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Output Controllability of Fractional-Order Systems with Generalized Caputo Proportional Operators

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Abstract

This paper investigates the output controllability of fractional-order systems governed by generalized Caputo proportional operators. For linear time-invariant systems, necessary and sufficient conditions are established via the fractional output controllability Gramian and a Kalman-type rank condition. For nonlinear fractional systems, sufficient conditions for output controllability are obtained using Schauder's fixed point theorem. A numerical example illustrates the theoretical results.

Keywords: Output controllability, Generalized fractional derivative, Kalman rank condition, Controllability Gramian, Nonlinear systems.
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1. Introduction

Control theory plays a fundamental role in the analysis and design of dynamical systems, aiming to steer a system from a given initial state to a desired final state through appropriate control inputs. Since its formalization by Kalman in the 1960s, the concept of controllability has been a cornerstone of modern systems theory, with broad applications in engineering, physics, and applied sciences.

In recent years, *fractional-order systems* have attracted increasing attention for their ability to capture memory and hereditary effects, which classical integer-order models often fail to describe [12, 14, 5, 16, 15]. These systems have proven effective in modeling phenomena such as viscoelasticity, electrical circuits, anomalous diffusion, and biological processes.

Among the various definitions of fractional derivatives, the *Caputo-type proportional fractional derivative* has emerged as a flexible and unifying framework, generalizing classical operators such as Riemann–Liouville, Hadamard, and Katugampola derivatives [2, 10, 11]. Defined via a shape function φ and parameters θ and ϱ, this operator allows a smooth transition between local and nonlocal behaviors, satisfying

$$\lim_{\theta\to 0^+} D^{\theta}_{\varrho,\varphi}\phi = \phi, \quad \lim_{\theta\to 1^-} D^{\theta}_{\varrho,\varphi}\phi = \phi',$$

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making it particularly suitable for systems exhibiting both static and dynamic responses.

Several recent studies have investigated controllability in fractional systems. For example, [6] established necessary and sufficient conditions for controllability and observability in linear proportional Caputo fractional systems, supported by numerical examples. Similarly, [13] analyzed tempered Caputo fractional systems and applied Kalman-type rank conditions to fractional versions of Chua's circuit and the Chua–Hartley oscillator. Other works extended classical Gramian and rank-based criteria to systems with distinct fractional derivatives or stochastic models [3, 1]. While these contributions represent significant progress, they mainly focus on *state controllability.*

In contrast, the problem of *output controllability*, relevant in practical applications where only certain system outputs are of interest, remains largely unexplored in the fractional-order setting [7]. To address this gap, the present work develops a rigorous framework for analyzing output controllability in both linear and nonlinear systems governed by the generalized proportional Caputo fractional derivative.

Specifically, we first consider linear time-invariant systems of the form

$$^{C}D_{\kappa}^{\alpha,\rho}x(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t),$$

and derive necessary and sufficient conditions for output controllability using a generalized fractional Gramian matrix and a Kalman-type rank criterion. We then extend the analysis to nonlinear systems

$$\begin{aligned} ^{C}D_{\kappa}^{\alpha,\rho}x(t) &= Ax(t) + Bu(t) + f(t, y(t), u(t)), \quad x(t_0) = x_0,\ t \geq t_0, && (1.1)\\ y(t) &= Cx(t), && (1.2)\end{aligned}$$

where f satisfies suitable continuity and boundedness assumptions. By leveraging Schauder's fixed point theorem and the output controllability of the associated linear system, sufficient conditions are established for the nonlinear case.

Overall, this study provides a rigorous theoretical framework for output controllability in fractional-order systems, bridging a gap in the literature and extending classical tools to generalized Caputo proportional operators.

2. Basic Notions

In this section, we present some basic definitions. For further details, the reader is referred to [11].

Definition 2.1. Let $\rho \in (0,1]$ and $\kappa \in C[a,b]$, where $a < b$, and assume that $\kappa'(t) > 0$. The *proportional differential operator* of order ρ of a function h, with respect to κ, is defined by

$$(D_{\kappa}^{\rho}h)(t) = (1-\rho)h(t) + \rho\frac{h'(t)}{\kappa'(t)}.$$

Definition 2.2. Let $\rho \in (0,1]$, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$, and $\kappa \in C[a,b]$, with $\kappa'(t) > 0$. The *left-sided fractional proportional integral* of a function h with respect to κ is defined as

$$({}_aI_{\kappa}^{\alpha,\rho}h)(t) = \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(\kappa(t)-\kappa(\tau))}(\kappa(t)-\kappa(\tau))^{\alpha-1}h(\tau)\kappa'(\tau)\,d\tau.$$

Definition 2.3. Let $\rho \in (0,1]$, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$, and $\kappa \in C[a,b]$, with $\kappa'(t) > 0$. Let h be an absolutely integrable function on $[a,b]$. The *generalized Caputo fractional proportional derivative* of h of order α with respect to κ, starting from a, is defined by

$$({}_a^CD_{\kappa}^{\alpha,\rho}h)(t) = \frac{1}{\rho^{m-\alpha}\Gamma(m-\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(\kappa(t)-\kappa(\tau))}(\kappa(t)-\kappa(\tau))^{m-\alpha-1}(D_{\kappa}^{m,\rho}h)(\tau)\kappa'(\tau)\,d\tau,$$

where $m = \lfloor \Re(\alpha) \rfloor + 1$, and

$$D_{\kappa}^{m,\rho}h = \underbrace{D_{\kappa}^{\rho}\cdots D_{\kappa}^{\rho}}_{m \text{ times}}h.$$

The Euler gamma function is given by

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t}\, dt, \quad \text{for } \Re(z) > 0.$$

Several well-known fractional derivatives can be obtained as special cases of Definition 2.3, summarized in Table 1.

Parameters	**Special case**	**Formula**
$\rho = 1,\ \kappa(t) = t$	Classical Caputo	$\dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_a^t (t-\tau)^{m-\alpha-1} h^{(m)}(\tau)\, d\tau$
$\rho = 1,\ \kappa(t) = \ln t$	Caputo–Hadamard	$\dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_a^t \left(\ln \frac{t}{\tau}\right)^{m-\alpha-1} \tau^{-m} h^{(m)}(\tau)\, d\tau$
$\rho \in (0,1),\ \kappa(t) = t$	Proportional Caputo	$\dfrac{1}{\rho^{m-\alpha}\Gamma(m-\alpha)} \displaystyle\int_a^t e^{\frac{\rho-1}{\rho}(t-\tau)} (t-\tau)^{m-\alpha-1} (D_t^{m,\rho} h)(\tau)\, d\tau$

Table 1: Special cases of the generalized proportional Caputo fractional derivative

Definition 2.4. Let $\alpha, \beta > 0$. The *two-parameter Mittag-Leffler function* of a matrix A is defined by

$$E_{\alpha,\beta}(A) = \sum_{n=0}^{\infty} \frac{A^n}{\Gamma(\alpha n + \beta)}.$$

For brevity, we write $E_\alpha(A) = E_{\alpha,1}(A)$.

3. Output Controllability Results

Let $\rho, \alpha \in (0,1]$, and let κ be a continuous, strictly increasing function on $[t_0, T]$, with $0 \le t_0 < T$. We consider the following linear time-invariant fractional system governed by the generalized Caputo fractional proportional derivative:

$$^{C}D_{\kappa}^{\alpha,\rho} x(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0, \tag{3.1}$$

$$y(t) = Cx(t), \tag{3.2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, and $y(t) \in \mathbb{R}^p$ is the output. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ define the system dynamics.

For simplicity, we denote the system (3.1)–(3.2) by the tuple (A, B, C).

Under the assumptions ensuring existence and uniqueness of solutions (see [6]), the solution of system (A, B, C) is given by

$$x(t) = e^{\frac{\rho-1}{\rho}(\kappa(t)-\kappa(t_0))} E_\alpha\Big(\rho^{-\alpha} A(\kappa(t)-\kappa(t_0))^\alpha\Big) x_0 + \int_{t_0}^{t} K(t,\tau) B u(\tau) \kappa'(\tau)\, d\tau, \tag{3.3}$$

where the kernel $K(t,\tau)$ is

$$K(t,\tau) = \rho^{-\alpha} e^{\frac{\rho-1}{\rho}(\kappa(t)-\kappa(\tau))} (\kappa(t)-\kappa(\tau))^{\alpha-1} E_{\alpha,\alpha}\Big(\rho^{-\alpha} A(\kappa(t)-\kappa(\tau))^\alpha\Big),$$

and $E_\alpha(\cdot)$, $E_{\alpha,\alpha}(\cdot)$ are the one- and two-parameter Mittag-Leffler matrix functions, respectively.

By substituting (3.3) into (3.2), the output is

$$\begin{aligned} y(t) &= e^{\frac{\rho-1}{\rho}(\kappa(t)-\kappa(t_0))} C E_\alpha\Big(\rho^{-\alpha} A(\kappa(t)-\kappa(t_0))^\alpha\Big) x_0 \\ &\quad + \rho^{-\alpha} C \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(\kappa(t)-\kappa(\tau))} (\kappa(t)-\kappa(\tau))^{\alpha-1} E_{\alpha,\alpha}\Big(\rho^{-\alpha} A(\kappa(t)-\kappa(\tau))^\alpha\Big) B u(\tau) \kappa'(\tau)\, d\tau \\ &\quad . \end{aligned} \tag{3.4}$$

Definition 3.1. The system (A,B,C) is *output controllable* on $[t_0,T]$ if, for any initial output $y(t_0)\in\mathbb{R}^p$ and any desired final output $y_d\in\mathbb{R}^p$, there exists a control function $u(t)$ such that $y(T)=y_d$.

The following theorems provide necessary and sufficient conditions for output controllability.

Theorem 3.2. *The system (A,B,C) is output controllable on $[t_0,T]$ if and only if the $p\times p$ matrix*

$$\begin{aligned}W(t_0,T)=\rho^{-\alpha}\int_{t_0}^{T} & e^{\frac{\rho-1}{\rho}(\kappa(T)-\kappa(\tau))}(\kappa(T)-\kappa(\tau))^{\alpha-1}\\ & \times CE_{\alpha,\alpha}\Big(\rho^{-\alpha}A(\kappa(T)-\kappa(\tau))^{\alpha}\Big)BB^{\top}E^{\top}_{\alpha,\alpha}\Big(\rho^{-\alpha}A(\kappa(T)-\kappa(\tau))^{\alpha}\Big)C^{\top}\kappa'(\tau)\,d\tau\end{aligned} \tag{3.5}$$

is nonsingular.

Proof.
- **Sufficiency.** If $W(t_0,T)$ is nonsingular, for any $(x_0,y_d)\in\mathbb{R}^n\times\mathbb{R}^p$, define

$$\begin{aligned}u(t)=B^{\top}&E^{\top}_{\alpha,\alpha}(\rho^{-\alpha}(\kappa(T)-\kappa(t))^{\alpha}A)C^{\top}W^{-1}(t_0,T)\\ &\times\Big(y_d-Ce^{\frac{\rho-1}{\rho}(\kappa(T)-\kappa(0))}E_{\alpha}(\rho^{-\alpha}(\kappa(T)-\kappa(0))^{\alpha}A)x_0\Big).\end{aligned}$$

Substituting this into (3.4) yields $y(T)=y_d$.

Necessity. Suppose, by contradiction, that $W(t_0,T)$ is singular while (A,B,C) is output controllable. Then there exists a nonzero $a\in\mathbb{R}^p$ such that $a^{\top}W(t_0,T)=0$. Premultiplying and postmultiplying $W(t_0,T)$ by $a^{\top}$ and a, we get

$$a^{\top}W(t_0,T)a=\rho^{-\alpha}\int_{t_0}^{T}e^{\frac{\rho-1}{\rho}(\kappa(T)-\kappa(\tau))}(\kappa(T)-\kappa(\tau))^{\alpha-1}(a^{\top}H(\tau))(a^{\top}H(\tau))^{\top}\kappa'(\tau)d\tau$$

,

where $H(\tau)=CE_{\alpha,\alpha}(\rho^{-\alpha}(\kappa(T)-\kappa(\tau))^{\alpha}A)B$.

Since $a^{\top}W_Da=0$ and the integrand is positive, it follows that $a^{\top}H(\tau)=0$ for all $\tau\in[t_0,T]$.

Choose $x_0=0$ and $y_d=a$. Then output controllability requires a control $u(t)$ such that

$$y(T)=\rho^{-\alpha}\int_{t_0}^{T}e^{\frac{\rho-1}{\rho}(\kappa(T)-\kappa(\tau))}(\kappa(T)-\kappa(\tau))^{\alpha-1}H(\tau)u(\tau)\kappa'(\tau)d\tau+Du(T)=a.$$

Premultiplying by $a^{\top}$ gives $a^{\top}a=0$, a contradiction. Hence $W(t_0,T)$ must be nonsingular. □

Theorem 3.3 (Kalman-type Rank Condition)**.** *The system (A,B,C) is output controllable on $[t_0,T]$ if and only if the $p\times((n+1)m)$ matrix*

$$\mathcal{C}_{\mathcal{O}}=[CB\ CAB\ CA^2B\ \dots\ CA^{n-1}B]$$

has full row rank, i.e.,

$$\operatorname{rank}(\mathcal{C}_{\mathcal{O}})=p.$$

Proof. By the Cayley–Hamilton theorem, the matrix function

$$(\kappa(t))^{\alpha-1}E_{\alpha,\alpha}(\rho^{-\alpha}(\kappa(t))^{\alpha}A)$$

can be expressed as a linear combination of $I,A,\dots,A^{n-1}$:

$$(\kappa(t))^{\alpha-1}E_{\alpha,\alpha}(\rho^{-\alpha}(\kappa(t))^{\alpha}A)=\sum_{k=0}^{n-1}c_k(\kappa(t))A^k,$$

where $c_k(\kappa(t))$ are scalar functions of $\kappa(t)$.

Substituting into the output expression at $t = T$, we have

$$y(T) - e^{\frac{\rho-1}{\rho}(\kappa(T)-\kappa(0))} C E_\alpha(\rho^{-\alpha}(\kappa(T)-\kappa(0))^\alpha A)x_0 = \sum_{k=0}^{n-1} CA^k B d_k,$$

where

$$d_k = \int_0^T c_k(\kappa(T)-\kappa(\tau)) e^{\frac{\rho-1}{\rho}(\kappa(T)-\kappa(\tau))} u(\tau)\kappa'(\tau)d\tau \quad (k = 0, \dots, n-1).$$

Because x_0 and y_d are arbitrary, the existence of a control $u(t)$ requires

$$\operatorname{rank}[CB\; CAB\; \dots\; CA^{n-1}B] = p.$$

Multiplying columns by nonzero scalars does not change the rank. This completes the proof. □

Remark 3.4. This theorem provides a purely algebraic test for output controllability, avoiding the need to compute the Gramian. In particular, any single-output system ($p = 1$) is automatically output controllable.

4. Output Controllability of Nonlinear Systems with Generalized Caputo Fractional Derivatives

In this section, we study the output controllability of the nonlinear system

$$ {}^C D_\kappa^{\alpha,\rho} x(t) = Ax(t) + Bu(t) + f(t, y(t), u(t)), \quad x(t_0) = x_0,\ t \geq t_0, \tag{4.1}$$
$$ y(t) = Cx(t), \tag{4.2}$$

where the matrices A, B, and C are as defined above, and the nonlinear function $f : J \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous, with $J = [t_0, T] \subset \mathbb{R}$ a fixed time interval.

Theorem 4.1 (Existence and Uniqueness)**.** *Let $J = [t_0, T]$. Assume that:*

(H1) $A \in \mathbb{R}^{n\times n}$ and B, C are constant matrices;

(H2) The function $f : J \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous;

(H3) There exists a constant $L > 0$ such that for all $(x_1, u_1), (x_2, u_2) \in \mathbb{R}^n \times \mathbb{R}^m$ and all $t \in J$,

$$\|f(t, Cx_1, u_1) - f(t, Cx_2, u_2)\| \leq L\left(\|x_1 - x_2\| + \|u_1 - u_2\|\right).$$

Then, for every $x_0 \in \mathbb{R}^n$ and $u \in C(J; \mathbb{R}^m)$, system (3.1) admits a unique solution $x \in C(J; \mathbb{R}^n)$.

To analyze system (4.1)–(4.2), we consider its linearized version:

$$ {}^C D_\kappa^{\alpha,\rho} x(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0,\ t \geq t_0, \tag{4.3}$$
$$ y(t) = Cx(t). \tag{4.4}$$

The solution of (4.1)–(4.2) can be expressed as

$$\begin{aligned}
y(t) &= e^{\frac{\rho-1}{\rho}(\kappa(t)-\kappa(t_0))} C E_\alpha\Big(\rho^{-\alpha} A(\kappa(t)-\kappa(t_0))^\alpha\Big)x_0 \\
&\quad + \rho^{-\alpha} C \int_{t_0}^t e^{\frac{\rho-1}{\rho}(\kappa(t)-\kappa(\tau))} (\kappa(t)-\kappa(\tau))^{\alpha-1} E_{\alpha,\alpha}\Big(\rho^{-\alpha} A(\kappa(t)-\kappa(\tau))^\alpha\Big) Bu(\tau)\kappa'(\tau)\, d\tau \\
&\quad + \rho^{-\alpha} C \int_{t_0}^t e^{\frac{\rho-1}{\rho}(\kappa(t)-\kappa(\tau))} (\kappa(t)-\kappa(\tau))^{\alpha-1} E_{\alpha,\alpha}(\rho^{-\alpha} A(\kappa(t)-\kappa(\tau))^\alpha) f(\tau, y(\tau), u(\tau))\kappa'(\tau) d\tau.
\end{aligned}$$

Let $\mathcal{C} := C(J; \mathbb{R}^p) \times C(J; \mathbb{R}^m)$ be a Banach space with norm

$$\|(y, u)\|_{\mathcal{C}} := \|y\| + \|u\|, \quad \|y\| = \sup_{t\in J} \|y(t)\|, \quad \|u\| = \sup_{t\in J} \|u(t)\|.$$

Theorem 4.2. *Assume that the continuous function f satisfies*

$$\lim_{\|(y,u)\|_{\mathcal{C}}\to\infty} \frac{\|f(t,y,u)\|_{\mathbb{R}^n}}{\|(y,u)\|_{\mathcal{C}}} = 0, \tag{4.5}$$

uniformly with respect to $t \in J$. Suppose furthermore that the linear system (4.3)*–*(4.4) *is output controllable. Then the nonlinear system* (4.1)*–*(4.2) *is also output controllable.*

Proof. Let $T > 0$ be such that the output controllability Gramian $W(0,T)$ is invertible. Let $x_0 \in \mathbb{R}^n$ be an arbitrary initial state and $y_d \in \mathbb{R}^p$ a desired final output.

Define an operator $\Phi : \mathcal{C} \to \mathcal{C}$ by

$$\Phi(z,v) = (y,u),$$

where

$$\begin{aligned} u(t) = {} & B^\top E_{\alpha,\alpha}^\top(\rho^{-\alpha}A(\kappa(T)-\kappa(t))^\alpha)C^\top W^{-1}(t_0,T)\Bigg(y_d - e^{\frac{\rho-1}{\rho}(\kappa(T)-\kappa(t_0))}CE_\alpha(\rho^{-\alpha}A(\kappa(T)-\kappa(t_0))^\alpha)x_0 \\ & - \rho^{-\alpha}C\int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(\kappa(T)-\kappa(\tau))}(\kappa(T)-\kappa(\tau))^{\alpha-1}E_{\alpha,\alpha}(\rho^{-\alpha}A(\kappa(T)-\kappa(\tau))^\alpha)f(\tau,y(\tau),u(\tau))\kappa'(\tau)d\tau\Bigg), \\ y(t) = {} & e^{\frac{\rho-1}{\rho}(\kappa(t)-\kappa(t_0))}CE_\alpha(\rho^{-\alpha}A(\kappa(t)-\kappa(t_0))^\alpha)x_0 \\ & + \rho^{-\alpha}C\int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(\kappa(t)-\kappa(\tau))}(\kappa(t)-\kappa(\tau))^{\alpha-1}E_{\alpha,\alpha}(\rho^{-\alpha}A(\kappa(t)-\kappa(\tau))^\alpha)Bu(\tau)\kappa'(\tau)d\tau \\ & + \rho^{-\alpha}C\int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(\kappa(t)-\kappa(\tau))}(\kappa(t)-\kappa(\tau))^{\alpha-1}E_{\alpha,\alpha}(\rho^{-\alpha}(\kappa(t)-\kappa(\tau))^\alpha A)f(\tau,y(\tau),u(\tau))\kappa'(\tau)d\tau. \end{aligned}$$

Using the growth condition (4.5), we can choose a ball $S(r) := \{(y,u) \in \mathcal{C} : \|(y,u)\| \le r\}$ such that Φ maps $S(r)$ into itself. Continuity of f ensures Φ is continuous, and by the Arzelà–Ascoli theorem, its image is relatively compact. Hence, Φ is completely continuous.

By Schauder's fixed-point theorem, Φ has a fixed point $(y,u) \in S(r)$. This fixed point corresponds to a control u steering the output from $y(0)$ to $y(T) = y_d$, proving output controllability of the nonlinear system. □

Example 4.3 (Chua–Hartley Oscillator)**.** Consider the generalized Caputo fractional proportional system for the Chua–Hartley oscillator [9]:

$$ {}^{C}D_{\kappa}^{\alpha,\rho}x_1(t) = \delta\left(x_2(t) + \frac{x_1(t) - y_1^3(t)}{7}\right), \tag{4.6}$$

$$ {}^{C}D_{\kappa}^{\alpha,\rho}x_2(t) = x_1(t) - x_2(t) + x_3(t) + u(t), \tag{4.7}$$

$$ {}^{C}D_{\kappa}^{\alpha,\rho}x_3(t) = -\frac{100}{7}x_2(t), \tag{4.8}$$

with output

$$y_1(t) = x_1(t) + x_3(t), \tag{4.9}$$

$$y_2(t) = 2x_1(t) - x_2(t). \tag{4.10}$$

The nonlinearity is

$$f(x_1,x_2,x_3) = \begin{pmatrix} \delta\frac{(x_1+x_3)^3}{7} \\ 0 \\ 0 \end{pmatrix},$$

which is continuous and satisfies the growth condition (4.5).

The linearized system around the origin is

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

with

$$A = \begin{pmatrix} \frac{\delta}{7} & \delta & 0 \\ 1 & -1 & 1 \\ 0 & -\frac{100}{7} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix}.$$

The output controllability matrix is

$$\mathcal{C}_O = \begin{pmatrix} 0 & \delta - \frac{100}{7} & \frac{\delta^2}{7} + \frac{100}{7} \\ -1 & 2\delta + 1 & \frac{2\delta^2}{7} - \delta + \frac{93}{7} \end{pmatrix},$$

with $\operatorname{rank}(\mathcal{C}_O) = 2 = \dim(\mathbb{R}^2)$. Thus, the linearized system is output controllable, and by the previous theorem, the nonlinear Chua–Hartley oscillator is output controllable.

Conclusion

We investigated the output controllability of fractional-order systems governed by generalized Caputo proportional operators. For linear systems, necessary and sufficient conditions were derived in terms of a fractional output controllability Gramian and a Kalman-type rank condition, thereby extending classical controllability results to the fractional-order setting. For nonlinear systems, under appropriate growth assumptions on the nonlinearity, sufficient conditions for output controllability were established using Schauder's fixed-point theorem.

These results contribute to a deeper theoretical understanding of control systems exhibiting memory effects and fractional dynamics. As a perspective for future research, the proposed output controllability framework may be extended to applied fractional-order models arising in real-world contexts. In particular, recent fractional epidemiological models incorporating vaccination, treatment, and memory effects—such as those developed for hepatitis B virus transmission [8] provide a natural motivation for studying controllability properties of biologically inspired systems under control interventions.

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