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Lie's classification of finite dimensional algebras of Vector Fields in $\mathbb{C}^N$

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Abstract

Brief proofs of classical results of Lie on finite dimensional subalgebras of vector fields in two and three variables are outlined. The results for algebras of maximal rank for vector fields in $\mathbb{C}^N$ — N arbitrary — are also given.

1. Introduction

The proofs of Lie's results on classification of finite dimensional subalgebras of vector fields in two and three variables [LiEn] continue to remain relatively inaccessible. The main reason seems to be that Lie did not make use of tools like representation theory and root systems because these tools were not available to him. They did become widely known a few years after Lie published his proofs but were not used by specialists in the subject.

If one uses these tools, the proofs simplify considerably and point a direction for the classification in higher number of variables.

We will give an outline of the main ideas and give references to detailed proofs. The applications of such classification are too many to be given in this paper. The interested reader could consult the books of Ibragimov [Ib1], [Ib2], Olver [Ol] and the article by Ibragimov "Sophus Lie and Harmony in Mathematical Physics" [Ib3].

2. Basic Definitions

A vector field defined on an open subset U of $\mathbb{R}^N$ is a vector valued function

$$V = (a_1, \cdots, a_N).$$

We assume that the functions a_i are C^∞. If these functions are real or complex analytic, then V is called a real analytic or a complex analytic vector field.

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We identify V with the directional derivative in the direction V. Thus if V is defined on an open subset U and f is a C^∞–function on U, then

$$V(f) = \sum a_i \frac{\partial f}{\partial x_i}.$$

If W is another vector field defined on U, then $[V, W](f)$ is, by definition,

$$V(W(f)) - W(V(f)).$$

With this definition the space of all vector fields on U becomes a Lie algebra.

If L_1, L_2 are two such algebras, Lie considered them to be equivalent if by a local change of variables, one can be transformed into the other.

Thus if L is an abelian algebra of dimension r defined on an open subset U of $\mathbb{R}^N$ and $X_1, \cdots, X_r$ is a basis of L and p a point in U with the tangent vectors $X_1(p), \cdots, X_r(p)$ linearly independent, then there is a change of variables $\widetilde{x}_1, \cdots, \widetilde{x}_N$ in which

$$X_i = \frac{\partial}{\partial \widetilde{x}_i}$$

on a neighbourhood of p.

Lie's classifications were based on the nature of generic orbits, whether the algebra was transitive or intransitive, whether it was primitive or imprimitive — meaning that it had an invariant foliation.

He classified complex analytic finite dimensional algebras of analytic vector fields in 2 variables completely and partially when $N = 3$.

A complete classification is claimed in Amaldi [Am1], [Am2], but to date no one has been able to verify this. Dubrovin [Du] has given a reason why the classification could not be complete for the solvable algebras.

It is therefore natural to classify semisimple and algebras with a proper Levi decomposition and compare with Lie and Amaldi.

We have attempted this in the papers [ABMS1], [ABMS2] and [AABMS]. Semisimple algebras of vector fields on $\mathbb{C}^N$ of maximal rank have been classified in [ABM2].

The method in all these classifications is algebraic and and it is based on the equality of algebraic and geometric ranks of Cartan subalgebras of semisimple Lie algebras of vector fields [ABM1].

2.1. Geometric Rank

Let L be a Lie algebra of vector fields defined on an open subset U of $\mathbb{R}^N$. The geometric rank of L is, by definition,

$$\max\{\dim(X(p)) \mid X \in L,\ p \in U\}.$$

Thus the rank of the algebra

$$\left\langle \frac{\partial}{\partial x},\ y\frac{\partial}{\partial x},\ \cdots,\ y^N \frac{\partial}{\partial x} \right\rangle$$

is 1 while its dimension is $(N + 1)$.

If L is a complex semisimple Lie algebra of vector fields and C is a Cartan subalgebra of L, then a basic fact is that its geometric rank and dimension coincide [ABM1].

2.2. Highest Weights

Equally we need to recall the definition of highest weights.

If B is a Borel subalgebra of a complex semisimple algebra L, and C is a Cartan subalgebra of L contained in B, then $B = C \oplus B'$.

If $\rho : L \longrightarrow \mathfrak{gl}(V)$ is a finite dimensional representation of L, a vector v in V is called a highest weight vector if $\rho(B')(v) = 0$ and v is a common eigenvector for $\rho(C)$.

The reader is referred to Kirillov [Ki] for all undefined terms in this paper.

3. Applications

Before giving applications, let us give for completeness, Lie's classification of finite dimensional subalgebras of vector fields on the line, up to local equivalence.

Sketch: If $X = f(x)\frac{d}{dx}$ commutes with $Y = g(x)\frac{d}{dx}$ and f is not identically zero, then Y must be a multiple of X. Therefore if L is a semisimple algebra of vector fields on the line, its Cartan subalgebras must be of dimension 1 and L must be isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. Hence L is generated by X, Y which are eigenvectors of $H = [X, Y]$ with nonzero and opposite eigenvalues. Therefore, in the coordinates in which the Cartan algebra is generated by $\frac{d}{dx}$, we may suppose that

$$X = \exp(x)\frac{d}{dx} \quad \text{and} \quad Y = \exp(-x)\frac{d}{dx}$$

(up to a constant). Moreover as the centralizer of X is X, up to constants, we see that there can be no highest weight vectors in any complement to L in any extension of L.

Now if L is nilpotent then it must be abelian of dimension 1 as a nilpotent algebra has a nonzero center.

If L is solvable and not nilpotent, then in the coordinate x in which the commutator of L is generated by $\frac{d}{dx}$ we see that if

$$Z = f(x)\frac{d}{dx}$$

is in L, then $f'(x)$ is a constant and therefore $f(x) = ax + b$.

This completes the local classification of finite dimensional algebras of vector fields on the line.

In the rest of the paper we give applications of the equality of geometric rank and dimension of Cartan subalgebra to the classification problem of finite dimensional algebra of vector fields.

Notation. A_n is the Lie algebra of the Lie group $\mathrm{SL}(n+1,\mathbb{C})$, B_n is the Lie algebra of the Lie group $\mathrm{SO}(2n+1,\mathbb{C})$ and D_n is the Lie algebra of the Lie group $\mathrm{SO}(2n,\mathbb{C})$. The algebra $\mathfrak{g}_2$ is a subalgebra of $\mathfrak{so}(8,\mathbb{C})$, given as fixed point locus of the automorphism of order 3, induced by the graph automorphism of the Dynkin diagram for D_4.

Proposition 3.1. *The only semisimple algebra of vector fields on $\mathbb{C}^2$ can only be of types A_1, A_2 or $A_1 \times A_1$.*

Proof. By the equality of the geometric and algebraic ranks of Cartan subalgebras of semisimple Lie algebras, the Cartan subalgebra of a semisimple algebra S of vector fields on $\mathbb{C}^2$ can be of dimension at most 2. Thus S can only be of type

$$A_1,\ A_1 \times A_1,\ A_2,\ B_2 \text{ or } \mathfrak{g}_2.$$

The algebra B_2 contains an algebra of type $A_1 \times A_1$ and $\mathfrak{g}_2$ contains an algebra of type A_2. The derived algebra of Borel subalgebras of both these types contains an abelian subalgebra of rank 2. Thus in suitable coordinates this algebra is

$$\langle \partial_x,\ \partial_y \rangle.$$

Therefore if B_2 could be realized as an algebra of vector fields on $\mathbb{C}^2$, then we would have $A_1 \times A_1 \subset B_2$ and $A_1 \times A_1$ could not have an invariant complement.

Similarly $A_2 \subset \mathfrak{g}_2 \subset V(\mathbb{C}^2)$ shows that A_2 could not have an invariant complement in $\mathfrak{g}_2$. □

Proposition 3.2. *If $L = S \ltimes R$ is a Levi decomposable algebra of vector fields on $\mathbb{C}^2$, then S must be isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.*

Proof. If S were of rank 2, then S must be isomorphic to $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$ or $\mathfrak{sl}(3,\mathbb{C})$ and the derived algebras of Borel subalgebras would have an abelian algebra of rank 2. Therefore, it would have no highest weight vectors in the radical $\mathcal{R}$. □

Proposition 3.3. *If L is a semisimple algebra of vector fields on $\mathbb{C}^N$ and it has a Cartan subalgebra of dimension N, then*

$$L \;\cong\; \bigoplus_{i=1}^{m} \mathfrak{sl}(k_i+1, \mathbb{C}), \quad k_1 + \cdots + k_m \;=\; N.$$

A proof of Proposition 3.3 is given in [ABM2].

3.1. Classification of semisimple algebras of vector fields on $\mathbb{C}^3$

Semisimple algebras of vector fields on $\mathbb{C}^3$ can only be of rank at most 3. Besides algebras of maximal rank, which are of type $A_1 \times A_1 \times A_1$, $A_1 \times A_2$ or A_3, we are left with types A_1, $A_1 \times A_1$, A_2, B_2 and $\mathfrak{g}_2$.

The main idea of canonical forms of these algebras is given in Section 2 of [AABMS].

The Lie algebra A_2 contains the Heisenberg algebra $[X, Y] \;=\; Z$, while B_2 contains $A_1 \times A_1$ and $\mathfrak{g}_2$ contains A_2.

Using the commutation relations given by the root system of $\mathfrak{g}_2$, we see that A_2 in $V(\mathbb{C}^3)$ cannot be extended to $\mathfrak{g}_2$ in $V(\mathbb{C}^3)$. The canonical forms of the Heisenberg algebra and $A_1 \times A_1$ together with the commutation relations given by the root systems are used in determining the fundamental root vectors and in finding all the embeddings of rank two algebra in $V(\mathbb{C}^3)$.

The reader is referred to [AABMS] for representations of all these algebras as vector fields in $V(\mathbb{C}^3)$. Clearly, if a semisimple algebra $\mathcal{S}$ has a representation in which the derived algebra of its Borel subalgebra has an abelian algebra of rank 3, then $\mathcal{S}$ can have no abelian extensions. It is by such considerations that representations are determined to be inequivalent or not.

3.2. Classification of Levi–decomposable subalgebras of vector fields on $\mathbb{C}^3$

Let L be a Levi–decomposable subalgebra of vector fields on $\mathbb{C}^3$, say

$$L \;=\; S \ltimes \mathcal{R},$$

where $S \subset V(\mathbb{C}^3)$ is semisimple. The semisimple part is known from Section 3.1. If the derived algebra of a Borel subalgebra of S has an abelian subalgebra of rank 3, then arguing as in the proof of Proposition 3.3 (given in [ABM2]) it follows that $\mathcal{R}$ has to be zero. Thus S can only be of types A_1, $A_1 \times A_1$, A_2 and B_2.

To find $\mathcal{R}$ one uses representation theory to build $\mathcal{R}$ and we must ensure that $\mathcal{R}$ is indeed a solvable algebra.

As there is no general theory to find solvable extensions of semisimple algebras, one has to argue case by case. This is work in progress.

In the preface of [LiEn] Lie writes that he has done all these classifications for applications in physics. He did not elaborate further on this.

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