{1}{p}} \left\| f(\tau, x^0(\tau), u(\tau)) - f(\tau, x^1(\tau), u(\tau)) \right\|_{L_q[a+\varepsilon,b]} =$$

$$= B \left\| f(\tau, x^0(\tau), u(\tau)) - f(\tau, x^1(\tau), u(\tau)) \right\|_{L_q[a+\varepsilon,b]} \leq B\alpha \left\| x^0 - x^1 \right\|_{L_p[a+\varepsilon,b]}.$$

Taking into account (3.9) or (3.10), we obtain:

$$0 \leq \rho_{L_p[a+\varepsilon,b]}(x^0, x^1) = \rho_{L_p[a+\varepsilon,b]}(IF(x^0,u), IF(x^1,u)) \leq$$

$$\leq B\alpha \left\| x^0 - x^1 \right\|_{L_p[a+\varepsilon,b]} \leq B\alpha \cdot 2\varepsilon_1 \to 0.$$

It is noted that the fixed small number $\varepsilon > 0$ and the number $\varepsilon_1 > 0$ satisfy $\varepsilon_1 \to 0$. Therefore, for this inequality to hold, it must be that $\rho_{L_p[a+\varepsilon,b]}(x^0, x^1) = 0$. Hence, $x^0(\tau) = x^1(\tau)$, i.e., the solution is unique.

Now, let us prove part *d)* of the theorem - **the solvability** of problem (2.5), (2.6)and the **equivalence** of its solution to the solution of problem (2.3), (2.4).

From the continuity of the norm and the existence of the solution to equation (2.6), it follows that the problem (2.5), (2.6) is solvable, since from relation (3.5) we have:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left| \int_a^{a+\varepsilon} \frac{K(t,\tau)}{\tau - a} f(\tau, x(\tau), u(\tau))\, d\tau \right| \leq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} k\gamma\varepsilon^2 = \lim_{\varepsilon \to 0} k\gamma\varepsilon \to 0.$$

It follows that problems (2.3), (2.4) and (2.5), (2.6) are equivalent in terms of their solution sets. Thus, the theorem is completely proven.

□

4. Analysis of the Obtained Results and Conclusion

The foundation of the obtained results lies in the formulated theorem concerning the study of an optimization problem for systems described by nonlinear integral equations with singularities at the boundary of integration. In the theorem's assumptions, the authors have, in their view, applied the weakest possible conditions that still allow for the verification of the kernel of the integrand and the nonlinear part. These conditions ensure the sufficiency of the theorem's assertions.

Optimization problems of this type arise in many applied areas, especially where the processes under investigation exhibit critical behavior: attaining large values, involving complex nonlinearities, being unstable, and undergoing abrupt changes under the influence of external and unaccounted (or immeasurable) factors - often resulting in inadequate model descriptions.

Such characteristics involving various types of singularities are diverse and demand in-depth research, the resolution of which is essential in applications [12, 13, 14]. It is noted that the results obtained here allow for directly computing an approximate solution to the singular nonlinear integral equation, and, within the set of solutions, determining a control function that ensures the minimum value of a given functional.

The proposed method offers several advantages over alternative approaches for solving nonlinear integral equations with boundary singularities. Unlike standard quadrature or collocation methods that may suffer from reduced accuracy near singular points, the combination of spectral collocation with orthogonal polynomials (e.g., Jacobi or Chebyshev) allows for high-order convergence even in the presence of singularities [8]. In comparison with classical B-spline collocation methods [6], the spectral approach achieves better approximation with fewer discretization points, which reduces computational cost while maintaining accuracy.

Furthermore, methods based on fractional splines or hp-version collocation [5, 9] provide flexibility in handling both weak and strong singularities, whereas standard iterative or projection methods may fail to converge or require fine discretization near the boundaries. The proposed approach also incorporates regularization techniques that stabilize the solution in ill-posed cases, which is a notable improvement over purely analytical or direct numerical schemes [1, 4].

Overall, the main advantages of the method can be summarized as follows:

- High accuracy near singularities without excessive mesh refinement.
- Reduced computational cost due to fewer required discretization points.
- Robust convergence for both weakly and strongly singular integral equations.
- Compatibility with regularization techniques for ill-posed problems.

These features make the method a reliable and efficient alternative to existing analytical and numerical approaches, particularly in the context of optimization problems governed by nonlinear integral equations with boundary singularities.

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